Short Questions

(1) Suppose that $\{X_t\}$ is a second order stationary time series with autocovariance function $\{c(r)\}$, with $\sum_r |rc(r)| < \infty$. Directly verify that the DFT of the covariances (you don’t need to approximate by a circulant matrix)

$$\text{cov}(J_n(\omega_{k_1}), J_n(\omega_{k_2})) = O\left(\frac{1}{n}\right) \quad k_1 \neq k_2,$$

where $J_n(\omega) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} X_t \exp(it\omega)$ and $\omega_k = \frac{2\pi k}{n}$.

(2) Derive conditions for the equation $X_t = (a + \eta_t)X_{t-1} + \varepsilon_t$, where $\{\eta_t\}$ and $\{\varepsilon_t\}$ are iid random variables with mean zero and finite variance, to have both a strictly stationary causal solution and be second order stationary.

(3) Let $\hat{c}_n(r) = \frac{1}{n} \sum_{t=1}^{n-|r|} Y_t Y_{t+|r|}$, by using the properties of the DFT and inverse DFT (you don’t need to approximate by a circulant matrix) show

$$\hat{c}_n(r) = \frac{2\pi}{n} \sum_{k=1}^{n} I_n(\omega_k) \exp(-ir\omega_k)$$

where $I_n(\omega) = |\frac{1}{\sqrt{2\pi}} X_t \exp(it\omega)|$ and $\omega_k = \frac{2\pi k}{n}$.

Long Questions (Do 2 of the questions out of 3)

(4) (i) State if and only if conditions based on the spectral density function for a sequence $\{c(k)\}$ to be positive definite.

(ii) Using (i), show that that the sequence

$$\{c(k); c(k) = \rho^{|k|}\cos(\omega_0 k), \text{ where } |\rho| < 1\},$$

is positive definite.

(iii) Using your answer in (ii) explain whether the sequence

$$\{c(k); c(k) = \sum_{s=1}^{p} \rho_s^{|k|}\cos(\omega_s k) \text{ where } |\rho_s| < 1\},$$

is positive definite.

(iv) Derive an AR(2) model with a causal solution which has the autocorrelation function $c(k) = \rho^{|k|}\cos(\omega_0 k)$ [Actually it does not have an AR(2) representation - see my solutions as to why!].
(v) Derive an AR(2) model with a non-causal solution which has the autocorrelation function \( c(k) = \rho^{|k|} \cos(\omega_0 k) \).

(5) Suppose that \( X_t \) satisfies the MA(\( q \)) representation \( X_t = \varepsilon_t + \sum_{j=1}^{q} \psi_j \varepsilon_{t-j} \) (assume \( \psi_j > 0 \)) and \( \sigma : [0,1] \rightarrow \mathbb{R} \) is a continuous positive function where for all \( u \) we have \( 0 < c_1 < \sigma(u) < c_2 < \infty \) for some constants \( c_1 \) and \( c_2 \).

(i) Show asymptotic normality of
\[
S_n = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \sigma\left(\frac{t}{n}\right) X_t,
\]
and obtain an expression for the asymptotic variance of \( S_n \).

[Hint: Derive asymptotic normality for the simplest case \( X_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} \) and extend the to higher orders].

(ii) Explain how the above argument can be extended to the MA(\( \infty \)) case where \( X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \) and \( \sum_{j=0}^{\infty} |j\psi_j| < \infty \).

(6) Suppose \( X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j} \), where \( \sum_{j} |\psi_j| < \infty \) and \( Y_t = \sum_{s=0}^{p} \theta_s X_{t-s} \).

(i) Obtain the Cramer-representation of \( \{Y_t\} \) in terms of \( A(\omega) = \sum_{j} \psi_j \exp(ij\omega) \) and \( \theta(\omega) = \sum_{s=0}^{p} \theta_s \exp(is\omega) \).

(ii) Using \( \hat{c}_n(r) = \frac{1}{n} \sum_{t=1}^{n-\lfloor |r| \rfloor} Y_t Y_{t+\lfloor |r| \rfloor} \) and
\[
\hat{c}_n(r) = \frac{2\pi}{n} \sum_{k=1}^{n} I_n(\omega_k) \exp(-ir\omega_k)
\]
where \( I_n(\omega) = |\frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} Y_t \exp(it\omega)|^2 \). Derive an asymptotic expression for the variance of \( \{\hat{c}_n(r)\} \).