9 Linear Filters

Time series models are often of the form of one series being a filtered version of another.

**Def:** Let \( \{X(t), t \in \mathbb{Z}\} \) be a time series and let \( \{\beta_j, j \in \mathbb{Z}\} \) be a set of real numbers. If the time series \( \{Y(t), t \in \mathbb{Z}\} \) is defined by

\[
Y(t) = \sum_{j=-\infty}^{\infty} \beta_j X(t - j), \quad t \in \mathbb{Z},
\]

then we say that \( Y \) is a linear filtered version of \( X \) having the \( \beta \)'s as the filter coefficients.

**Ex1:** The moving average smoother

\[
Y(t) = \sum_{j=-K}^{K} \frac{1}{2K+1} X(t - j),
\]

has \( Y \) as a filtered version of \( X \) with coefficients

\[
\beta_j = \begin{cases} 
1/(2K + 1), & \text{if } |j| \leq K \\
0, & \text{if } |j| > K.
\end{cases}
\]

**Ex2:** The \( d \)th differencing operator

\[
Y(t) = X(t) - X(t - d), \quad t \in \mathbb{Z}
\]
has $Y$ as a filtered version of $X$ with coefficients

$$
\beta_j = \begin{cases} 
1, & \text{if } j = 0 \\
-1, & \text{if } j = d \\
0, & \text{otherwise}
\end{cases}
$$

**Ex3:** The $MA(1, \beta, \sigma^2)$ process

$$
X(t) = \epsilon(t) + \beta \epsilon(t - 1), \quad t \in \mathbb{Z}
$$

has $X$ as a filtered version of white noise with coefficients

$$
\beta_j = \begin{cases} 
1, & \text{if } j = 0 \\
\beta, & \text{if } j = 1 \\
0, & \text{otherwise}
\end{cases}
$$
9.1 The Filter Theorem

It is easy to prove a theorem that lets us find the properties of the output of a filter from the properties of the input to the filter and the coefficients of the filter:

Filter Theorem: If $Y$ is a filtered version of $X$ with filter coefficients \{$\beta_j, j \in Z$\}, then if $X$ is covariance stationary, so is $Y$ and

1. The spectral density $f_Y$ of the output is related to the spectral density $f_X$ of the input by

$$f_Y(\omega) = \left| g(e^{2\pi i \omega}) \right|^2 f_X(\omega), \quad \omega \in [0, 1],$$

where the complex valued polynomial $g(z)$ is given by

$$g(z) = \sum_{j=-\infty}^{\infty} \beta_j z^j, \quad z \in \mathbb{C}.$$

Note that $g(e^{2\pi i \omega})$ is called the frequency transfer function of the filter.

2. The autocovariance function $R_Y$ of the output is related to the autocovariance function $R_X$ of the input by

$$R_Y(v) = \sum_{k=-\infty}^{\infty} R_{\beta}(k) R_X(v - k), \quad v \in \mathbb{Z},$$
where the sequence $\{R_{\beta}(k), k \in Z\}$ is given by

$$R_{\beta}(k) = \sum_{j=-\infty}^{\infty} \beta_j \beta_{j+k}, \quad k \in Z.$$
9.2 Using the Filter Theorem

We can derive a large number of facts from this simple theorem:

1. Since white noise is covariance stationary with spectral density function $f_\epsilon(\omega) = \sigma^2$ and an MA process $X$ is a filtered version of white noise, we know immediately that $X$ is covariance stationary with spectral density function

$$f_X(\omega) = \left| 1 + \beta e^{2\pi i \omega} \right|^2 \sigma^2,$$

which a little algebra will show is the same as what we found in the previous topic by first principles. We can use the filter theorem also to find $R_X$. Further, we can easily find $R_X$ and $f_X$ for an MA process of order $q$

$$X(t) = \epsilon(t) + \beta_1 \epsilon(t - 1) + \cdots + \beta_q \epsilon(t - q),$$

where $\epsilon \sim WN(\sigma^2)$ as

$$f_X(\omega) = \left| 1 + \beta_1 e^{2\pi i \omega} + \cdots + \beta_q e^{2\pi i q \omega} \right|^2 \sigma^2,$$

and

$$R_X(v) = \begin{cases} \sigma^2 \sum_{k=0}^{q-|v|} \beta_k \beta_{k+|v|}, & \text{if } |v| \leq q \\ 0, & \text{if } |v| > q \end{cases}$$

2. If we think of each realization of a time series as a sum of sinusoids and $f(\omega)$ as the average value of the amplitude of the sinusoid of
frequency $\omega$ in the various realizations, then the function $g(e^{2\pi i \omega})$ is useful in determining what the filter does to the sinusoid of frequency $\omega$ in the input series. For example, for $d$th differencing, we have

$$g(e^{2\pi i \omega}) = 1 - e^{2\pi i (d) \omega}, \quad \omega \in [0, 1],$$

which is zero whenever $d\omega$ is an integer since $e^{2\pi i k} = 1$ for any integer $k$. Thus at any frequency that is a multiple of $1/d$, we have that $g$ is zero and thus so is $|g|^2$ and so is the spectral density of the output series. Thus the $d$th difference operator annihilates the sinusoids of frequencies that are multiple of $1/d$ in the input (which is what we intend it to do).

3. Another example of what $g$ tells us is the moving average smoother, for which

$$g_K(e^{2\pi i \omega}) = \frac{1}{2K + 1} \sum_{j=-K}^{K} e^{2\pi i j \omega},$$

where we placed the subscript $K$ on $g$ to indicate the moving average smoother parameter. This summation is a famous one and is called the Dirichlet kernel

$$D_K(\omega) = \sum_{j=-K}^{K} e^{2\pi i j \omega} = \frac{\sin\left((K + \frac{1}{2})2\pi \omega\right)}{\sin \pi \omega}.$$ 

Thus

$$f_Y(\omega) = \left(\frac{1}{2K + 1}\right)^2 |D_K(\omega)|^2 f_X(\omega).$$
Graphs of $|g_K|^2$ are given in the figure below for $K = 2, 4, 6, \ldots, 20$ (note that $g$ gets narrower as $K$ increases).
9.3 Low Pass, High Pass, and Band Pass Filters

We can categorize filters by their effect in the frequency domain. A filter whose frequency transfer function is zero for frequencies greater than some frequency $\omega_0$ only allows sinusoids of frequencies less than $\omega_0$ to “pass through” the filter. Thus the filter is called a low pass filter. A filter that allows only high frequencies to pass through is called a high pass filter, while if only frequencies in the band of frequencies $\omega_0 \pm \tau$ are allowed to pass through the filter, it is called a band pass filter.

Since $g$ is the Fourier transform of the $\beta$’s, we can find the $\beta$’s for a given $g$ (such as a band pass filter) by

$$\beta_j = \int_0^1 g(e^{2\pi i \omega}) e^{-2\pi j \omega} d\omega.$$ 

**Ex:** If we want a band pass filter at $\omega_0 \pm \tau$ we must integrate $g(e^{2\pi i \omega}) = 1$ for $\omega \in [\omega_0 - \tau, \omega_0 + \tau]$ or $\omega \in [1 - (\omega_0 + \tau), 1 - (\omega_0 - \tau)]$ and 0 otherwise since $f(\omega) = f(1 - \omega)$ for $\omega \in [0.5, 1]$. If you do the integral, you find

$$\beta_j = \begin{cases} 
4\tau, & \text{if } j = 0 \\
\frac{2}{\pi j} \cos 2\pi j \omega_0 \sin 2\pi j \tau, & \text{if } j \neq 0
\end{cases}$$
9.4 Truncating Filters

If we want to actually apply a filter to a data set we have a problem since we must truncate the filter

\[ Y(t) = \sum_{j=-M}^{M} \beta_j X(t-j), \]

since we don’t have an infinitely long stretch of data. So how do the true frequency transfer function \( g(e^{2\pi i \omega}) \) and that of the truncated filter \( g_M(e^{2\pi i \omega}) \) compare? One example is the band pass filter example above where basically what we are doing is approximating the function that is 1 for \( \omega \in \omega_0 \pm \tau \) and zero otherwise (and thus has a jump discontinuity at the end points of the band) by the first \( M \) terms of its Fourier series representation. A famous result in Fourier series is that the representation converges to the average of the two points at the discontinuities and that the representation becomes very wiggly within the frequency band. The usual solution to this problem is to apply a set of standard weights to the \( \beta \)'s that improve how well the resulting approximate transfer function fits the desired \( g \) (see page 83 of the text for more details).