Homework 3 – Solutions

Problem 1.

a) Consider $f(x) = cx^{-3}1_{(1,\infty)}(x)$ with $c \in \mathbb{R}$. Determine c such that f is a probability density function and find the corresponding cumulative distribution function F.

$1 = \int_1^\infty c/x^3\,dx = -c/(2x^2)|_1^\infty = c/2 \Rightarrow c = 2.$
$F(x) = 0$ for $x < 1$; $x \geq 1$: $F(x) = \int_1^x 2/y^3\,dy = -1/y^2|_1^x = 1 - 1/x^2.$

b) Consider $f(x) = c\sin(x)1_{(0,\pi)}(x)$ with $c \in \mathbb{R}$. Determine c such that f is a probability density function and find the corresponding cumulative distribution function F.

$1 = c\int_0^\pi \sin x\,dx = c(-\cos x)|_0^\pi = 2c \Rightarrow x = 1/2.$
$x \in (0,\pi):\ F(x) = \int_0^x 1/2\,\sin y\,dy = 1/2 - 1/2\cos x.$

Problem 2. Consider a distribution P given by a pdf $f(\cdot)$ with $f(x) = 0$ for all $x \leq 0$. Assume F is a continuous antiderivative of f with $F'(x) = f(x)$ for all $x > 0$ and $F(x) = 0$ for all $x \leq 0$. Then, by definition,

$$P((a,b]) = \int_a^b f(x)\,dx = F(b) - F(a) \quad \text{for } 0 \leq a \leq b < \infty.$$

Further assume that there is a constant $\lambda > 0$ with

$$(*) \qquad \lim_{0<h\to 0} \frac{P((t,t+h]\,|\,(t,\infty))}{h} = \frac{1}{\lambda} \quad \text{for all } t > 0.$$

a) Show that $f(x) = \frac{1}{\lambda}e^{-x/\lambda}$ for all $x > 0$ (exponential distribution).

The right-hand side is

$$\begin{aligned}
\lim_{0<h\to 0} \frac{P((t,t+h]\,|\,(t,\infty))}{h} &= \lim_{0<h\to 0} \frac{1}{h}\frac{P((t,t+h])}{P((t,\infty))} \\
&= \lim_{0<h\to 0} \frac{P((t,t+h])}{h}\frac{1}{1-F(t)} \\
&= \lim_{0<h\to 0} \frac{F(t+h)-F(t)}{h}\frac{1}{1-F(t)} \\
&= \frac{f(t)}{1-F(t)},
\end{aligned}$$

which is $1/\lambda$ by assumption. Therefore

$$\begin{aligned}
\frac{1}{\lambda} = \frac{f(t)}{1-F(t)} = -\frac{d}{dt}\log(1-F(t)) \quad &\iff \quad \frac{d}{dt}\log(1-F(t)) = -\frac{1}{\lambda} \\
&\iff \quad \log(1-F(t)) = -\frac{1}{\lambda}t + c \\
&\iff \quad 1 - F(t) = e^{-t/\lambda}\tilde{c} \\
&\iff \quad F(t) = 1 - e^{-t/\lambda}\tilde{c}
\end{aligned}$$

and the derivative is $f(t) = F'(t) = \frac{1}{\lambda}e^{-t/\lambda}\tilde{c}$ with $\tilde{c} = 1$ ($\int f = 1$).

b) If P is the distribution of the lifetime of an individual or object, then $P((t,\infty))$ is the probability that the individual survives at least t time units. Given this, how would you interpret equation $(*)$? Is it reasonable to assume that humans' lifetimes are exponentially distributed?

No, the rate should depend on t (current age).

Problem 3. Consider a random variable X with a Poisson(1) distribution and the transformation $Y = (X+1)^{-1}$. Specify the probability mass function $f_Y(y)$ of Y. Check your answer by verifying that $f_Y(y)$ satisfies the criteria for a probability mass function.

$P(X = x) = e^{-1}1^x/x! = e^{-1}/x!$, $x \in \mathbb{N}$; possible values for Y: $y \in I = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$.

$$P(Y = y) = P(\frac{1}{X+1} = y) = P(X+1 = \frac{1}{y}) = P(X = \frac{1}{y} - 1) = \frac{e^{-1}}{(1/y-1)!}.$$

$$\sum_{y \in I} P(Y = y) = \sum_{i=0}^{\infty} e^{-1}\frac{1}{i!} = e^{-1}\sum_{i=0}^{\infty}\frac{1}{i!} = e^{-1}e = 1.$$

Problems 4 - 6

1.53

a) $F_Y(y) = 1 - y^{-2}1_{[1,\infty)}(y)$ which is constant for $y \leq 1$. For $y > 1$ we have $F'(y) = 2/y^3 > 0$. F_Y is therefore non-decreasing. Further, $F_y(y) \to 0/1$ as $y \to -\infty/+\infty$.

b) The pdf is $2/y^3 1_{(1,\infty)}(y)$ (see above).

c) For $Z = 10(Y-1)$ we have $F_Z(z) = P(10(Y-1) \leq z) = \{1 - (z/10+1)^{-2}\}1_{(0,\infty)}(z)$.

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$$P(V \leq v) = \begin{cases} 0 & \text{if } v < 5 \\ 1 - e^{-2} & \text{if } 5 \leq v < 6 \\ 1 - e^{-v/3} & \text{if } 6 \leq v \end{cases}$$

2.1

a) $f_Y(y) = 14y(1 - y^{1/3})1_{(0,1)}(y)$,

b) $f_Y(y) = \frac{7}{4}\exp(-\frac{7}{4(y-3)})1_{(3,\infty)}(y)$,

c) $f_Y(y) = 15\sqrt{y}(1-\sqrt{y})^2 1_{(0,1)}(y)$.