Homework 1 – Solutions

Problem 1: Prove the finite version of DeMorgan's Laws. Let $A_1, \ldots, A_n \subset \mathcal{S}$. Prove that

$$\text{(i)}\ (\cup_{i=1}^n A_i)^c = \cap_{i=1}^n A_i^c, \qquad \text{(ii)}\ (\cap_{i=1}^n A_i)^c = \cup_{i=1}^n A_i^c.$$

Proof of (i):

$x \in (\cup_{i=1}^n A_i)^c \iff x \notin \cup_{i=1}^n A_i \iff x \notin A_i$ for any $i \iff x \in A_i^c$ for every i
$\iff x \in \cap_{i=1}^n A_i^c$.

Proof of (ii):

Let $A_1^c, \ldots, A_n^c \subset \mathcal{S}$. By (ii), $(\cup_{i=1}^n A_i^c)^c = \cap_{i=1}^n A_i^{cc} = \cap_{i=1}^n A_i$. Taking the complement on both sides gives (ii).

Problems 2-5:

1.13

Assume A and B disjoint. Then $P(A \cup B) = P(A) + P(B) = \frac{1}{3} + \frac{3}{4} > 1$, which is a contradiction.

1.24

(b) $P(\text{A wins}) = \sum_{i=1}^\infty P(E_i) = \sum_{i=1}^\infty (1-p)^{2i-1}p = p\sum_{i=0}^\infty (1-p)^{2i} = p\frac{1}{1-(1-p)^2} = \frac{1}{2-p}$.

(a) $P(\text{A wins}) = 2/3$ follows from (b) with $p = 1/2$.

(c) $P(\text{A wins}) = \frac{1}{2-p}$ is increasing in p, $p \in (0,1)$. Therefore $\frac{1}{2-p} > \frac{1}{2-0} = 1/2$.

1.26

Set $X =$ "number of rolls until 6 occurs". Then "$X > 5$" means "no 6 on the first five positions, i.e. there are 5 possibilities (1, 2, 3, 4 or 5) to fill each of these positions. Since all results are equally likely, we obtain the probability by counting:

$$P(X > 5) = \frac{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5}{6 \cdot 6 \cdot 6 \cdot 6 \cdot 6} \approx 0.41$$

1.34

Litter 1: $L_1 = \{BBG\}$, litter 2: $L_2 = \{BBBGGG\}$. Select litter and offspring at random.

(a) $P(B) = P(BL_1) + P(BL_2) = P(B|L_1)P(L_1) + P(B|L_2)P(L_2) = 2/3 \cdot 1/2 + 3/5 \cdot 1/2 = 19/30$.

(b) Use the Bayes formula and part (a) to obtain

$$P(L_1|B) = \frac{P(L_1 B)}{P(B)} = \frac{P(B|L_1)P(L_1)}{P(B)} = \frac{2/3 \cdot 1/2}{19/30} = \frac{10}{19}.$$

Problem 6. You roll two fair dice hoping for a total of 7 (probability 1/6). After the roll one die is hidden, but you see that the other is a 4. What is the (updated) chance that you have a total of 7? What is the (updated) chance that the total is n $(n = 1, 2, \ldots)$?

Solution:

Set $X =$ total, E is the event "one die shows 4". That $P(X = 7|E) = 2/11$ can be obtained by counting:

11	12	13	**14**	15	16
21	22	23	**24**	25	26
31	32	33	<u>**34**</u>	35	36
41	**42**	<u>**43**</u>	44	**45**	**46**
51	52	53	**54**	55	56
61	62	63	**64**	65	66

More generally:

$$P(X = n|E) = \frac{P(X = n \text{ and } E)}{P(E)} = \frac{2/36}{11/36} = 2/11 \quad \text{for } n = 5, 6, 7, 9, 10,$$

$$P(X = 8|E) = 1/11.$$