1. Let $X_1, \ldots, X_n$ be independent random variables from the distribution with density
\[ f(x|\theta) = \frac{1}{\theta} x^{\frac{1}{\theta} - 1}, \quad 0 < x < 1, \quad \theta > 0. \]

(a.) Obtain the level $\alpha$ likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Show that the rejection region is of the form
\[ R = \{ x : -\sum \log(x_i)/\theta_0 < a_1 \text{ or } -\sum \log(x_i)/\theta_0 > a_2 \} \]
and discuss how to obtain $a_1$ and $a_2$.

(b.) Obtain the level $1 - \alpha$ confidence interval found by inverting the likelihood ratio test.

(c.) Suppose now that $\theta$ has an inverse gamma distribution with density
\[ \pi(\theta) = \frac{1}{\Gamma(a)b^a} \left( \frac{1}{\theta} \right)^{a+1} e^{-b/(\theta)}, \quad \theta > 0 \]
where $a > 0$ and $b > 0$ are known. Show how to find a $1 - \alpha$ Bayes HPD credible set for $\theta$.

(d.) Find the general form of a level $1 - \alpha$ confidence interval for $\theta$ based on the pivot $-\sum \log(x_i)/\theta$. Then derive the shortest such interval.
Recall the hint from Test 2: The distribution of $Y_i = -\log(X_i)$ is exponential ($\theta$).

2. Let $X_1, \ldots, X_n$ be a random sample from the distribution with density
\[ f(x|\theta) = \frac{2x}{\theta^2}, \quad 0 \leq x \leq \theta. \]

(a.) Obtain a moment estimator of $\theta$.

(b.) Obtain the m.l. estimator of $\theta$.

(c.) Obtain the U.M.V.U. estimator of $\theta$.

(d.) Suppose now that $\theta$ has a Pareto distribution with density
\[ \pi(\theta) = \frac{1}{\theta^2}, \quad \theta \geq 1. \]
Obtain the Bayes estimator of $\theta$ for squared error loss.
3. Let $X_1, X_2$ be two independent observations from the uniform $(0, \theta)$ distribution. Consider the problem of testing $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

(a.) Find the values of $c_1$ and $c_2$ so that the following tests have level 0.02.

$$
\phi_1(x_1, x_2) = 1 \quad \text{if } x_1 > c_1 \\
= 0 \quad \text{otherwise}
$$

$$
\phi_2(x_1, x_2) = 1 \quad \text{if } x_1 + x_2 > c_2 \\
= 0 \quad \text{otherwise}
$$

Also, compute the powers of the two tests.

(b.) Are either of the two tests in part (a.) most powerful level $\alpha = 0.02$ for testing $H_0 : \theta = 1$ versus $H_1 : \theta = 2$? If so, prove it. If not, obtain the most powerful level $\alpha = 0.02$ test and compute its power.