Homework 2 – Solutions

Problems 1-3:

1.36
We have 10 independent Bernoulli trials with success probabilty $p = 1/5$. The number of successes "X" therefore has a binomial distribution, $X \sim \text{bin}(10, 1/5)$. This gives

$$P(X \geq 2) = 1 - P(X < 2) = 1 - \sum_{k=0}^{1} \binom{10}{k} \left(\frac{1}{5}\right)^k \left(\frac{4}{5}\right)^{10-k} = 1 - \left(\frac{4}{5}\right)^{10} - 10\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^9$$
$$\approx 0.624,$$

$$P(X \geq 2 | X \geq 1) = \frac{P(X \geq 2 \text{ and } X \geq 1)}{P(X \geq 1)} = \frac{P(X \geq 2)}{P(X \geq 1)} = \frac{0.624}{1 - (4/5)^{10}} \approx 0.699.$$

1.39
Let $P(A) > 0$ and $P(B) > 0$. Suppose $A \cap B = \emptyset$ and A, B independent. This implies $0 = P(\emptyset) = P(A \cap B) = P(A)P(B) > 0$, which is a contradiction.

1.44 A student guessing the answers to 20 multiple choice questions can be regarded as a sequence of 20 independent Bernoulli trials with success probability $p = 1/4$. Then $X =$ "number of successes" (again) has a binomial distribution and

$$P(10 \leq X \leq 20) = \sum_{k=10}^{20} \binom{20}{k} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{20-k}.$$

Problem 4. Suppose there were seven road accidents in one week. What is the probability that they all happened on different days?

The number of ways to draw a sample of size 7 (with replacement) out of the population $\{1, 2, \ldots, 7\}$ is 7^7. There are 7! ways to permute the numbers $1, 2, \ldots, 7$, which is the event of interest. The probability of ths event is therefore $7!/7^7$ (which is approximately 0.006).

Problem 5. Consider

$$\mathcal{S} = \{1, 2, 3, 4\}; \; P = \text{discrete uniform distribution on } \mathcal{S};$$
$$A = \{1, 2\},\ B = \{1, 3\},\ C = \{1, 4\}.$$

Are the events A, B and C independent? If yes, in which sense? Explain your answer.

The events are pairwise independent but not mutually independent:
$P(AB) = 1/4 = P(A)P(B)$, $P(AC) = 1/4 = P(A)P(C)$, $P(BC) = 1/4 = P(B)P(C)$, $P(ABC) = 1/4 \neq P(A)P(B)P(C) = 1/8$.

Problem 6. Consider the probability mass function of the binomial distribution

$$f(j) = b_j(n, p) = \binom{n}{j} p^j (1-p)^{n-j}, \quad j = 0, 1, \ldots, n,\ n \in \mathbb{N},\ 0 < p < 1.$$

Show that the binomial distribution can sometimes be approximated by the Poisson distribution. Formally, for $p = p_n \in (0, 1)$, $\lim_{n\to\infty} p_n = 0$, $\lim_{n\to\infty} np_n = \lambda > 0$ holds

$$\lim_{n\to\infty} b_j(n, p_n) = e^{-\lambda} \frac{\lambda^j}{j!} \quad \text{for every } j \in \{0, 1, \ldots\}.$$

Proof. We will use the hint provided with the question as follows: $\lim_{n\to\infty}\left(1+\frac{-\lambda+\varepsilon_n}{n}\right)^n = e^{-\lambda}$, with $\varepsilon_n = \lambda - np_n \to 0$. Then

$$\begin{aligned}
\binom{n}{j} p^j (1-p)^{n-j} &= \frac{n(n-1)\cdots(n-j+1)}{j!}\frac{1}{n^j}(np)^j(1-p)^n(1-p)^{-j} \\
&= \frac{n}{n}\cdot\frac{n-1}{n}\cdot\dots\cdot\frac{n-j+1}{n}\cdot\frac{1}{j!}(np)^j\left(1-\frac{np-\lambda+\lambda}{n}\right)^n(1-p)^{-j} \\
&\to 1\cdot 1\cdot\dots\cdot 1\cdot\frac{1}{j!}\cdot\lambda^j\cdot e^{-\lambda}\cdot 1.
\end{aligned}$$

□