Homework 5 – Solutions

Problem 1: Consider a r.v. X and the transformation $Y = e^X$ with $E(Y) = 1$. Can we determine the sign of $E(X)$? If no: why not? If yes: find it.

Yes, EX is negative. Apply Jensen's inequality to $Y = g(X) = e^X$ with g convex:

$$1 = EY = Ee^X = Eg(X) \geq g(EX) = e^{EX} \iff EX \leq \log 1 = 0.$$

Problem 2: Suppose you toss a coin about which you know that, on average, in one third of all cases it shows "Tails" and in two thirds of all cases "Heads". Consider the following situation: each time you observe "Heads" you roll a fair die and note the number of dots x. If the coin shows "Tails" you do *not* roll the die but, instead, note $x = 0$ for the observed number of dots. Find the moment generating function of $X =$ "number of dots".

$P(X = 0) = 1/3$, $P(X = i) = \frac{2}{3} \cdot \frac{1}{6} = \frac{1}{9}$ for $i = 1, \ldots, 6$,

$$Ee^{tX} = e^{t0}\frac{1}{3} + \frac{1}{9}\sum_{i=1}^{6} e^{ti} = \frac{1}{3} + \frac{1}{9}(e^t + e^{2t} + \ldots + e^{6t}).$$

Problems 3 to 6:

2.30 (a) $Ee^{tX} = \int_0^c e^{tx}\frac{1}{c}\,dx = \frac{1}{ct}e^{tx}\Big|_0^c = \frac{e^{tc}-1}{ct}$.

(b) $Ee^{tX} = 2(cte^{tc} - e^{tc} + 1)/(c^2t^2)$.

(c) $Ee^{tX} = e^{\alpha t}/(1 - t^2\beta^2)$ for $-1/\beta < t < 1/\beta$.

2.34

The odd moments are the same for both distributions, namely zero, since both distributions are symmetric around zero.

Second and fourth moments of the standard normal distribution: $EX^2 = 1$, $EX^4 = 3$ (use, for example, the mgf).

$EY^2 = (\sqrt{3})^2 \cdot \frac{1}{6} + (-\sqrt{3})^2 \cdot \frac{1}{6} + 0 \cdot \frac{2}{3} = 1$, $EY^4 = (\sqrt{3})^4 \cdot \frac{1}{6} + (-\sqrt{3})^4 \cdot \frac{1}{6} + 0 \cdot \frac{2}{3} = 2 \cdot 9/6 = 3$.

3.3

Write B for "busy", F for "free".
Last 4 seconds: $\underline{\text{B}}$ $\underline{\text{F}}$ $\underline{\text{F}}$ $\underline{\text{F}}$ (The "busy" is necessary because otherwise you could walk earlier.)
First 3 seconds: _ _ _ Anything but F F F is okay. (F F F would allow you to walk earlier.)
$\Rightarrow$ P(wait exactly four seconds) = _ _ _ $\underline{\text{B}}$ $\underline{\text{F}}$ $\underline{\text{F}}$ $\underline{\text{F}}$ $= \{1 - (1-p)^3\} \cdot p \cdot (1-p)^3$.

3.6 (a) $X =$ "sum of successes" $\sim$ binomial(n, p) distribution with $n = 2000$ and $p = 0.01$.

(b) $P(X < 100) = \sum_{k=0}^{99} \binom{99}{k} 0.01^k 0.99^{99-k}$.

(c) Poisson approximation with $\lambda = np = 20$: $P(X < 100) = \sum_{k=0}^{99} = e^{-20}20^k/k! \doteq 1$.
(> `ppois(99,2000)` in R yields "1".)