Let $X_i$ denote the outcome of the $i$th randomly chosen individual. Therefore $X_i$ can take the value 0 or 1, we write $X_i \in \{0, 1\}$.

Usually we are not interested in the response of $X_i$, but the collective response.

Suppose we observe the response of two individuals, $X_1, X_2$. The list of all possible outcomes are:

<table>
<thead>
<tr>
<th>Outcome</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>Person 1 said 'no', Person 2 said 'no'.</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>Person 1 said 'yes', Person 2 said 'yes'.</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>Person 1 said 'yes', Person 2 said 'no'.</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>Person 1 said 'no', Person 2 said 'yes'.</td>
</tr>
</tbody>
</table>
Usually were not interested in the actual outcomes, but we are often interested in the number of people who said yes.

The number of people who said yes can be described by the random variable $S_2$, where

$$S_2 = X_1 + X_2$$

<table>
<thead>
<tr>
<th>Outcome</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$S_2$</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>no one said 'yes'.</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>two people said 'yes'.</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>one person said 'yes'.</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>one person said 'yes'.</td>
</tr>
</tbody>
</table>

We see that $S_2$ can take one of three possible outcomes: $S_2 \in \{0, 1, 2\}$.

When:

- $S_2 = 0$ Both people said 'no'.
- $S_2 = 1$ One person out of the two said 'yes'.
- $S_2 = 2$ Both people said 'yes'.

\[\]
Calculating the Binomial Probabilities

We want to calculate the probability:
\[ P(S_2 = 0), \quad P(S_2 = 1), \quad P(S_2 = 2). \]

\(^(*)\)

To do this we will assume that \( X_1 \) and \( X_2 \) are independent random variables. That is, the outcome of \( X_1 \) has no influence on the outcome of \( X_2 \).

To calculate \(^(*)\) we use that:
\[ P(X_i = 1) = 0.8 \quad \left[ \text{The chance that a randomly chosen individual says 'yes' is 0.8} \right] \]
\[ P(X_i = 0) = 0.2 \quad \left[ \text{The chance that a randomly chosen individual says 'no' is 0.2} \right] \]

\( \text{(k) } P[S_2 = 0] \)?

The only outcome that gives \( S_2 = 0 \) is:

<table>
<thead>
<tr>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( S_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
so the event \( \{ S_2 = 0 \} = \{ X_1 = 0 \text{ and } X_2 = 0 \} \)

\[ \Rightarrow P(S_2 = 0) = P\{ X_1 = 0 \text{ and } X_2 = 0 \} \]

Now \( X_1 \) and \( X_2 \) are independent random variables.

This means that \( X_1 \) and \( X_2 \) have no influence on each other. Therefore

\[ P\{ X_1 = 0 \text{ and } X_2 = 0 \} = P(X_1 = 0) \times P(X_2 = 0) \]

\[ = (0.2)^2 \]

Therefore

\[ P(S_2 = 0) = (0.2)^2 \]

(ii) \( P\{ S_2 = 1 \} \)?

Two different outcomes can lead to \( S_2 = 1 \):

<table>
<thead>
<tr>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( S_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The events \( A = \{ X_1 = 1 \text{ and } X_2 = 0 \} \) and \( B = \{ X_1 = 0 \text{ and } X_2 = 1 \} \) are mutually exclusive (only one can occur, not both).
Therefore
\[
P[S_2 = 1] = P[(X_1 = 1 \text{ and } X_2 = 0) \text{ or } (X_1 = 0 \text{ and } X_2 = 1)]
\]
\[
= P[(X_1 = 1 \text{ and } X_2 = 0)] + P[(X_1 = 0 \text{ and } X_2 = 1)]
\]
[since the events are mutually exclusive].

We calculate \(P[(X_1 = 1 \text{ and } X_2 = 0)]\) by using the independence of \(X_1\) and \(X_2\).
\[
P[X_1 = 1 \text{ and } X_2 = 0] = P[X_1 = 1] \times P[X_2 = 0]
\]
\[
= 0.8 \times 0.2.
\]

Using the same argument we can show that
\[
P[X_1 = 0 \text{ and } X_2 = 1] = P[X_1 = 0] \times P[X_2 = 1]
\]
\[
= 0.2 \times 0.8.
\]

Altogether this means that:
\[
P[S_2 = 1] = P[X_1 = 1 \text{ and } X_2 = 0] + P[X_1 = 0 \text{ and } X_2 = 1]
\]
\[
= 2 \times \frac{0.2 \times 0.8}{2}\]
\[\text{two different outcomes which give } S_2 = 1\]
\[\text{The probability for each outcome.}\]
Example 2: Suppose we randomly select 4 people and we are interested in the number of people who say yes.

How do we calculate \( P[S_4 = 3] \)?

(Recall \( S_4 = X_1 + X_2 + X_3 + X_4 \), so the event \( S_4 = 3 \) is the event the number of people who say yes is 3)

To calculate \( P[S_4 = 3] \), we need to find all the outcomes which give \( S_4 = 3 \), then calculate the probability for each outcome.

<table>
<thead>
<tr>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( X_3 )</th>
<th>( X_4 )</th>
<th>( S_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

We see that each event is mutually exclusive. (If one outcome occurs, the others cannot).
Therefore mutually exclusive means:

\[ P[S_4 = 3] = P[X_1 = 1 \text{ and } X_2 = 1 \text{ and } X_3 = 1 \text{ and } X_4 = 0] \]
\[ + P[X_1 = 1 \text{ and } X_2 = 0 \text{ and } X_3 = 1 \text{ and } X_4 = 1] \]
\[ + P[X_1 = 1 \text{ and } X_2 = 1 \text{ and } X_3 = 0 \text{ and } X_4 = 1] \]
\[ + P[X_1 = 0 \text{ and } X_2 = 1 \text{ and } X_3 = 1 \text{ and } X_4 = 1] \]

By independence

\[ P[X_1 = 1 \text{ and } X_2 = 1 \text{ and } X_3 = 1 \text{ and } X_4 = 0] \]
\[ = P[X_1 = 1] \cdot P[X_2 = 1] \cdot P[X_3 = 1] \cdot P[X_4 = 0] \]
\[ = 0.8 \times 0.8 \times 0.8 \times 0.2 = 0.8^3 \times 0.2 \]

Using the same argument we get the same probability for the different outcomes. Therefore:

\[ P[S_4 = 3] = 4 \times 0.8^3 \times 0.2 \]

4 different outcomes give \( S_4 = 3 \)
Using the same set of arguments as those given above, we can show that:

\[ P(S_4 = 0) = 0.2^4 = 0.0016 \]
\[ P(S_4 = 1) = 4 \times 0.2^3 \times 0.8 = 0.0256 \]
\[ P(S_4 = 2) = 6 \times 0.2^2 \times 0.8^2 = 0.1536 \]
\[ P(S_4 = 3) = 4 \times 0.2 \times 0.8^3 = 0.4096 \]
\[ P(S_4 = 4) = 0.8^4 = 0.4096 \]

\[ \frac{0.0016 + 0.0256 + 0.1536 + 0.4096}{1} = 1 \]

**Remember** you cannot count the number of outcomes which give \( S_4 = k \) (say \( k=0,1,2,3,4 \)) and divide this by the total number of outcomes.

\[ P(S_4 = k) \neq \frac{\text{Number of outcomes which give } S_4 = k}{\text{Total number of different outcomes}} \]

This is because the probability of different outcomes can be different. For example, in the above case,

\[ P\left( x_1 = 1, x_2 = 1 \text{ and } x_3 = 1 \text{ and } x_4 = 1 \right) \neq P\left( x_1 = 0 \text{ and } x_2 = 0 \text{ and } x_3 = 0 \text{ and } x_4 = 0 \right) \]

\[ = 0.8^4 = 0.24 \]