Big and little oh

Big and little oh in mathematics

(a) **Big oh** If we write \( S_n = O(a_n) \), then this implies there exists a finite constant \( C \) such that for all \( n, \frac{|S_n|}{a_n} \leq C \).

(b) **Little Oh** If we write \( S_n = o(a_n) \), then this implies the sequence \( \frac{|S_n|}{a_n} \to 0 \) as \( n \to \infty \).

Example: Often we write \( S_n = \mu + b_n + o(a_n) \). This implies that the reminder \( S_n - \mu - b_n \) is such that \( |S_n - \mu - b_n|/a_n \to 0 \) as \( n \to \infty \). Thus \( a_n \) dominates the remainder \( S_n - \mu - b_n \).

Big and little oh in probability

Often we are given an estimator, \( S_n \) of \( \mu \). Sometimes we can evaluate \( E[S_n - \mu]^2 \). Typically we can show \( E[S_n - \mu]^2 = O(n^{-\gamma}) \) (usually \( \gamma = 1/2 \)).

(a) **Big oh** If we write \( S_n = O_p(a_n) \). This means for every \( \varepsilon > 0 \) there exists a finite constant \( C_\varepsilon \) such that for all \( n \)

\[
P \left( \left| \frac{S_n}{a_n} \right| > C_\varepsilon \right) \leq \varepsilon.
\]

Example: Suppose \( E|S_n - \mu|^2 \leq \frac{K}{n} \), then we can say that \( S_n = O_p(n^{-1/2}) \). This is due to Chebyshev’s inequality. That is

\[
P \left( \left| \frac{S_n}{n^{-1/2}} \right| > C_\varepsilon \right) \leq \frac{nE|S_n|^2}{C_\varepsilon^2} \leq \frac{K}{C_\varepsilon^2}.
\]

Thus, for a given \( \varepsilon \), if we choose \( C_\varepsilon = (K/\varepsilon)^{1/2} \), we have that

\[
P\left( \left| \frac{S_n}{n^{-1/2}} \right| > C_\varepsilon \right) \leq \varepsilon.
\]

This implies \( S_n = O_p(n^{-1/2}) \).

(b) **Little Oh** If we write \( S_n = \mu + X_n + o_p(a_n) \), then this implies the sequence \( \frac{|S_n - \mu - X_n|}{a_n} \to 0 \) as \( n \to \infty \). In other words, for every \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} P \left( \left| \frac{S_n - \mu - X_n}{a_n} \right| > \varepsilon \right) = 0.
\]

(c) Suppose \( a_n \to 0 \) as \( n \to \infty \). If \( |S_n - \mu| = O_p(a_n) \), then \( |S_n - \mu|^2 = O_p(a_n^2) = o_p(a_n) \). If \( |S_{1,n} - \mu_1| = O_p(a_n) \) and \( |S_{2,n} - \mu_2| = O_p(a_n) \), then \( |S_{1,n} - \mu_1||S_{2,n} - \mu_2| = O_p(a_n^2) = o_p(a_n) \).