14 Nonparametric Spectral Density Estimation

One of the major aims of time series analysis, particularly in the physical and geo- sciences, is the estimation of the spectral density function f. We saw in the previous section that the sample spectral density function $\hat{f}$ is asymptotically unbiased (so it has the correct basic shape as f) but is very wiggly (this is a consequence of the facts that $\hat{f}$'s at adjacent natural frequencies are asymptotically uncorrelated and that the variance of $\hat{f}$ does not decrease as n increases so that $\hat{f}$ gets no closer to f as the length of a realization increases).

There are two basic methods for getting an improved estimator of f:

1. **Parametric** or **Model-based** methods find an $ARMA$ type model for the data, then estimate the parameters of the model, and then substitute the estimates of the parameters into the formula for the true f, thus giving an estimate of f.

2. **Nonparametric** or **Non Model-based** methods smooth the sample spectral density $\hat{f}$ without making any assumption about a model for the data.

In this section we consider the nonparametric method.

14.1 Weighted Averages of $\hat{f}$

If we want to estimate f at one of the natural frequencies ω_k, a natural thing to do would be to use $\tilde{f}_M(\omega_k) =$ the average of $\hat{f}(\omega_k)$ and the M values of $\hat{f}$ on either side of $\hat{f}(\omega_K)$. This would give

$$\begin{aligned} \mathrm{E}(\tilde{f}_M(\omega_k)) &= \frac{1}{2M+1} \sum_{j=-M}^{M} f(\omega_{k-j}) \\ \mathrm{Var}(\tilde{f}_M(\omega_k) &= \frac{1}{(2M+1)^2} \sum_{j=-M}^{M} f^2(\omega_{k-j}). \end{aligned}$$

In any frequency band where f is not changing much, the expectation will be approximately f, while for places where f is changing, a bias will be introduced. Note that

$$\mathrm{Var}(\tilde{f}(\omega_k)) \leq \max_{-M \leq j \leq M} f^2(\omega_{k-j})/(2M+1) \to 0,$$

if if $M \to \infty$ as $n \to \infty$.

Of course, just doing straight moving averages is not the most efficient way to do the averaging. We need to do weighted averages

$$\tilde{f}_w(\omega_k) = \sum_{j=-M}^{M} w_j \hat{f}(\omega_{k-j}),$$

where the weight applied to $\hat{f}(\omega_j)$ for ω_j close to ω_k is larger than for one far awy from ω_k. (Note that we want to use symmetric weights, that is, $w_{-j} = w_j$, since we want equal weight applied to $\hat{f}$'s equally distant from the "target frequency" ω_k.)

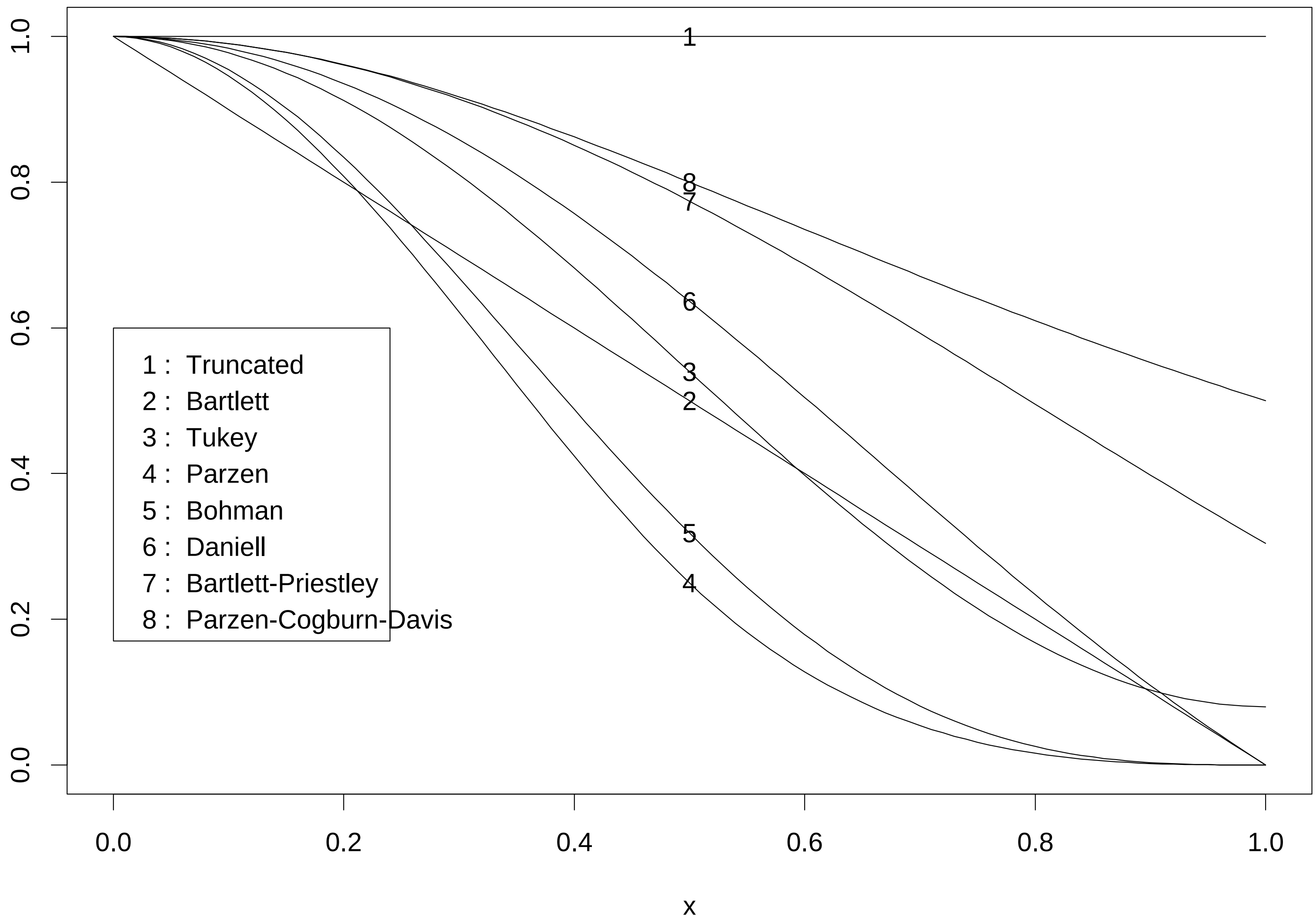
A useful way to determine a set of weights $w_0, \ldots, w_M$ is to "sample" a continuous function that looks like a weight function (high in the middle, falls off on either side of the middle) $\lambda(x)$, where $\lambda(x)$ is defined only on the interval $[-1, 1]$. Thus we let

$$w_j = \lambda(j/M), \qquad j = 0, 1, \ldots, M.$$

The table below gives eight standard weight functions (the first five are zero outside $[-1, 1]$ while the others are not).

Name	$\lambda(x)$
Truncated	1
Bartlett	$1-\|x\|$
Tukey	$0.54+0.46\cos\pi x$
Parzen	$\begin{cases} 1-6x^2+6\|x\|^3, & \|x\|\le 0.5 \\ 2(1-\|x\|)^3, & 0.5\le\|x\|\le 1 \end{cases}$
Bohman	$(1-\|x\|)\cos\pi x+\frac{1}{\pi}\sin\pi\|x\|$
Daniell	$\dfrac{\sin\pi x}{\pi x}$
Bartlett-Priestley	$\dfrac{3}{(\pi x)^2}\left(\dfrac{\sin\pi x}{\pi x}-\cos\pi x\right)$
Parzen-Cogburn-Davis	$\dfrac{1}{1+x^{2r}}$

Eight Weight Functions



14.2 Applying Kernels to Sample Autocovariances

Another way to smooth the periodogram is to notice that the sample spectral density can be written as

$$\hat{f}(\omega) = \sum_{v=-(n-1)}^{n-1} \hat{R}(v) e^{-2\pi i v \omega},$$

which is estimating the true spectral density function

$$f(\omega) = \sum_{v=-\infty}^{\infty} R(v) e^{-2\pi i v \omega}.$$

Thus there are two problems with $\hat{f}$:

1. **Truncation error:** Truncating the sum at $-(n-1)$ to $(n-1)$.
2. **Estimation error:** Having to use the estimator $\hat{R}(v)$ for $R(v)$.

A great deal of study has been done on “speeding up” the convergence of Fourier series by applying weights to the $\hat{R}(v)$’s which would give a new spectral estimator:

$$\tilde{f}_{n,k} = \sum_{v=-(n-1)}^{n-1} k_n(v) \hat{R}(v) e^{-2\pi i v \omega},$$

where the $k_n(v)$’s are a set of weights.

If we sample the same weight functions we looked before, that is, use $k_n(v) = \lambda(v/M)$, then we get

$$\hat{f}_{\lambda,M}(\omega) = \sum_{v=-\infty}^{\infty} \lambda(v/M)\hat{R}(v)e^{-2\pi i v\omega}.$$

This estimator can also be viewed as smoothing $\hat{f}$ as a little algebra shows

$$\hat{f}_{\lambda,M}(\omega) = \int_0^1 \Lambda_M(\omega - \tau)\hat{f}(\tau)d\tau,$$

where

$$\Lambda_M(\omega) = \sum_{v=-\infty}^{\infty} \lambda(v/M)e^{-2\pi i v\omega}.$$

Thus $\hat{f}_{\lambda,M}(\omega)$ is an integrated weighted average of $\hat{f}$.

Def: The function λ is called a **lag window generator**, the function $k_n(v) = \lambda(v/M)$ is called a **lag window**, and the function Λ_M is called a **spectral window**. If $\lambda(x) = 0$ outside of $[-1, 1]$, then

$$\hat{f}_{\lambda,M}(\omega) = \sum_{v=-M}^{M} \lambda(v/M)\hat{R}(v)e^{-2\pi i v\omega},$$

and the “scale parameter” M is called a **truncation point**.

14.3 Properties of Nonparametric Spectral Estimators

If $M \to \infty$ and $M/n \to 0$ as $n \to \infty$, then $\hat{f}_{\lambda,M}(\omega)$ is asymptotically unbiased and

$$\text{Var}(\hat{f}_{\lambda,M}(\omega)) \approx \frac{M}{n} f^2(\omega) \int \lambda^2(x)dx.$$

Further, the values of $\hat{f}_{\lambda,M}(\omega)$ at a set of fixed frequencies are asymptotically independent and if we let

$$\nu = \frac{2n}{M \int \lambda^2(x)dx},$$

then

$$\nu \hat{f}_{\lambda,M}(\omega)/f(\omega) \sim \chi^2_\nu.$$

