Homework 4 – Solutions

Problems 1 - 5:

2.8 (a) $F(x) = (1-e^{-x})1_{[0,\infty)}$ has limits $0/1$ (as $x \to -\infty/\infty$) and is increasing and continuous. For $x \geq 0$ we set $y = 1 - e^{-x}$ and solve this with respect to x. This gives $x = \log \frac{1}{1-y}$.

2.11 (b) $Y = |X|$, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $F_Y(y) = P(|X| \leq y) = F_X(y) - F_X(-y)$,

$$f_Y(y) = F'_X(y) - F'_X(-y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} + \frac{1}{\sqrt{2\pi}} e^{-y^2/2} = \sqrt{\frac{2}{\pi}} e^{-y^2/2},$$

$$EY = \int_0^\infty y \sqrt{\frac{2}{\pi}} e^{-y^2/2}\, dy = \sqrt{\frac{2}{\pi}}, \quad EY^2 = 1, \quad Var(Y) = EY^2 - (EY)^2 = 1 - \frac{1}{\pi}.$$

2.14 (a) Let X continuous and non-negative. Then

$$\begin{aligned} \int_0^\infty \{1 - F_X(x)\}\, dx &= \int_0^\infty P(X > x)\, dx = \int_0^\infty \int_x^\infty f_X(y)\, dy\, dx \\ &= \int \int f_X(y) 1_{(x,\infty)}(y) 1_{(0,\infty)}(x)\, dy\, dx = \int_0^\infty \int_0^y f_X(y)\, dx\, dy \\ &= \int_0^\infty \int_0^y dx\, f_X(y)\, dy = \int_0^\infty y\, f_X(y)\, dy = EX. \end{aligned}$$

2.16

$$ET = \int_0^\infty \{1 - F_T(x)\}\, dx = \int_0^\infty \{\alpha e^{-\lambda x} + (1-\alpha) e^{-\mu x}\}\, dx = \frac{\alpha}{\lambda} + \frac{1-\alpha}{\lambda}.$$

2.17

a) $\frac{1}{2} = \int_{-\infty}^m f(x)\, dx = \int_{-\infty}^m 3x^2 1_{(0,1)}(x)\, dx = x^3 \Big|_0^m = m^3 \Rightarrow m = 2^{-1/3}$.

b) $m = 0$ since f is symmetric about 0.

Problem 6: Let X be a random variable with range $\{0, 1, 2, \ldots\}$. The discrete version of the formula from problem 2.14 in the textbook, i.e. 2.14 (b), can be written

$$EX = \sum_{k=1}^\infty P(X \geq k).$$

(The proof is analogous to that of 2.14 (a).) Use this formula to solve the following problem.

A fair dice is thrown n times. The sample space is $S = \{1, 2, \ldots, 6\}^n$, the outcomes are of the form $s = (s_1, \ldots, s_n) \in S$. Let Y_n denote the largest of the results thrown, i.e. Y_n is a r.v. with $Y_n(s_1, \ldots, s_n) = \max_{1 \leq k \leq n} s_k$.

a) Find EY_n and show

$$\lim_{n\to\infty} E(Y_n) = 6.$$

We have $EY_n = \sum_{k=1}^6 P(Y_n \geq k) = \sum_{k=1}^6 \{1 - P(Y_n < k)\} = 6 - \sum_{k=1}^5 P(Y_n \leq k) = 6 - \sum_{k=1}^5 (\frac{k}{6})^n \to 6 - 0 = 6$ as $n \to \infty$.

b) Show

$$\lim_{n\to\infty} Var(Y_n) = 0.$$

From (a) we know $0 \le P(Y_n \le k) \to 0$ for $k = 1, \ldots, 5$ and therefore $P(Y_n = k) \to 0$ for $k = 1, \ldots, 5$. Also

$$6 = \lim_{n\to\infty} EY_n = \lim \sum_{k=1}^{5} kP(Y_n = k) + 6 \lim P(Y_n = 6) = 6 \lim P(Y_n = 6)$$

which yields $\lim P(Y_n = 6) = 1$. We use this and the above to obtain

$$Var\, Y_n = E(Y_n - EY_n)^2 = \sum_{k=1}^{5} (k - EY_n)^2 P(Y_n = k) + (6 - EY_n)^2 P(Y_n = 6) \to 0.$$

Problem 7: Let X be a discrete random variable that takes on values $0, 1, 2$ with probability 1/2, 3/8, 1/8, respectively.

a) Find $E(X)$: $EX = \sum_x xp(x) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{1}{8} = \frac{5}{8}$.

b) Find the pmf of $Y = X^2$ and use it to find $E(Y)$: pmf: $p_Y(0) = \frac{1}{2}$, $p_Y(1) = \frac{3}{8}$, $p_Y(4) = \frac{1}{8}$,

$$EY = \sum_y y p_Y(y) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{3}{8} + 4 \cdot \frac{1}{8} = \frac{7}{8}.$$

c) Use the definition of $E\{g(X)\}$, where $g(X)$ is a function of X, to find $E(X^2)$ and compare to your answer in part (b).

$$EX^2 = \sum_x x^2 p_X(x) = 0^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{1}{8} = \frac{7}{8}.$$

d) Find $Var(X)$: $Var(X) = EX^2 - (EX)^2 = \frac{7}{8} - (\frac{5}{8})^2 = \frac{31}{64}$.