STAT 613 Problems 1

These problems cover Section 5.2 (The exponential family) and Section 4.4 (The maximum likelihood and Fisher information).

(1) Suppose that $X_t$ are iid Weibull random variables with density $f(x) = \frac{\alpha y^{\alpha-1}}{\theta^\alpha} \exp\left(-\frac{y}{\theta}\right)^\alpha$, where parameters $\theta$ and $\alpha$ are unknown.

(i) Evaluate the log-likelihood, the derivative of the likelihood and the second derivative of the likelihood (often called the Hessian).

(ii) Using the class notes derive the limiting distribution of the maximum likelihood estimators $\hat{\theta}_T$ and $\hat{\alpha}_T$.

We want to investigate the finite sample behaviour of the estimator through simulations and compare how the parameters behave for different parameter values and for the asymptotic theory. You will need to use R.

(iii) Plot the Weibull density for 3 different pairs of $(\alpha, \theta)$ (such that they have different shapes).

(iv) Using and adapting the code written by Group 1 sample the Weibull for the three different Weibulls given in part (iii). For each distribution draw 10 independent samples, and evaluate the MLE estimator (there are various ways to this - you could use the function nlm in R).

Repeat the above 1000 times and plot the estimated density of $\hat{\theta}_T$ and $\hat{\alpha}$. You can do this using the command plot(density(data)) in R.

(v) Estimate Fisher information (you can do this by taking the average of the Hessian over all the 1000 samples), evaluate the inverse and see how well this approximates the variance of your estimators.

(vi) Repeat the above for sample size $n = 50$ and $n = 100$.

(vii) Suppose you draw 100 observations from the Weibull distribution where $\alpha = 2$ and $\theta = 3$. Using the R output suggest a method for constructing a 95% CI for $\alpha$.

(2) Question 5.2.5 page 182, Davison (2002).

(3) Question 4.4.4 page 125, Davison (2002).

(OLD1) Suppose that $X_t$ are iid normal variables with mean zero unknown variance $\theta$.

(i) Construct the maximum likelihood estimator of the variance $\theta$ (denote the mle as $\hat{\theta}_T$).

(ii) Using the class notes derive the limiting distribution of $\hat{\theta}_T$ when $\theta > 0$.

(iii) We want to investigate the finite sample behaviour of the estimator through simulations and compare how the parameters behave for different parameter values. You will need to use R.

Do the following simulation study for normal distributions with mean zero and variance $\theta = 0.01$, $\theta = 0.2$, $\theta = 1$ and $\theta = 5$. 
Sample from a normal distribution with mean 0 and variance \( \theta \) (specified above) \( n = 15 \) times. Calculate the sample variance.

Repeat the above algorithm 500 times. Now plot the estimated density of \( \theta \) based on your 500 simulations. You can do this using the command \texttt{plot(density(data))} in R.

Do this for the different values of \( \theta \).
Repeat the above for sample size \( n = 100 \).
Detail your observations (include plots in your HW).