Example. The simplest r.v. is the **Bernoulli** r.v. For any event A set $X(s) = 1_A(s)$, $p = P(A)$. Then $P(X = 1) = P(\{s : 1_A(s) = 1\}) = P(A) = p$ and $P(X = 0) = 1 - p$. The pmf for X is

$$f_X(x) = (1-p)1_{\{0\}}(x) + p1_{\{1\}}(x) = p^x(1-p)^{1-x}1_{\{0,1\}}(x).$$

X is sometimes called "Bernoulli trial" with "$X = 1$" success,"$X = 0$" failure.

Definition. A **degenerate** r.v. X is one such that $P(X = x_0) = 1$ for some real x_0. (X is a constant on a set with probability one.)

Example. Consider n independent Bernoulli trials, that is independent events $A_1, \ldots, A_n$ each with probability p, and $Y =$ "number of successes", $Y = 1_{A_1}(s) + \ldots + 1_{A_n}(s)$. Then

$$f_Y(y) = P(Y = y) = p^y(1-p)^{n-y}\binom{n}{y}1_{\{0,1,\ldots,n\}}(y) \quad \text{("\textbf{binomial} distribution");}$$

$\binom{n}{y}$ gives the number of ways to place the y successes. The variable Y is a binomial random variable.

The number of successes in n independent Bernoulli trials with success probability p has a binomial(n,p) distribution.

Example. Consider independent flips of a coin with $A_n =$ "H on n^{th} flip" and $P(A_n) = p$. This is a sequence of Bernoulli trials. Write $T = \min\{n : 1_{A_n}(s) = 1\}$ for the number of flips heads appears first. Then

$$f_T(t) = P(T = t) = p(1-p)^{t-1}1_{\{1,2\ldots\}}(t) \quad \text{("\textbf{geometric} distribution").}$$

Definition. Let X be a r.v. The **cumulative distribution function** (cdf) of X is

$$F_X(x) = P(X \le x) \quad \text{for any } x \in \mathbb{R}.$$

We often write $X \sim F$ if F is the cdf of X or $X \sim$ binomial (geometric, normal, ...) distribution (if we study a specific distribution).

Example. The cdf of $X \sim$ Bernoulli(p) is a step function with two jumps, namely one at $x = 0$ (from 0 to $1 - p$) and one at $x = 1$ (from $1 - p$ to 1),

$$F_X(x) = (1-p)1_{[0,\infty)}(x) + p1_{[1,\infty)}(x).$$

Example (cont.) Consider $T \sim$ geometric(p) with pmf $f_T(t) = P(T = t) = p(1-p)^{t-1}$, $t = 1, 2, \ldots$. Find $F_T(t)$ and show that $\lim_{t\to\infty} F_t(t) = 1$.

Solution 1: $P(T \le t) = P(T = 1) + P(T = 2) + \ldots + P(T = t) = p(1-p)^0 + p(1-p) + p(1-p)^2 + \ldots + p(1-p)^{t-1} = p\sum_{i=0}^{t-1}(1-p)^i = p\frac{1-(1-p)^t}{1-(1-p)}$ (geometric series). The last term is $1 - (1-p)^t$ with $1 - p \in (0, 1)$ and therefore tends to 1 as $t \to \infty$.

Solution 2 (using the complement): for $t = 1, 2, \ldots$ we have $F_T(t) = P(T \le t) = 1 - P(T > t)$ with $P(T > t) = P$("no heads in the first t flips") $= (1-p)^t$ which yields $F_T(t) = 1 - (1-p)^t$.

Note that in both examples the cdf's are step functions.

Remark (without proof). X is a discrete r.v. $\iff$ its cdf is a step function.

Theorem. $F(x)$ is a cdf for some r.v. X if and only if

(i) $x \leq y \Rightarrow F(x) \leq F(y)$ (non-decreasing),
(ii) $F(x+h) \to F(x)$ as $0 < h \to 0$ (right continuous),
(iii) $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$ (probability measure).

Proof. "$\Leftarrow$" See any book on measure and probability theory.
"$\Rightarrow$"
(i) $x \leq y \Rightarrow$ "$X \leq x$" = "$\{s : X(s) \in (-\infty, x]\}$" $\subset$ "$X \leq y$".

(ii) Consider A_n = "$X \leq x + \frac{1}{n}$" which is decreasing in n. Applying the continuity theorem gives $\lim_{n\to\infty} P(A_n) = P(\cap A_n) = P(X \leq x)$.
Note that F is, in general, not left-continuous: $P(X \leq x - \frac{1}{n}) \to P(X < x)$ where, in general, $P(X < x) \neq P(X \leq x)$.

(iii) Use continuity and events "$X \leq -n$" (decreasing), "$X \leq n$" (increasing). $\square$

Example: Uniform(0,1) or **U(0,1)** distribution.
Consider $\mathcal{S} = (0, 1]$, the probability measure P on $\mathcal{S}$ given by the length of the interval, $P(a, b] = b - a$, and the random variable $U(s) = s$ for $s \in \mathcal{S}$. Note that $P((a, b]) = P((a, c] \cup (c, b]) = P((a, c]) + P((c, b])$ (additivity) and that $P \geq 0$ and $P(\mathcal{S}) = 1$. Then, for $0 < u \leq 1$

$$F_U(u) = P(U \leq u) = P(U(s) \in (0, u]) = P((0, u]) = u.$$

F_U is continuous and an antiderivative, $F_U(u) = \int_0^u 1\,dx$ for $u \in (0, 1]$. The derivative is $F_U'(u) = \frac{d}{du}u = 1$ on $(0, 1]$ and zero outside $(0, 1]$.

Definition. X is a **continuous** r.v. if its cdf is continuous. X is **absolutely continuous** if there exists a *probability density function* (pdf) $f_X(x)$ such that

$$F_X(b) - F_X(a) = \int_a^b f_X(x)\,dx \quad \text{for all}\, a < b.$$

Therefore, $F_X(x) = \int_{-\infty}^x f_X(t)\,dt$, $f_X(x) = \frac{d}{dx}F_X(x)$ for those x at which F' exists.

Example (uniform; cont.). If $0 \leq a < b \leq 1$ then $F_U(b) - F_U(a) = \int_a^b 1\,du$. For arbitrary $a < b$ we have $F_U(b) - F_U(a) = \int_a^b 1_{(0,1]}(u)\,du$. U is therefore absolutely continuous with pdf $f_U(u) = 1_{(0,1]}(u)$.

Example (lifetime; cont.). Let (again) $P(T > t) = e^{-t/10}$ for $t > 0$. Then, for $t > 0$, $F_T(t) = P(T \leq t) = 1 - P(T > t) = 1 - e^{-t/10}$, and $F_T(t) = 0$ for $t \leq 0$. This gives the pdf

$$f_T(t) = \frac{1}{10}e^{-t/10}1_{(0,\infty)}(t) \quad (\text{“}\textbf{exponential}\text{ distribution”}).$$

Note that F_X is continuous everywhere but f_X is not continuous at 0.

Remark. A function $f(x)$ is a pmf (or pdf) $\iff$ $f(x) \geq 0$ for all $x \in \mathbb{R}$ and $\sum_x f(x) = 1$ (or $\int_{-\infty}^{\infty} f(x)\,dx = 1$).

Notation. $\{x : f(x) > 0\}$ "**support** of f".