with $\int_{\mathbf{y}}^{y+\delta} f_Y(u)\,du \approx \delta\cdot f_Y(\mathbf{y})$, and an analogous approximation for the numerator, i.e.

$$\begin{aligned} P(a \le X \le b \mid y \le Y \le y+\delta) &\approx \frac{\delta \cdot \int_a^b f_{X,Y}(x,\mathbf{y})\,dx}{\delta \cdot f_Y(\mathbf{y})} = \int_a^b \frac{f_{X,Y}(x,y)}{f_Y(y)} dx \\ &= \int_a^b f_{X|Y}(x|y)\,dx \end{aligned}$$

Theorem. Consider (X,Y) with joint pdf $f_{X,Y}(x,y)$. Then
(i) X and Y are independent $\iff f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all x,y;
(ii) X and Y are independent $\iff f_{X|Y}(x|y) = f_X(x)$ for all x,y;
(ii) X and Y are independent $\iff f_{X,Y}(x,y) = g(x)h(y)$ for all x,y.

Proof. See discrete case.

Example. What is $f_{X,Y}(x,y) = c\cdot e^{-(x^2+y^2)/2}$ for some suitable $c>0$?

$f_{X,Y}(x,y) = c\cdot e^{-x^2/2}e^{-y^2/2} = \frac{1}{\sqrt{2\pi}}\cdot e^{-x^2/2}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}$ is the product of two independent N(0,1) pdf's. (Choose $c = 1/(2\pi)$.)

Theorem. If $X_1,\ldots,X_k$ independent, then $g_1(X_1),\ldots,g_k(X_k)$ are independent as well, for any (measurable) functions $g_1,\ldots,g_k$.

Proof. Let $B_i \subset \mathbb{R}$ and $A_i = g_i^{-1}(B_i)$, $i=1,\ldots,k$. By definition of g_i^{-1}: $X_i \in A_i \iff g_i(X_i) \in B_i$. Now use the assumed independence: $P(g_1(X_1) \in B_1,\ldots,g_k(X_k) \in B_k) = P(X_1 \in A_1,\ldots,X_k \in A_k) = \prod_{i=1}^k P(X_i \in A_i) = \prod_{i=1}^k P(g_i(X_i) \in B_i)$. □

Theorem. Suppose (X,Y) has joint pdf $f_{X,Y}(x,y)$ and let $T = X+Y$.
(i) The pdf for T is given by $f_T(t) = \int_{-\infty}^{\infty} f_{X,Y}(x,t-x)\,dx = \int_{-\infty}^{\infty} f_{X,Y}(t-y,y)\,dy$.
(ii) (**Convolution formula**) If X and Y are independent then

$$f_T(t) = \int_{-\infty}^{\infty} f_X(x)f_Y(t-x)\,dx = \int_{-\infty}^{\infty} f_X(t-y)f_Y(y)\,dy.$$

Proof of (i). We show that $P(T\le t) = \int_{-\infty}^{t} f_T(u)\,du$ with f_T from above. Integrate $f_{X,Y}(x,y)$ on $\{(x,y) : x+y \le t\}$ (and use the substitution $u = y+x$ with $dy = du$):

$$P(T\le t) = \int_{-\infty}^{\infty}\int_{-\infty}^{t-x} f_{X,Y}(x,y)\,dy\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{t} f_{X,Y}(x,u-x)\,du\,dx.$$

Changing the order of integration yields the desired form,

$$P(T\le t) = \int_{-\infty}^{t}\underbrace{\int_{-\infty}^{\infty} f_{X,Y}(x,u-x)\,dx}_{f_T(u)}\,du.$$

Example. Let $X \sim$ expo(β) (= gamma(1, β)), $Y \sim$ gamma(α,β), X,Y independent, $T = X+Y$.

$$\begin{aligned} f_T(t) &= \int_{-\infty}^{\infty} \frac{1}{\beta}e^{-\frac{x}{\beta}}\underbrace{1_{(0,\infty)}(x)}_{=1 \text{ if } x>0}\frac{1}{\Gamma(\alpha)\beta^\alpha}(t-x)^{\alpha-1}e^{-\frac{t-x}{\beta}}\underbrace{1_{(0,\infty)}(t-x)}_{=1 \text{ if } x<t}\,dx \\ &= \frac{e^{-\frac{t}{\beta}}}{\Gamma(\alpha)\beta^{\alpha+1}}\int_0^t (t-x)^{\alpha-1}\,dx. \end{aligned}$$

Substitute $t-x=u$, $x=t-u$, $dx=-du$, $x=0\to u=t$, $t\to 0$.

$$\begin{aligned} f_T(t) &= \frac{e^{-\frac{t}{\beta}}}{\Gamma(\alpha)\beta^{\alpha+1}}\int_t^0 u^{\alpha-1}\,(-du) = \frac{e^{-\frac{t}{\beta}}}{\Gamma(\alpha)\beta^{\alpha+1}}\int_0^t u^{\alpha-1}\,du = \frac{e^{-\frac{t}{\beta}}}{\Gamma(\alpha)\beta^{\alpha+1}}\frac{u^\alpha}{\alpha}\Big|_0^t \\ &= \frac{e^{-\frac{t}{\beta}}}{\Gamma(\alpha)\beta^{\alpha+1}}\frac{t^\alpha}{\alpha}. \end{aligned}$$

Since $\alpha\Gamma(\alpha)=\Gamma(\alpha+1)$, this is the pdf of the gamma$(\alpha+1,\beta)$ distribution.

Theorem. Consider X,Y with pdf $f_{X,Y}$, $U=g(X,Y)$, $V=h(X,Y)$ and assume that the transformation $(x,y)\mapsto\{g(x,y),h(x,y)\}$ is one-to-one and differentiable on a set A with $P\{(X,Y)\in A\}=1$. Then (U,V) has a pdf satisfying

$$f_{U,V}(u,v)=f_{X,Y}(x,y)\cdot\Big|\det\underbrace{\begin{pmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\end{pmatrix}}_{\text{Jacobian matrix}}\Big|$$

"$f_{U,V}(u,v)\partial u\partial v = f_{X,Y}(x,y)\partial x\partial y$" with $\dfrac{\partial x\partial y}{\partial u\partial v}=|\det(\text{Jacobian})|$.

(Recall: $\det\begin{bmatrix}a & b\\ c & d\end{bmatrix}=ad-bc$.)

Proof. This is change-of-variable in multivariable calculus. We want these integrals to match:

$$\int_{-\infty}^{b}\int_{-\infty}^{a} f_{U,V}(u,v)\,du\,dv=\int\int_{g^{-1}(-\infty,a]\cap h^{-1}(-\infty,b]} f_{X,Y}(x,y)\,dx\,dy.$$

Example (continued). Consider (X,Y) with joint pdf

$$f_{X,Y}(x,y)=\frac{1}{2}(\lambda^2e^{-\lambda(x+y)}+\mu^2e^{-\mu(x+y)})1_{(0,\infty)}(x)1_{(0,\infty)}(y).$$

Set $T=X+Y$, $W=\frac{X}{X+Y}$, then $W=\frac{X}{T}\iff X=TW$, $Y=T-X=T(1-W)$ and

$$\frac{\partial x\partial y}{\partial t\partial w}=\Big|\det\begin{pmatrix}\partial x/\partial t & \partial x/\partial w\\ \partial y/\partial t & \partial y/\partial w\end{pmatrix}\Big|=\Big|\det\begin{pmatrix}w & t\\ 1-w & -t\end{pmatrix}\Big|=|-t|=t,$$

$$\begin{aligned} f_{T,W}(t,w) &= t\cdot f_{X,Y}\{wt,(1-w)t\}\\ &= t\cdot\frac{1}{2}[\lambda^2e^{-\lambda\{wt+(1-w)t\}}+\mu^2e^{-\mu t}]1_{(0,\infty)}(tw)1_{(0,\infty)}\{t(1-w)\} \end{aligned}$$

with $1_{(0,\infty)}(tw)=1$ if $t>0,w>0$, and $1_{(0,\infty)}\{t(1-w)\}=1$ if $t>0$, $1-w>0\iff w<1$. Therefore, $1_{(0,\infty)}(tw)1_{(0,\infty)}\{t(1-w)\}=1_{(0,\infty)}(t)1_{(0,1)}(w)$, and

$$f_{T,W}(t,w)=f_T(t)f_W(w)=\frac{t}{2}\{\lambda^2e^{-\lambda t}+\mu^2e^{-\mu t}\}1_{(0,\infty)}(t)\cdot 1_{(0,1)}(w)$$

i.e. T and $W\sim$ U$(0,1)$ are independent since $f_{T,W}$ factors.