

Homework 6 – Solutions

Problems 1 - 3:

3.18 Let $Y \sim \text{NB}(r, p)$ and $G \sim \text{gamma}(r, 1)$.

$$M_{pY}(t) = \left(\frac{p}{1 - (1-p)e^{pt}} \right)^r \rightarrow \left(\frac{1}{1-t} \right)^r \quad (p \rightarrow 0)$$

holds since $\frac{f(p)}{g(p)} = \frac{p}{1 - (1-p)e^{pt}} \rightarrow \frac{1}{1-t}$ as $p \rightarrow 0$, which can be proved with L'Hôpital:

$$\frac{f'(p)}{g'(p)} = \frac{-1}{-e^{pt} + (1-p)te^{pt}} = \frac{1}{e^{pt} - (1-p)te^{pt}} \rightarrow \frac{1}{e^0 - 1 \cdot t \cdot e^0} = \frac{1}{1-t}.$$

3.28 (c) $f(x) = B(\alpha, \beta)^{-1} e^{(\alpha-1)\log x} e^{(\beta-1)\log(1-x)} 1_{(0,1)}(x)$

(d) $f(x) = e^{-\lambda} \frac{1}{x!} 1_{\{0,1,\dots\}}(x) e^{x \log \lambda}$

(e) $f(x) = \binom{r+x-1}{x} p^r e^{x \log(1-p)} 1_{\{0,1,\dots\}}(x)$, $p \in (0, 1)$

3.39 (a) $P(Z \geq 0) = \int_0^\infty \frac{1}{\pi(1+x^2)} dx = \frac{1}{\pi} \tan^{-1}(x) \Big|_0^\infty = \frac{1}{\pi} \left(\frac{\pi}{2} - 0 \right) = \frac{1}{2}$

$$\stackrel{3.38}{\Rightarrow} P(X \geq x_{1/2}) = P(X \geq \sigma z_{1/2} + \mu) = P(X \geq \mu) = \frac{1}{2}.$$

(b) $1/4 = P(Z \geq z_{1/4}) = P(X \geq \sigma z_{1/4} + \mu) = P(X \geq \sigma + \mu)$ if $z_{1/4} = 1$. This holds since

$$P(Z \geq 1) = \int_1^\infty \frac{1}{\pi(1+x^2)} dx = \frac{1}{\pi} \tan^{-1}(x) \Big|_1^\infty = \frac{1}{\pi} \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{1}{4}.$$

Problem 4: Consider a lifetime T with hazard function

$$h(t) = 1 - \frac{e^{-t}}{(1+e^{-t})^2}, \quad t \geq 0.$$

Find the cdf $F(t)$ of T and check your answer by showing that $F(t)$ is indeed a distribution function.

$$\int_0^t h(x) dx = \int_0^t \left\{ 1 - \frac{e^{-x}}{(1+e^{-x})^2} \right\} dx = t - \int_0^t \frac{e^{-x}}{(1+e^{-x})^2} dx = t - \frac{1}{1+e^{-x}} \Big|_0^t = t - \frac{1}{1+e^{-t}} + \frac{1}{2},$$

$$F(t) = 1 - \exp\left(-\int_0^t h(x) dx\right) = 1 - \exp\left(-t + \frac{1}{1+e^{-t}} - \frac{1}{2}\right), \quad t \geq 0.$$

$F(t)$ is continuous, increasing, $F(t) = 0$ for $t \leq 0$ and $\lim_{t \rightarrow \infty} F(t) = 1$.

Problem 5: Consider the random vector (X, Y) with distribution given by the below table.

		X		
		1	2	3
Y	1	0.1	0.2	0.0
	2	0.3	0.0	0.1
	3	0.2	0.1	

(a) Complete the table by filling in the missing value and by specifying the marginal distributions of X and Y .

		X			
		1	2	3	Σ
Y	1	0.1	0.2	0.0	0.3
	2	0.3	0.0	0.1	0.4
	3	0.2	0.1	0	0.3
		0.6	0.3	0.1	1

(b) Show that X and Y are dependent: $P(X = 1, Y = 1) = 0.1 \neq P(X = 1)P(Y = 1) = 0.18$.

(c) Determine the distribution of random variables \tilde{X} and \tilde{Y} that have the same marginal distributions as X and Y but are independent.

		\tilde{X}			
		1	2	3	Σ
\tilde{Y}	1	0.18	0.09	0.03	0.3
	2	0.24	0.12	0.04	0.4
	3	0.18	0.09	0.03	0.3
		0.6	0.3	0.1	1