Instructions: Answer all the following questions. This exam is closed book. You may use the table of distributions from your book. Be sure to justify all your answers (including yes or no questions). Start each problem on a separate sheet of paper. Good luck.

1. Let $X$ be a random variable with probability density function $f(x|\theta)$ given below:

\[
\begin{align*}
f(x|0) &= 2x, \quad \text{if } 0 \leq x \leq 1, \\
&= 0, \quad \text{otherwise},
\end{align*}
\]

\[
\begin{align*}
f(x|1) &= 3x^2, \quad \text{if } 0 \leq x \leq 1, \\
&= 0, \quad \text{otherwise},
\end{align*}
\]

\[
\begin{align*}
f(x|2) &= xe^{x}, \quad \text{if } 0 < x < 1, \\
&= 0, \quad \text{otherwise}.
\end{align*}
\]

(a.) (16 points) Find the most powerful level 0.0975 test of $H_0: \theta = 0$ versus $H_1: \theta = 1$. Obtain its power.

(b.) (16 points) Is the test obtained in part (a.) the uniformly most powerful level 0.0975 test for $H_0: \theta = 0$ versus $H_1: \theta \in \{1, 2\}$? Justify your answer.

Note: If you could not find the MP test in part (a.), use the test with rejection region $(0, \sqrt{0.0975})$ to answer part (b.).

2. Let $X_1, \ldots, X_n$ be a random sample from the distribution with p.d.f.

\[
f(x|\theta) = \frac{1}{\theta} x^{\frac{1}{\theta} - 1}, \quad 0 < x < 1, \quad \theta > 0.
\]

(a.) (16 points) Obtain the UMP test of $H_0: \theta \leq 1$ versus $H_1: \theta > 1$ and show that it has a one-sided rejection region in terms of the sufficient statistic (found in Test 1), $\sum_{i=1}^{n} \log(X_i)$.

(b.) (14 points) Use the Central Limit Theorem to determine the sample size $n$ so that the level 0.05 UMP test of $H_0: \theta \leq 1$ versus $H_1: \theta > 1$ has power of 0.90 when $\theta = 1.2$.

Hint: You may use the fact that the distribution of $-\log(X_i)$ is exponential ($\theta$).
3. Let $X_1, \ldots, X_n$ be a random sample from the Weibull distribution with density

$$f(x|\theta) = \frac{2x}{\theta} \exp\left\{-\frac{x^2}{\theta}\right\}, \quad x > 0, \quad 0 < \theta < \infty.$$ 

Consider testing $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$.

(a.) (16 points) Obtain likelihood ratio statistic for testing $H_0$ versus $H_1$.

(b.) (10 points) Obtain the rejection region for the level $\alpha$ likelihood ratio test of $H_0$ versus $H_1$. Express the rejection region in terms of a commonly used distribution. (Try to do this using an exact distribution. If you cannot, use an approximate distribution.)

(c.) (12 points) Does a UMP level $\alpha$ test exist for testing $H_0$ versus $H_1$? Explain why or why not.