Data Analysis and Statistical Methods
Statistics 651

http://www.stat.tamu.edu/~suhasini/teaching.html

Lecture 26 (MWF) Tests and CI based on two proportions

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Comparing proportions in two populations

• Just as it is interesting to compare the means in two different populations to see whether and effect has an influence or to see whether the overall behaviour is the same, it is also extremely useful to compare the proportions in two different populations.

• We will go through some examples and give the analysis. Later we will explain how we did the analysis.
The Thai HIV Vaccine Trials

In 2006, drug trials in Thailand were done for the vaccine against HIV.

<table>
<thead>
<tr>
<th></th>
<th>number of people infected</th>
<th>No affected</th>
<th>Sample size</th>
</tr>
</thead>
<tbody>
<tr>
<td>HIV vaccine</td>
<td>51</td>
<td>7949</td>
<td>8000</td>
</tr>
<tr>
<td>Placebo</td>
<td>72</td>
<td>7928</td>
<td>8000</td>
</tr>
</tbody>
</table>

We want to test the hypothesis that the vaccine gave some protection. This would mean that the proportion of people in the entire population who take the vaccine and go on to develop HIV is less than the proportion of the entire population who do not take the vaccine and go on to develop HIV. Hence we want to test $H_0: p_V - p_P \geq 0$ against $H_A: p_V - p_P < 0$ (ie. the people who took the vaccine are at less risk of infection). The data estimates $\hat{p}_V = 0.0065$ and $\hat{p}_P = 0.009$ and $\hat{p}_V - \hat{p}_P = -0.002625$, is this slight difference (of 0.26%) statistically significant?
The analysis

The main point is that the standard error is relatively small, 0.000138 (we calculate this later). This z-value

\[ z = \frac{-0.002625}{0.00138} = -1.90. \]

The area to the LEFT of -1.9 on the z-tables is 2.87%. If we use the significance level of 5% this result is statistically significant, however, if we use 1% as the significance level it’s not.
• Therefore, there is some evidence to suggest that the vaccination may offer some protection against HIV.

• If want to measure the degree of protection we can construct a CI for the difference

<table>
<thead>
<tr>
<th>Difference</th>
<th>n1</th>
<th>_N_1</th>
<th>n2</th>
<th>_N_2</th>
<th>Sample Diff.</th>
<th>Std. Err.</th>
<th>L. Limit</th>
<th>U. Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>p1 - p2</td>
<td>51</td>
<td>8000</td>
<td>72</td>
<td>8000</td>
<td>-0.002625</td>
<td>0.0013808222</td>
<td>-0.0053132617</td>
<td>0.130176E-5</td>
</tr>
</tbody>
</table>
The SA HIV Vaccine Trials

In 2005, drug trials in SA (amongst males) were done for the vaccine against HIV.

<table>
<thead>
<tr>
<th></th>
<th>number of people infected</th>
<th>No affected</th>
<th>Sample size</th>
</tr>
</thead>
<tbody>
<tr>
<td>HIV vaccine</td>
<td>49</td>
<td>865</td>
<td>914</td>
</tr>
<tr>
<td>Placebo</td>
<td>33</td>
<td>889</td>
<td>922</td>
</tr>
</tbody>
</table>

We want to test the hypothesis that the vaccine gave some protection. This would mean that the proportion of people in the entire population who take the vaccine and go on to develop HIV is less than the proportion of the entire population who do not take the vaccine and go on to develop HIV. Hence we want to test $H_0 : p_V - p_P \geq 0$ against $H_A : p_V - p_P < 0$ (i.e. the people who took the vaccine are at less risk of infection). The data estimates $\hat{p}_V = 0.053$ and $\hat{p}_P = 0.035$ and $\hat{p}_V - \hat{p}_P = 0.0178$. It is immediately clear that there is no evidence in the data that the vaccine works (in this cohort).
The main point $\hat{p}_V - \hat{p}_P = 0.0178$ is positive and we are looking for a significant negative difference. We wanted to do the test. The z-value is

$$z = \frac{0.0178}{0.0096} = 1.848.$$ 

The area to the LEFT of 1.848 on the z-tables is 96.7%. There is no evidence to against the null. We cannot reject the null.
• If want to measure the degree of difference we can construct a CI for the difference
Hair remedies

The FDA approved the drug Minodixil as a remedy for male pattern baldness. They did a study and this is what they found:

<table>
<thead>
<tr>
<th></th>
<th>new hair growth</th>
<th>no hair growth</th>
<th>Sample size</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minodixil</td>
<td>99</td>
<td>211</td>
<td>310</td>
</tr>
<tr>
<td>Placebo</td>
<td>60</td>
<td>242</td>
<td>302</td>
</tr>
</tbody>
</table>

Let $\pi_M$ be the probability a person has new hair growth and uses Minodixil and $\pi_P$ be the probability a person has new hair growth and doesn’t use Minodixil. Our estimates are $\hat{p}_M = 0.32$ and $\hat{p}_P = 0.2$. Use this data to test $H_0 : p_M - p_P \leq 0$ against the alternative $H_A : p_M - p_P > 0$. The data gives an estimated difference $\hat{p}_M - \hat{p}_P = 0.12$, this this 12% difference seen in the data statistically significant?
The analysis

The main point $\hat{p}_M - \hat{p}_P = 0.12$ is positive and we are looking for a significant positive difference. We do a two sample test on proportions. The z-value is

$$z = \frac{0.12}{0.035} = 3.40.$$  

The area to the RIGHT of 3.40 on the z-tables is 0.03%. As this is less than 5%, there IS evidence that minidoxil reduces hair loss.
• If want to measure how much better Minoxidil is over the placebo we can construct the CI

![Two sample Proportion with summary](chart.png)

95% confidence interval results:
- \( p_1 \): proportion of successes for population 1
- \( p_2 \): proportion of successes for population 2
- \( p_1 - p_2 \): difference in proportions

<table>
<thead>
<tr>
<th>Difference</th>
<th>Count1</th>
<th>Total1</th>
<th>Count2</th>
<th>Total2</th>
<th>Sample Diff.</th>
<th>Std. Err.</th>
<th>L. Limit</th>
<th>U. Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_1 - p_2 )</td>
<td>99</td>
<td>210</td>
<td>60</td>
<td>302</td>
<td>0.12067334</td>
<td>0.05604781</td>
<td>0.05185986</td>
<td>0.18937188</td>
</tr>
</tbody>
</table>

• Compare the standard errors of the test statistic with those use in constructing the confidence interval and we see that they are different.

This is because like the one sample proportion procedures, they are constructed under different conditions. We explain why below.
The normality result

- The p-values and confidence intervals constructed above were derived under the assumption that the estimate for the difference in proportions is normally distributed

\[
(\hat{p}_1 - \hat{p}_2) \xrightarrow{\mathcal{D}} \mathcal{N}\left(p_1 - p_2, \frac{p_1(1 - p_1)}{n} + \frac{p_2(1 - p_2)}{m}\right).
\]

- The above result holds ‘roughly’ true if the number of successes and failures in both groups is greater than 5 (this basically ensures the distributions are not too asymmetric about the mean of the distribution - just as in the one sample case).

- We then can almost plug and chug. There is however, one problem \( p_1 \) and \( p_2 \) are unknown.
• These need to be estimated - but the eagle eyed may have noticed that the standard errors are different for testing and the confidence intervals.

• Importantly, we do not use the t-distribution in any of the calculations, we only use the normal distribution.
Let us return to the hair remedy example. We want to test $H_0 : p_M - p_P \leq 0$ against $H_A : p_M - p_P > 0$.

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Remember we want to calculate the chance of the data giving a difference of 12% (0.12) when placebo and Minodixil have exactly the same effect of hair. The standard error is

$$\sqrt{\frac{p_M(1-p_M)}{310} + \frac{p_P(1-p_P)}{302}}$$
but the placebo and Minidoxil have same effect \((p_M = p_P = p)\), then

\[
\sqrt{\frac{p_M(1 - p_M)}{310} + \frac{p_P(1 - p_P)}{302}} = \sqrt{p(1 - p) \left( \frac{1}{310} + \frac{1}{302} \right)}.
\]

• Now we need to find the ‘best estimator’ of \(p\). The larger the sample size the the more reliable (and better) and estimator.

• Since under the null there is NO difference between the placebo or Minodixil we can ‘pool’ the data.

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<td>60</td>
<td>242</td>
<td>302</td>
</tr>
<tr>
<td>Pooled</td>
<td>159</td>
<td>453</td>
<td>612</td>
</tr>
</tbody>
</table>
• If there is no gain from using Minodixil, the proportion of people who notice a difference by simply massaging the head would be \( \hat{p} = \frac{159}{612} = 0.260 \). We use this as our best estimator of \( p \) (under the null). The standard error under the null:

\[
\text{s.e} = \sqrt{0.26 \times 0.74 \left( \frac{1}{310} + \frac{1}{302} \right)} = 0.0354.
\]

• This is quite important. If we have more information about the data we need to pool it to obtain a better estimator - as we have done above.
The standard error for constructing confidence intervals

- We return to the normality result:
  \[(\hat{p}_1 - \hat{p}_2) \xrightarrow{D} \mathcal{N}(p_1 - p_2, \left(\frac{p_1(1 - p_1)}{n} + \frac{p_2(1 - p_2)}{m}\right))\].

- Using this result the 95% CI for the difference in proportions is
  \[
  \left[\hat{p}_1 - \hat{p}_2 \pm 1.96 \times \sqrt{\frac{p_1(1 - p_1)}{m} + \frac{p_2(1 - p_2)}{n}}\right].
  \]

Application to hair remedy data:

\[
\left[0.12 \pm 1.96 \times \sqrt{\frac{p_M(1 - p_M)}{310} + \frac{p_P(1 - p_P)}{302}}\right].
\]
• But we do not know $p_M$ or $p_P$. Unlike testing we do not make any assumptions about it under the null. Therefore we simply replace $p_M$ and $p_P$ with their estimators $\hat{p}_M = 0.32$ and $\hat{p}_P = 0.2$ to give

$$\left[0.12 \pm 1.96 \times \sqrt{\frac{0.32 \times 0.68}{310} + \frac{0.2 \times 0.8}{302}} \right] = [0.051, 0.189] = [5.1, 18.9]\%.$$
Relative Risk

• In medical data and other applications the relative risk is often considered. We illustrate this through the HIV example:

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<tbody>
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<td>HIV vaccine</td>
<td>51</td>
<td>8000</td>
<td>0.006375</td>
</tr>
<tr>
<td>Placebo</td>
<td>72</td>
<td>8000</td>
<td>0.009</td>
</tr>
</tbody>
</table>

• We may ask how much more risk is there is taking the Placebo over the vaccine, this is best measured by the ratio

\[
RR = \frac{0.009}{0.006375} = 1.4.
\]
• I.e. The data suggests that you are 1.4 times more likely to develop HIV if you don’t take the vaccine than if you do.

• However, caution needs to be used when interpreting 1.4. 1.4 has been calculated from the sample. We want to relative risk based on the population. Therefore confidence intervals need to obtained for the population RR. If these were constructed, one found that the interval is wide with the left hand side end overlapping 1. This would be mean we have to be very cautious about interpreting the factor 1.4 as a gain in using a vaccine.