Example (capture/recapture sampling).
The fishery department wants to know how many fish of a certain type (N) are in a lake. In order to do so, they catch 200, tag them and release them back to the lake. A few weeks later they again catch 200, 23 are tagged. Let Y = "# tagged fish in the sample", $Y \sim$ hypergeometric with $n = 200$, $M = 200$ (tagged fish in the lake), N.
In order to estimate N we use $p = M/N \iff N = M/p$ and plug in an estimator for p: $\hat{p} = Y/n = 23/200 \Rightarrow \hat{N} = M/\hat{p} = 200 \cdot 200/23 = 1739$ (our estimate for N). Can we safely rule out that there are at least 2,000 in the lake?

No: if there are 2,000 (or more) fish of the relevant type in the lake then it is quite likely to have 23 or more tagged fish in the sample: $P_{2000}(Y \geq 23) = 1 - \sum_{y=0}^{22} \binom{200}{y}\binom{1800}{200-y}/\binom{2000}{200} \approx 1 - \sum_{y=0}^{22} \binom{200}{y} 0.1^y \cdot 0.9^{200-y} \approx 0.27$. [We used the binomial approximation and $M/N = 200/2,000 = 0.1$.]

3.3 Gamma and related distributions

The integral $\int_0^\infty x^{z-1}e^{-x}dx$ usually cannot be expressed. For $z > 0$ we call it **gamma function**,

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx.$$

From calculus we know:

Theorem. (i) $\Gamma(z+1) = z \cdot \Gamma(z)$,
(ii) $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \ldots$,
(iii) $\Gamma(1/2) = \sqrt{\pi}$.

Note: $\Gamma(z)$ can be evaluated for $z = 0.5, 1, 1.5, 2, 2.5, \ldots$, e.g. $\Gamma(4) = 3\Gamma(3) = 3 \cdot 2 \cdot \Gamma(2) = 3 \cdot 2 \cdot 1 \cdot \Gamma(1) = 3!$ since $\Gamma(1) = \int_0^\infty e^{-x}dx = 1$. Similarly, $\Gamma(1.5) = 0.5 \cdot \Gamma(0.5) = \sqrt{\pi}/2$.

Definition.
(i) The **gamma**(α, β) distribution ($\alpha > 0, \beta > 0$) has pdf

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} 1_{(0,\infty)}(x).$$

Special case: $\alpha = m/2$ ($m = 1, 2, \ldots$), $\beta = 2$: χ^2_m distribution
(**chi-square** distribution with m degrees of freedom).

(ii) The **beta**(α, β) distribution ($\alpha > 0, \beta > 0$) has pdf

$$f(x) = B(\alpha,\beta)^{-1} x^{\alpha-1}(1-x)^{\beta-1} 1_{(0,1)}(x), \quad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \text{ (beta function)}.$$

Theorem. $Z \sim N(0,1) \Rightarrow Z^2 \sim \chi^2_1$. (Proof: Section 2.1).

Theorem. Let $X \sim$ gamma(α, β).
(i) $EX = \alpha\beta$, $EX^2 = \alpha(\alpha+1)\beta^2$ ($\Rightarrow Var(X) = EX^2 - E^2X = \alpha\beta^2$),
$EX^m = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+m-1)\beta^m$ for positive integer m.
(ii) The mgf is $M_X(t) = (1-\beta t)^{-\alpha}$.

Proof. (ii) exercise.

(i) Use (ii), or directly,

$$\begin{aligned}
EX^m &= \int_0^\infty x^m \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{m+\alpha-1} e^{-x/\beta} dx \quad [t = \frac{x}{\beta}, dt = 1/\beta\, dx] \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty (\beta t)^{m+\alpha-1} e^{-t} \beta\, dt \;=\; \frac{\beta^m}{\Gamma(\alpha)} \int_0^\infty t^{m+\alpha-1} e^{-t} dt \\
&= \frac{\beta^m}{\Gamma(\alpha)} \Gamma(m+\alpha) = \frac{\beta^m}{\Gamma(\alpha)} (m+\alpha-1)\Gamma(m+\alpha-1) \;=\; \ldots \\
&= \frac{\beta^m}{\Gamma(\alpha)} (m+\alpha-1)(m+\alpha-2)\cdots\alpha\Gamma(\alpha). \quad \square
\end{aligned}$$

Remark. $E\chi^2_m = \alpha\beta = m$ $(\alpha = m/2, \beta = 2)$; $\;Var\,\chi^2_m = \alpha\beta^2 = m/2 \cdot 2^2 = 2m$.

Example. The beta(1,1) distribution is the uniform(0,1) distribution: we have, for $x \in (0,1)$,

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} x^0 (1-x)^0 = \frac{1\Gamma(1)}{\Gamma(1)^2} = \frac{1}{\Gamma(1)} = 1.$$

Remark. If $U \sim$ beta(α,β) then $V = 1-U \sim$ beta(β,α). To see this let $u, v \in (0,1)$. Then

$$\begin{aligned}
f_V(v) &= \left|\frac{du}{dv}\right| f_U(u) = \left|\frac{d(1-v)}{dv}\right| f_U(1-v) \\
&= B(\alpha,\beta)^{-1}(1-v)^{\alpha-1}\{1-(1-v)\}^{\beta-1} = B(\alpha,\beta)^{-1} v^{\beta-1}(1-v)^{\alpha-1}.
\end{aligned}$$

3.4 Lifetimes and reliability

Definition. The **hazard** rate of a life with lifetime X is the instantaneous rate of failure given the current length, namely

$$h(t) = \lim_{\delta\to 0} \frac{1}{\delta} P(t < X \le t+\delta \mid X > t), \quad t > 0.$$

Theorem. Suppose X is a lifetime with hazard rate $h(t)$, $t > 0$, cdf F_X, pdf f_X. Then $h(t) = f_X(t)/\{1 - F_X(t)\}$ and

$$F_X(t) = 1 - \exp\{-\int_0^t h(x)\, dx\}, \quad t > 0.$$

Proof. $\frac{1}{\delta} P(t < X \le t+\delta \mid X > t) = \frac{\frac{1}{\delta}\{F_X(t+\delta) - F_X(t)\}}{1 - F_X(t)} \overset{\delta\to 0}{\longrightarrow} \frac{f_X(t)}{1-F_X(t)}$.
For the second statement use $F_X(0) = 0$ and $h(t) = -\frac{d}{dt}\log(1 - F_X(t))$ (chain rule): $-\int_0^t h(x)\, dx = \log\{1 - F_X(x)\}\big|_0^t = \log\{1 - F_X(t)\} \Leftrightarrow \exp\{-\int_0^t h(x)\, dx\} = 1 - F_X(t)$. $\square$

Example. Some objects (e.g. people) have a decreasing–increasing failure rate, i.e. a "U-shaped" hazard function. Choose, for example, $h(t) = \frac{\gamma_1}{\beta_1} t^{\gamma_1 - 1} + \frac{\gamma_2}{\beta_2} t^{\gamma_2 - 1}$ with $\gamma_1 < 1 < \gamma_2$. Then

$$F_X(t) = 1 - \exp[-\int_0^t \{\frac{\gamma_1}{\beta_1} x^{\gamma_1 - 1} + \frac{\gamma_2}{\beta_2} x^{\gamma_2 - 1}\} dx] = 1 - \exp(-\frac{1}{\beta_1} t^{\gamma_1} - \frac{1}{\beta_2} t^{\gamma_2}).$$