A note on general quadratic forms of nonstationary time series

Junbum Lee and Suhasini Subba Rao
Department of Statistics, Texas A&M University
College Station, U.S.A.
`jlee@stat.tamu.edu` and `suhasini@stat.tamu.edu`

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Abstract

In this paper general quadratic forms of nonstationary, α-mixing time series are considered. Under relatively weak mixing and moment assumptions, asymptotic normality of these forms are derived. The results are applied to the weighted covariance of the Discrete Fourier Transforms of a time series, which is an important example of a quadratic form. In order to show asymptotic normality of the generalised quadratic form, bounds for their moments are obtained using martingale and Near-Epoch Dependent methods.

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1 Introduction

The study of the asymptotic theory of statistics often involves quadratic forms which have the general form

$$W_T = \frac{1}{T}\sum_{t,\tau=1}^{T} G_{t,\tau} h(X_t, X_\tau), \tag{1}$$

where $\{X_t\}$ is a stochastic process, $h(\cdot)$ is a function and $\{G_{t,\tau}\}$ are weights, which vary according to the application. Various statistical methods depend on the asymptotic sampling distribution of above statistic.

In view of its importance, several authors have studied W_T for the particular case $h(X_t, X_\tau) = X_t X_\tau$ under various assumptions on the stochastic process $\{X_t\}$. For example, Mikosch (1990), Götze and Tikhomirov (1999) and the references therein, analysis W_T under the assumption that $\{X_t\}$ are iid random variables. Kokoszka and Taqqu (1997) and Bhansali, Giraitis, and Kokoszka (2007) relax the independence assumption and establish asymptotic normality of W_T under the assumption that $\{X_t\}$ is a realisation from stationary, linear time series. Rosenblatt (1984) allows for nonlinear time series, by assuming that $\{X_t\}$ are α-mixing. In particular, he shows asymptotic normality of W_T under the assumption that $\{X_t\}$ is a strictly stationary α-mixing time series and has absolutely summability eight order cumulants. The generalising to mixing random variables, allow $\{X_t\}$ to be a non-linear time series, but the cumulant assumptions are quite strong. Recently, Gao and Anh (2000) relax the moment assumptions by considering geometric mixing $\{X_t\}$ and Lin (2009) considers the case $\{X_t\}$ is the sum of stationary α-mixing random variables. It should be mentioned, that there are other methods for measuring dependence. For example, Wu and Shao (2007) show asymptotic normality when $\{X_t\}$ can be written as a function of the innovations and satisfies the assumption of physical dependence. The study of the general quadratic form given in (1) can also arise in several applications, including nonparametric estimators, but has received less attention. One reason for this is that techniques used in the articles mentioned above cannot be directly applied to (1). Moreover, the underlying assumption in all the above references is that the process $\{X_t\}$ is strictly stationary.

In the analysis of nonstationary time series (which is possibly nonlinear), quadratic forms of the above type do occur, for example estimators of the time-varying spectral density involve quadratic forms (see, for example, Dahlhaus (2000) and Dwivedi and Subba Rao (2011)). In this paper, our objective is to study the asymptotic theory of general quadratic forms for nonstationary processes.

In Section 2 we show asymptotic normality of the general quadratic form under some moment assumptions and α-mixing of the stochastic process (which includes both nonstationary and nonlinear processes). By using Ibragimov-type inequalities (see Statulevicius and Jakimavicius (1988)) which link cumulants to the mixing rate, we avoid direct assumptions on the summability of the cumulants. The assumptions allow the weights $G_{t,\tau}$ to also depend on T, thus including the case of spectral density estimators. To understand how quadratic forms of stationary and nonstationary processes may differ, in Section 3, we consider the sampling properties of quadratic forms of locally stationary processes, which are a subclass of nonstationary time series (see Priestley (1965) and Subba Rao (1970), Dahlhaus (1997), Dahlhaus and Subba Rao (2006), Subba Rao (2006) and Zhou and Wu (2009) for examples of locally stationary time series). We extend the classical results on cumulants of DFTs (see Brillinger (1981)) to general locally stationary processes (note that Paparoditis (2009) considers cumulants of DFTs of time-varying MA(∞) processes). These results are used to show that the sampling properties of certain quadratic forms behave differently for stationary and locally stationary processes. From a statistical perspective, these results are of interest, as they can be used as an alternative characterisation of nonstationary time series. In Section 4 we derive some results on cumulants and moments of the quadratic form. We use mixingale and near-epoch dependent methods to prove the results in this section, these techniques may also be of independent interest. To prove the central limit theorem we use a similar Bernstein blocking argument, and this proof can be found in Section 5. The technical details are given in the appendix.

2 The quadratic form

Let us suppose that $\{X_{t,T}; 1 \leq t \leq T\}$ is a time series which we do not assume to be stationary. By allowing $X_{t,T}$ to depend on T, the results below cover the case of triangular arrays, and in particular allows for locally stationary time series. We will assume that for all t, $\mathbb{E}(X_{t,T}) = 0$ and for all t, T $0 < \mathrm{var}(X_{t,T}) < \infty$. This condition excludes degenerate cases by ensuring that $\{X_{t,T}\}$ does not converge to a non-random sequence but always has a bounded variance. In this paper we consider general quadratic forms of the type

$$Q_T = \frac{1}{T}\sum_{t,\tau=1}^{T} G_{t,\tau} h(X_{t,T}, X_{\tau,T}), \tag{2}$$

where we do not impose any conditions on the function $h : \mathbb{R}^2 \to \mathbb{R}$. By allowing this amount of generality on $h(\cdot)$, we need to assume that the weights $G_{t,\tau}$ decay to zero, in the sense that $\sup_{t,T}\sum_{\tau}|G_{t,\tau}| < \infty$. For example, if $h(X_{t,T}, X_{\tau,T}) = (X_{t,T} + X_{\tau,T})$, then for the variance of Q_T to decay to zero as $T \to \infty$, we require such a condition on the weights. In order to relax this condition on the weights $\{G_{t,\tau}\}$, a stronger condition on $h(\cdot)$ is required. Therefore, in addition to the above, we will also consider quadratic forms which have the multiplicative form $h(X_{t,T}, X_{\tau,T}) = X_{t,T}X_{\tau,T}$:

$$Q_{T,M} = \frac{1}{T}\sum_{t,\tau=1}^{T} G_{t,\tau,M} X_{t,T} X_{\tau,T} \tag{3}$$

where for some $0 < \alpha < 1$, $M := M(T) = T^{\alpha}$, and for $|t-\tau| > M$, then $G_{t,\tau,M} = 0$.

We now state some conditions, which we use to prove asymptotic normality of Q_T and $Q_{T,M}$.

Assumption 2.1 *(i) Let us suppose that $\{X_{t,T}\}$ is an α-mixing time series such that*

$$\sup_{k} \sup_{\substack{A\in\sigma(X_{t+k,T}, X_{t+1+k,T},\ldots)\\ B\in\sigma(X_{k,T}, X_{k-1,T},\ldots)}} |P(A\cap B) - P(A)P(B)| \leq \alpha(t),$$

where $\alpha(t)$ are the mixing coefficients which satisfy $\alpha(t) \leq K|t|^{-s}$ for some $s > 0$.

(ii) *(a)* *For* Q_T *defined in (2), we suppose* $|G_{t,\tau}| \leq C|t-\tau|^{-\delta}$ *(*$\delta > 2$*) and* $c_1\frac{G}{T} \leq var(Q_T) \leq c_2\frac{G}{T}$ *(for some* $0 < c_1 \leq c_2 < \infty$*), were* $G = \sup_t \sum_\tau |G_{t,\tau}| < \infty$*.*

(b) *For* $Q_{T,M}$ *defined in (3), we suppose that* $G_{t,\tau,M} = 0$ *for* $|t-\tau| > M$ *and for all* T*,* $c_1\frac{G_M}{T} \leq var(Q_{T,M}) \leq c_2\frac{G_M}{T}$ *(for some* $0 < c_1 \leq c_2 < \infty$*), where* $G_M = \sup_t \sum_\tau |G_{t,\tau,M}|^2$ *and* $\inf_M G_M > 0$*.*

(iii) *(a)* *For some* $r > 2s/(s-2) > 0$*, we have* $\sup_{t,\tau,T} \mathbb{E}|h(X_{t,T}, X_{\tau,T})|^r < \infty$*.*

(b) *For some* $r > 4s/(s-6) > 0$*, we have* $\sup_{t,T} \mathbb{E}|X_{t,T}|^r < \infty$*.*

Before stating the asymptotic sampling properties of the the quadratic forms, some comments on the assumptions are in order. To prove asymptotic normality of Q_T and $Q_{T,M}$ we have to treat the cases differently and use a slightly different set of conditions. This is primarily because we need to obtain moment bounds for each of these terms (see Lemmas 4.2 and 4.3 below). The details can be found in the appendix, but to give a flavour of the methods, to bound Q_T we treat $\{\sum_\tau G_{t,\tau} h(X_{t,T}, X_{\tau,T})\}_t$ as a stochastic process with decaying dependence structure and use the notion of L_2-NED together with martingale methods to obtain the moment bounds. However, in the case of $Q_{T,M}$, despite Assumption 2.1(i,iib) (in particular the mixing and $G_{t,\tau,M} = 0$ for $|t-\tau| > M$) implying that the dependence in the sequence $\{\sum_\tau G_{t_1,\tau,M} X_{t_1,T} X_{\tau,T}\}$ decays the further apart the ts, the same methods used to bound Q_T, when applied to $Q_{T,M}$ gives sub-optimal bounds. Instead we use iterative martingale methods to obtain the optimal moment bounds for $Q_{T,M}$. We observe that in the case that $|G_{t,\tau}| \leq C|t-\tau|^{-\delta}$ ($\delta > 2$) and $g(X_{t,T}, X_{\tau,T}) = X_{t,T}X_{\tau,T}$, then Assumption 2.1(iiia) is slightly weaker than Assumption 2.1(iiib). As the assumptions on $Q_{T,M}$ allow $\sum_\tau |G_{t,\tau,M}| \to \infty$ as $T \to \infty$, we require that the fourth order cumulants are absolutely summable, see Remark **??** below. Finally, several time series, both stationary and nonstationary, satisfy the α-mixing conditions given in Assumption 2.1(i), see, for example, Tjostheim (1990), Doukhan (1994), Cline and Pu (1999), Bradley (2007) and Fryzlewicz and Subba Rao (2011).

We now derive the limiting distribution of Q_T and $Q_{T,M}$.

Theorem 2.1 *Suppose Assumption 2.1(i,ii(a),iii(a)) is satisfied. Let* $var(Q_T) = V_T$*, then we have* $V_T^{-1/2}\big(Q_T - \mathbb{E}(Q_T)\big) \xrightarrow{D} \mathcal{N}(0,1)$ *as* $T \to \infty$*.*

PROOF. In the appendix. □

Theorem 2.2 *Suppose Assumption 2.1(i,ii(b),iii(b)) is satisfied. Let* $var(Q_{T,M}) = V_T$*, then we have* $V_T^{-1/2}\big(Q_{T,M} - \mathbb{E}(Q_{T,M})\big) \xrightarrow{D} \mathcal{N}(0,1)$ *as* $T \to \infty$*.*

PROOF. In the appendix, □

The above results are for quadratic forms of univariate time series. As multivariate time series arise in several applications we now give an analogous result for multivariate time series, noting that the proof is almost identical to the univariate case.

Corollary 2.1 *Let us suppose that* $\{\underline{X}_{t,T}\}$ *is a d-dimensional vector time series, which is mixing*

$$\sup_{k} \sup_{\substack{A\in\sigma(\underline{X}_{t+k,T},\underline{X}_{t+1+k,T},\ldots)\\ B\in\sigma(\underline{X}_{k,T},\underline{X}_{k-1,T},\ldots)}} |P(A\cap B)-P(A)P(B)| \leq \alpha(t),$$

where $\alpha(t)$ *are the mixing coefficients and are such that* $\alpha(t)\leq K|t|^{-s}$ *where* $s>0$*, and suppose there exists some* $r>\frac{4s}{s-6}$*, such that* $\sup_{t,T}\mathbb{E}(\sum_{j=1}^{d}|X_{t,T,j}|)^{r}<\infty$ *(where* $|\cdot|$ *denotes the Euclidean norm of a vector or matrix). Define the quadratic form*

$$Q_T \;=\; \frac{1}{T}\sum_{t,\tau=1}^{T} \underline{X}'_{t,T}\boldsymbol{G}_{t,\tau,M}\underline{X}_{\tau,T},$$

where $\{\boldsymbol{G}_{t,\tau,M}\}$ *is a* $d\times d$ *matrix which satisfies* $\boldsymbol{G}_{t,\tau,M}=0$ *(for* $|t-\tau|>M$*). We assume there exists* $0<c_1\leq c_2<\infty$ *such that* $c_1G_M/T\leq var(Q_T)\leq c_2G_M/T$ *(*$G_M=\sup_t\sum_\tau|G_{t,\tau,M}|$*). Then we have* $V_T^{-1/2}(Q_T-\mathbb{E}(Q_T))\xrightarrow{D}\mathcal{N}(0,1)$*, where* $V_T=var(Q_T)$*.*

PROOF. The proof is exactly the same as the proof of Theorem 2.2, hence we omit the details.
□

3 The weighted discrete Fourier transform covariance

In this section we consider the weighted discrete Fourier transform covariance, which is an example of a quadratic form. The weighted DFT covariance is defined as

$$H_T=\frac{1}{T}\sum_{k=1}^{T}H(\omega_k)J_T(\omega_k)\overline{J_T(\omega_{k+r})}, \tag{4}$$

where $J_T(\omega_k)=\frac{1}{\sqrt{2\pi T}}\sum_{t=1}^{T}X_{t,T}\exp(it\omega_k)$ and $\omega_k=\frac{2\pi k}{T}$ and $H:[0,2\pi]\rightarrow\mathbb{R}$ is a function with bounded second derivatie. We start by considering the properites of H_T under the assumption of second order stationarity and contrast these with the case that $\{X_{t,T}\}$ is nonstationary. Under the assumption that $\{X_{t,T}\}$ is a stationary time series with $\sum_r|r|\cdot|\mathrm{cov}(X_0,X_r)|<\infty$, it can be shown that

$$\mathbb{E}(H_T)=\left\{\begin{array}{cc} O(\frac{1}{T}) & r\neq 0 \\ \int_0^{2\pi}H(\omega)f(\omega)d\omega+O(\frac{1}{T}) & r=0 \end{array}\right., \tag{5}$$

where f denotes the spectral density of the time series. An explicit expression for the asymptotic variance of $T\mathrm{var}(H_T)$ can easily be obtained and by using Theorem 2.2 we can prove asymptotic normality of H_T. Thus in the case of stationarity the weighted DFT covariance for $r \neq 0$, does not appear to be of interest. Indeed, frequency domain methods for linear time series models mostly deal with $r = 0$, and in this case H_T is called the weighted periodogram. On the other hand, the case $r \neq 0$ can be very useful in the study of nonstationary processes; as H_T may contain interesting information about nonstationary properties of $\{X_{t,T}\}$ (see, for example, Dwivedi and Subba Rao (2011) and Subba Rao (2011)). In the remainder of this section we will focus on the sampling properties of H_T, when $\{X_{t,T}\}$ is nonstationary.

In order to derive an asymptotic expression for the expectation and variance of H_T we will place some structure on the nonstationarity, and assume the nonstationarity changes slowly over time. More precisely, let us suppose that $\{X_{t,T}\}$ is a nonstationary time series, which can 'locally' be approximated by a stationary time series. In order to make this definition precise, we define a time series $\{X_t(u)\}$, which for fixed u is strictly stationary. Then $\{X_{t,T}\}$ is called locally stationary if it satisfies

$$\big|X_{t,T} - X_t(u)\big| \leq \big(|\frac{t}{T} - u| + \frac{1}{T}\big)V_t,$$

where $\{V_t\}$ is a positive stationary time series and $\mathbb{E}|V_t| < \infty$. The asymptotics in this case are done in the rescaled sense. A simple example of a time series $\{X_{t,T}\}$ which satisfies the above is the time-varying AR process $X_{t,T} = a(\frac{t}{T})X_{t-1,T} + \varepsilon_t$ where $\sup_u |a(u)| < 1$. In this case, $X_t(u) = a(u)X_{t-1}(u) + \varepsilon_t$ and $V_t = aV_{t-1} + |\varepsilon_t|$, where $a = \sup_u |a(u)|$ (see Subba Rao (1970) and Dahlhaus (1997)).

In order to study H_T we will derive some expressions for the cumulants of the DFT. This will extend the results on cumulants of DFTs derived in Brillinger (1981) to locally stationary time series. We recall that Brillinger (1981) showed that if $\{X_t\}$ is a stationary time series under certain conditions on the cumulants of $\{X_t\}$ and for $\omega_{j_k} = \frac{2\pi j_k}{T}$ the DFTs are such that

$$\begin{aligned} &\mathrm{cum}(J_T(\omega_{j_1}), \ldots, J_T(\omega_{j_n})) \\ &= \begin{cases} O(\frac{1}{T^{n/2}}) & \text{if} \quad \sum_{s=1}^{n} \omega_{k_s} \neq 2\pi r, \quad r \in \mathbb{Z} \\ \frac{(2\pi)^{n/2-1}}{T^{n/2-1}} f_n(\omega_{j_1}, \ldots, \omega_{j_{n-1}}) + O(\frac{1}{T^{n/2}}) & \text{otherwise} \end{cases} \end{aligned} \tag{6}$$

where $f_n(\omega_1, \ldots, \omega_{n-1}) = \frac{1}{(2\pi)^{n-1}} \sum_{t_1,\ldots,t_{n-1}=-\infty}^{\infty} \kappa_n(t_1, \ldots, t_{n-1}) \exp(i \sum_{j=1}^{n-1} t_j\omega_j)$, with $\kappa_n(t_1, \ldots, t_{n-1}) = \mathrm{cum}(X_0, X_{t_1}, \ldots, X_{t_{n-1}})$, denotes the nth order spectra of $\{X_t\}$. In the case that $\{X_t\}$ is nonstationary the above result does not hold true, however for locally stationary processes an interesting analogous result can be obtained. In order to derive this, we need to define the localised higher order spectra for locally stationary time series.

Definition 3.1 *Define the local covariance and nth order cumulant as* $c(u, r) = cov(X_t(u), X_{t+r}(u))$

and $\kappa_n(u; t_2 - t_1, \ldots, t_n - t_1) = cum\big(X_{t_1}(u), \ldots, X_{t_n}(u)\big)$ *respectively. Using this notation we define the local spectral density* $f(u; \omega) = \frac{1}{2\pi} \sum_{r=-\infty}^{\infty} c(u; r) \exp(ir\omega)$ *and the* n*th order local spectra*

$$f(u; \omega_1, \ldots, \omega_{n-1}) = \frac{1}{(2\pi)^{n-1}} \sum_{t_1, \ldots, t_{n-1} = -\infty}^{\infty} \kappa_n(u; t_1, \ldots, t_{n-1}) \exp(i \sum_{j=1}^{n-1} t_j \omega_j), \tag{7}$$

where $\omega_1, \ldots, \omega_{n-1} \in [0, 2\pi]$*.*

Using the above definition we can now state the following assumptions.

Assumption 3.1 (Lipschitz conditions on the local cumulants) *For a given* n*, there exists a sequence* $\{\kappa_n(t_1, \ldots, t_{n-1})\}$ *such that* $\sum_{t_1, \ldots, t_{n-1}} (1 + |t_j|) |\kappa_n(t_1, \ldots, t_{n-1})| < \infty$ *(for all* $1 \le j \le (n-1)$*), and the cumulants corresponding to* $\{X_t(u)\}$ *satisfy*

- *(i)* $\big|cum(X_{t_1,T}, \ldots, X_{t_n,T}) - cum(X_{t_1}(\frac{t_1}{T}), \ldots, X_{t_n}(\frac{t_n}{T}))\big| \le T^{-1} \kappa_n(t_2 - t_1, \ldots, t_n - t_1)$*.*
- *(ii) For all* $1 \le j \le (n-1)$*,* $u_j, v_j \in [0, 1]$*, we have*

$$\begin{aligned} &\big|cum(X_{t_1}(u_1), \ldots, X_{t_{j-1}}(u_{j-1}), X_{t_j}(v_j), X_{t_{j+1}}(u_{j+1}), \ldots, X_{t_n}(u_n) - \\ &cum(X_{t_1}(u_1), \ldots, X_{t_{j-1}}(u_{j-1}), X_{t_j}(u_j), X_{t_{j+1}}(u_{j+1}), \ldots, X_{t_n}(u_n)\big| \\ &\le |u_j - v_j| \cdot |\kappa_n(t_2 - t_1, \ldots, t_n - t_1)|. \end{aligned}$$

- *(iii) For all* $1 \le j \le n - 1$*, we have* $\sup_u \sum_{r_1, \ldots, r_{n-1}} (1 + |t_j|) |\kappa_n(u; t_1, \ldots, t_{n-1})| < \infty$*.*
- *(iv)* $\sup_{u, \omega_1, \ldots, \omega_{n-1}} |\frac{\partial^2 f_n(u; \omega_1, \ldots, \omega_{n-1})}{\partial u^2}| < \infty$*.*

The above assumption appears a little unwielding, however if the nth moment exists, then it can be shown that several locally stationary time series satisfy these assumptions (see for example, Dahlhaus and Polonik (2006) for the time varying MA(∞) model and Dahlhaus and Subba Rao (2006) for the time-varying ARCH process).

We will use the following lemma to derive the cumulants of the DFT of a locally stationary process.

Lemma 3.1 *Suppose Assumption 3.1 holds and define the Fourier coefficients of the local higher order spectra as*

$$F_n(k; \omega_1, \ldots, \omega_{n-1}) = \int_0^1 f_n(u; \omega_1, \ldots, \omega_{n-1}) \exp(i2\pi k u) du. \tag{8}$$

Then we have

$$\sup_{\omega_1,\ldots,\omega_{n-1}} |F_n(k;\omega_1,\ldots,\omega_{n-1})| \leq C \sup_{u,\omega_1,\ldots,\omega_{n-1}} |\frac{\partial f_n(u;\omega_1,\ldots,\omega_{n-1})}{\partial u}| \frac{1}{|k|^2} \tag{9}$$

were C is a finite constant.

PROOF. The proof follows from Assumption 3.1(iv), and is a straightforward application of Briggs and Henson (1997), Theorem 6.2 for bounds on Fourier coefficients. □

Using the following lemma, we derive a generalisation of Brillinger (1981), Theorem 4.3.2, (see (6) for a summary).

Lemma 3.2 *Suppose Assumption 3.1 holds, then we have*

$$cum(J_T(\omega_{j_1}),\ldots,J_T(\omega_{j_n})) = \frac{(2\pi)^{(n/2)-1}}{T^{(n/2)-1}} F_n\big[(j_1+\ldots+j_n);\omega_{j_1},\ldots,\omega_{j_{n-1}}\big] + O(\frac{1}{T^{n/2}}). \tag{10}$$

where $\omega_{j_k} = \frac{2\pi j_k}{T}$ and $j_k \in \mathbb{Z}$.

PROOF. Expanding the cumulant term we have

$$\mathrm{cum}(J_T(\omega_{j_1}),\ldots,J_T(\omega_{j_n})) = \frac{1}{(2\pi T)^{n/2}} \sum_{t_1,\ldots,t_n=1}^{T} \mathrm{cum}(X_{t_1,T},\ldots,X_{t_n,T}) \exp(it_1\omega_{j_1}+\ldots+it_n\omega_{j_n}).$$

We now replace $\mathrm{cum}(X_{t_1,T},\ldots,X_{t_n,T})$ with $\kappa_n(\frac{t_1}{T}, t_2-t_1,\ldots,t_n-t_1)$ and use Assumption 3.1 to obtain

$$\begin{aligned} &\mathrm{cum}(J_T(\omega_{j_1}),\ldots,J_T(\omega_{j_n})) \\ = &\frac{1}{(2\pi T)^{n/2}} \sum_{t_1,\ldots,t_n=1}^{T} \kappa_n(\frac{t_1}{T}, t_2-t_1,\ldots,t_n-t_1) \exp(it_1\omega_{j_1}+\ldots+it_n\omega_{j_n}) + O(\frac{1}{T^{n/2}}). \end{aligned}$$

Replacing the above summand with $f_n(\frac{t}{T};\omega_{j_1},\ldots,\omega_{j_n})$ (see (7)) we have

$$\mathrm{cum}(J_T(\omega_{j_1}),\ldots,J_T(\omega_{j_n})) = \frac{(2\pi)^{(n/2)-1}}{T^{n/2}} \sum_{t=1}^{T} f_n(\frac{t}{T},\omega_{j_1},\ldots,\omega_{j_{n-1}}) \exp(it\sum_{s=1}^{n}\omega_{j_s}) + O(\frac{1}{T^{n/2}}) \tag{11}$$

and replacing the sum with the integral gives

$$\begin{aligned} &\mathrm{cum}(J_T(\omega_{j_1}),\ldots,J_T(\omega_{j_n})) \\ = &\frac{(2\pi)^{(n/2)-1}}{T^{n/2-1}} \int_0^1 f_n(u;\omega_{j_1},\ldots,\omega_{j_{n-1}}) \exp(iu\sum_{s=1}^{n} j_s) du + O(\frac{1}{T^{n/2}}). \end{aligned}$$

Finally, substituting (9) into the above gives (10). $\square$

It is interesting to note that Paparoditis (2009), Lemma 6.2, derives a similar result to (11) for time-varing MA(∞) processes.

Comparing the above result to the cumulants of DFTs of stationary time series in (6) leads to some interesting conclusions. In the case that $\{X_{t,T}\}$ is second order stationary we have $\mathrm{cov}(J_T(\omega_{k_1}), J_T(\omega_{k_2})) = o(1)$, whereas if $\{X_{t,T}\}$ were (second order) nonstationary there is an 'ordering' in correlation between the DFTs. More precisely, $\big|\mathrm{cov}(J_T(\omega_{k_1}), J_T(\omega_{k_2}))\big| \leq C|k_1 - k_2|^{-2}$, where C is a finite constant. Hence the correlation between the DFTs decay the further apart the frequencies. It should be mentioned if $\{X_{t,T}\}$ is nonstationary but not locally stationary it is not clear whether the correlation structure between the DFTs decay the further apart the frequencies. It is possible, that the differences in correlation of the DFT, could be a way of discriminating locally stationary from general nonstationary behaviour.

In the lemma below we derive the mean and variance of H_T under the assumption of local stationarity. Furthermore, we show asymptotic normality, to do this we show that H_T can be rewritten as a quadratic form defined in (3):

$$H_T = \frac{1}{T}\sum_{t,\tau} X_{t,T}X_{\tau,T}\exp(-i\omega_r\tau)\Big(\frac{1}{2\pi T}\sum_{k=1}^{T} H(\omega_k)\exp(i\omega_k(t-\tau))\Big) = \frac{1}{2\pi T}\sum_{t,\tau} G_{t,\tau,T}X_{t,T}X_{\tau,T},$$

where $G_{t,\tau,T} = \exp(-i\omega_r)h_{t-\tau,M}$, with $h_{t-\tau,T} = \frac{1}{2\pi T}\sum_{k=1}^{T} H(\omega_k)\exp(i\omega_k(t-\tau))$. To obtain a bound on $G_{t,\tau,T}$ (such that we can use Theorem 2.2) to prove asymptotic normality, we use that since $\sup_\omega |H''(\omega)| < \infty$, then $|h_{t-\tau,T} - h(t-\tau)| = O(T^{-2})$ and $|h(s)| \leq C|s|^{-2}$, where where $h(s) = \int H(\omega)\exp(-is\omega)d\omega$. Thus $H_T = \tilde{H}_T + O_p(T^{-1/2-\gamma})$, where

$$\tilde{H}_T = \frac{1}{T}\sum_{t=1}^{T}\sum_{\tau=1}^{T} X_{t,T}X_{\tau,T}I(\frac{t-\tau}{T^{1/2+\gamma}})h(t-\tau)\exp(-i\omega_r\tau) \qquad (12)$$

and $I(x) = 1$ for $x \in [-1, 1]$ and zero elsewhere. Now we can apply Theorem 2.2 to obtain the following result.

Lemma 3.3 *Suppose Assumption 3.1 holds with $n = 1, \ldots, 4$, and $H(\omega)$ has a bounded second derivative. Then we have*

$$\mathbb{E}(H_T) = \int_0^1\int_0^{2\pi} H(\omega)f(u,\omega)\exp(-i2\pi ru)d\omega du + O(\frac{1}{T}) \qquad (13)$$

and

$$var(\sqrt{T}H_T) = \frac{1}{T}\sum_{k_1,k_2=1}^{T} H(\omega_{k_1})H(\omega_{k_2})\Big[F_2(k_1-k_2,\omega_{k_1})F_2(k_2-k_1,-\omega_{k_1+r}) +$$
$$F_2(-(k_1+k_2+r),-\omega_{k_1+r})F_2(k_1+k_2+r,\omega_{k_1}) + \frac{(2\pi)^{1/2}}{T}F_4(0,\omega_{k_1},-\omega_{k_1+2},-\omega_{k_2})\Big]. \quad (14)$$

If, in addition, Assumption 2.1(i,ii(b),iii(b)) holds, then we have

$$V_T^{-1/2}\Big(H_T - \int_0^1\int_0^{2\pi} H(\omega)f(u,\omega)\exp(-i2\pi r u)d\omega du\Big) \xrightarrow{D} \mathcal{N}(0,1), \quad (15)$$

where $V_T = var(H_T)$, and $var(\sqrt{T}H_T)$ is given in (14).

PROOF. Now by using Lemma 3.2 we have

$$\mathbb{E}(H_T) \;=\; \frac{1}{T}\sum_{k=1}^{T} H(\omega_k)\mathbb{E}(J_T(\omega_k)\overline{J_T(\omega_{k+r})}) = \frac{1}{T}\sum_{k=1}^{T} H(\omega_k)F_n(-r,\omega_k) + O(\frac{1}{T}).$$

Since $H(\cdot)$ has a bounded first derivative, we can exchange summands with integrals we obtain (13). To prove (14) we expand var(H_T) using the expansion of cov(Y_1Y_2,Y_3Y_4) in terms of cumulants and use Lemma 3.2 to get the result. To prove (15) we use the expansion in (12) and apply Theorem 2.2. □

4 Some bounds on cumulants and moments

In this section we state some bounds on the sums of moments and cumulants. These results will be used to prove Theorems 2.1 and 2.2. We mention that the techniques used in the proof of the results may also be of independent interest.

The following two lemmas concern summability of the higher order cumulants of a stochastic process. These results guarantee that Assumption 3.1(iii) is satisfied. We first state a bound for the sum of cumulants based on the mixing rate. This result is motivated by Neumann (1996), Remark 3.1. Let $\|X\|_p = (\mathbb{E}(|X|^p))^{1/p}$ and K denote a finite generic constant.

Lemma 4.1 *Let us suppose that $\{X_{t,T}\}$ is a α-mixing time series with rate $\{\alpha(t)\}$. If $t_1 \leq t_2 \leq \ldots \leq t_k$, then we have $|cum(X_{t_1,T},\ldots,X_{t_k,T})| \leq C_k \sup_{t,T} \||X_{t,T}\|_r^k \prod_{i=2}^{k}\alpha(t_i-t_{i-1})^{\frac{1-k/r}{k-1}}$,*

$$(i)\ \sup_{t_1}\sum_{t_2,\ldots,t_k=1}^{\infty} |cum(X_{t_1,T},\ldots,X_{t_k,T})| \leq C_k \sup_{t,T}\|X_{t,T}\|_r^k\big(\sum_t \alpha(t)^{\frac{1-k/r}{k-1}}\big)^{k-1} < \infty, \quad (16)$$

and for all $2 \leq j \leq k$ *we have*

$$\text{(ii)} \sup_{t_1} \sum_{t_2,\ldots,t_k=1}^{\infty} (1+|t_j|)|cum(X_{t_1,T},\ldots,X_{t_k,T})| \leq C_k \sup_{t,T} \|X_{t,T}\|_r^k \Big(\sum_t \alpha(t)^{\frac{1-k/r}{k-1}}\Big)^{k-1} < \infty, \quad (17)$$

where C_k *is a finite constant which depends only on* k.

PROOF. See Appendix. □

Using the lemma above, the following corollary on the absolute summability of the fourth order cumulants immediately follows. This corollary gives sufficient conditions, based on moments and mixing rates, for Assumption 3.1 to hold.

Corollary 4.1 *Suppose that* $\{X_{t,T}\}$ *is a* α*-mixing time series which satisfies Assumption 2.1(i), where* $\alpha(t) \leq K \cdot |t|^{-s}$.

(i) Let us suppose that $r > 4s/(s-3)$ *and* $\sup_{t,T} \mathbb{E}|X_{t,T}|^r < \infty$*, then we have* $|cov(X_{t,T},X_{\tau,T})| \leq C|t-\tau|^{-\frac{(s+3)}{2}}$ *and*
$\sup_{t_1} \sum_{t_2,t_3,t_4=-\infty}^{\infty} |cum(X_{t_1,T},X_{t_2,T},X_{t_3,T},X_{t_4,T})| < \infty$.

(ii) Let us suppose that $r > 4s/(s-6)$ *and* $\sup_{t,T} \mathbb{E}|X_{t,T}|^r < \infty$*, then we have* $|cov(X_{t,T},X_{\tau,T})| \leq C|t-\tau|^{-\frac{(s+6)}{2}}$ *and for all* $2 \leq j \leq 4$,
$\sup_{t_1} \sum_{t_2,t_3,t_4=-\infty}^{\infty} (1+|t_j|)|cum(X_{t_1,T},X_{t_2,T},X_{t_3,T},X_{t_4,T})| < \infty$.

It is worth mentioning that Assumption 2.1(iia) is weaker than those in Corollary 4.1, this is because we do not require absolute summability of the fourth order cumulants in order for $\text{var}(Q_T) = O(T^{-1})$.

In order to use a blocking argument to prove Theorems 2.1 and 2.2, we need to partition the data such that Q_T can be written as a sum of random variables which are non-intersecting. This is immediately possible with $Q_{T,M}$ but not Q_T, Thus we now define a close approximation of Q_T which satisfies this condition. Let

$$\tilde{Q}_{T,M} = \frac{1}{T} \sum_{t,\tau=1}^{T} I(\frac{t-\tau}{M}) G_{t,\tau} h(X_{t,T},X_{\tau,T}) \quad (18)$$

where $M = T^{1/2+\gamma}$ for some $0 < \gamma < 1/2$ and I is the indicator variable as defined in (12). Since $|G_{t,\tau}| \leq K|t-\tau|^{-\delta}$ ($\delta > 2$) we have

$$Q_T = \tilde{Q}_{T,M} + O_p(T^{-1/2-\gamma}), \quad (19)$$

and $\text{var}(\sqrt{T}Q_T) = \text{var}(\sqrt{T}\tilde{Q}_{T,M}) + O(T^{-\gamma})$. We will show that $\text{var}(\sqrt{T}Q_T) = O(1)$, thus Q_T and the truncated $\tilde{Q}_{T,M}$ are asymptotically equivalent. The results concerning $\tilde{Q}_{T,M}$ and $Q_{T,M}$ are largely the same, the only difference are the proofs, thus to unify notation, we let $Q_{T,M} := \tilde{Q}_{T,M}$ and $G_{t,\tau,M} = I(\frac{t-\tau}{M})G_{t,\tau}$, and state under what conditions we obtain the each result.

We now define sub-blocks of $Q_{T,M}$, which will be used to prove Theorem 2.2. Let

$$Y_{t,T} = \sum_{\tau<t} G_{t,\tau,M} h(X_{t,T}, X_{\tau,T}) + \sum_{\tau\leq t} G_{\tau,t,M} h(X_{\tau,T}, X_{t,T}). \tag{20}$$

In the case that $h(X_{t,T}, X_{\tau,T}) = X_{t,T}X_{\tau,T}$ the above is

$$Y_{t,M} = \sum_{\tau=1}^{t} F_{t,\tau,M} X_{t,T} X_{\tau,T} \quad \text{where} \quad F_{t,\tau,M} = \left\{ \begin{array}{cc} G_{t,t,M} & t=\tau \\ (G_{t,\tau,M} + G_{\tau,t,M}) & t \neq \tau \end{array} \right. .$$

To use the Bernstein blocking argument we define a sub-block of S_T. Let

$$B^{(u)}_{T,S_T} = \frac{1}{T} \sum_{t=u+1}^{S_T+u} Y_{t,T}, \tag{21}$$

noting that $B^{(0)}_{T,T} = Q_{T,M}$. Lemma 4.1 can be used to obtain bounds for $\mathrm{var}(B^{(u)}_{T,S_T})$ and other integer moments of $B^{(u)}_{T,S_T}$. However, in order prove asymptotic normality under relatively weak assumptions we will require bounds on non-integer moments of $B^{(u)}_{T,S_T}$, which use more subtle arguments. The actual proof used to obtain the bounds differs, depending on whether we use Assumption 2.1(iia,iiia) or 2.1(iib,iiib). Thus we state the results separately.

Lemma 4.2 *Suppose Assumption 2.1(i,ii(a), iii(a)) holds. Let* $\mathcal{F}_t = \sigma(X_{t,T}, X_{t-1,T}, \ldots)$. *If for some* $r > q$ *we have* $\sup_{t,\tau,T} \|h(X_{t,T}, X_{\tau,T})\|_r < \infty$, *then*

$$\left\|Y_{t,T} - \mathbb{E}(Y_{t,T}|\mathcal{F}_{t-j})\right\|_q \leq K\left(j^{-(\delta-1)} \sup_{\tau} \|h(X_{t,T}, X_{\tau,T})\|_q + \sup_{\tau} \|h(X_{t,T}, X_{\tau,T})\|_r j^{-s(\frac{1}{q}-\frac{1}{r})}\right). \tag{22}$$

and almost surely $Y_{t,T} = \sum_j N_{j,T}(t-j)$ *where* $N_{j,T}(t-j) = \mathbb{E}(Y_{t,T}|\mathcal{F}_{t-j}) - \mathbb{E}(Y_{t,T}|\mathcal{F}_{t-j-1})$. *Let* $q \geq 2$ *and* $B^{(u)}_{T,S_T}$ *be defined as in (21). Suppose the above conditions are satisfied, then we have*

$$\left\|B^{(u)}_{T,S_T}\right\|_q \leq KT^{-1} S_T^{1/2} \sum_{j=1}^{\infty} \left(\frac{1}{j^{\delta-1}} + \frac{1}{j^{s(\frac{1}{q}-\frac{1}{r})}}\right). \tag{23}$$

PROOF. See the appendix. □

Lemma 4.3 *Suppose Assumption 2.1(i,ii(b),iiib) holds. Let* $\mathcal{F}_{t,T} = \sigma(X_{t,T}, X_{t-1,T}, \ldots)$ *and denote* $\mathbb{E}(Z|\mathcal{F}_{j,T}) = \mathbb{E}_j(Z)$. *If for some* $r > q$ *we have* $\sup_{t,T} \|X_{t,T}\|_r < \infty$, *then we obtain the* *bound*

$$\|\mathbb{E}_{t-j}(X_{t,T}) - \mathbb{E}_{t-j-1}(X_{t,T})\|_q \leq 4(2^{1/q}+1)\alpha(j)^{\frac{1}{q}-\frac{1}{r}}\|X_{t,T}\|_r, \tag{24}$$

and $X_{t,T}$ *almost surely admits the representation* $X_{t,T} = \sum_{j=0}^{\infty}\big(\mathbb{E}_{t-j}(X_{t,T}) - \mathbb{E}_{t-j-1}(X_{t,T})\big)$*. Let* $M_j(t-j) = \mathbb{E}_{t-j}(X_{t,T}) - \mathbb{E}_{t-j-1}(X_{t,T})$*. If for some* $\tilde{r}/2 > r > q$ *we have* $\sup_{t,T}\|X_{t,T}\|_{\tilde{r}} < \infty$*, then*

$$\begin{aligned}&\big\|\mathbb{E}_{t-j_1-i}(M_{j_1}(t-j_1)M_{j_2}(t-j_1)) - \mathbb{E}_{t-j_1-i-1}(M_{j_1}(t-j_1)M_{j_2}(t-j_1))\big\|_q \\ \leq\;& K\|X_{t,T}\|_{\tilde{r}}^2\alpha(j_1)^{\frac{1}{2r}-\frac{1}{\tilde{r}}}\alpha(j_2)^{\frac{1}{2r}-\frac{1}{\tilde{r}}}\alpha(i)^{\frac{1}{q}-\frac{1}{r}}.\end{aligned} \tag{25}$$

Let $q \geq 2$ *and* $B_{T,S_T}^{(u)}$ *be defined as in (21). If there exists, an* $\tilde{r}$*, such that* $\sup_{t,T}\|X_{t,T}\|_{\tilde{r}} < \infty$*, where* $\tilde{r}/2 > r > q$*, then we have*

$$\big\|B_{T,S_T}^{(u)}\big\|_q \;\leq\; KT^{-1}S_T^{1/2}\Big[G_M^{1/2}\big(\sum_{j=1}^{\infty}\frac{1}{j^{s(\frac{1}{2q}-\frac{1}{r})}}\big)^2 + \big(\sum_{j=1}^{\infty}\frac{1}{j^{s(\frac{1}{q}-\frac{1}{r})}}\big)\big(\sum_{j=1}^{\infty}\frac{1}{j^{s(\frac{1}{2r}-\frac{1}{\tilde{r}})}}\big)^2\Big]. \tag{26}$$

PROOF. See the appendix. □

A simple application of the lemmas above is to derive bounds for the moments of the quadratic form $Q_{T,M}$ (since $Q_{T,M}$ is a special case of $B_{T,S_T}^{(u)}$, with $u=0$ and $S_T = T$). By using the arguments in Lemma 5.1, below, it can be shown that for some $\epsilon > 0$, we have $\|Q_{T,M}\|_{2+\epsilon} \leq K/T^{1/2}$ (under Assumption 2.1(i,iia,iiia)) and $\|Q_{T,M}\|_{2+\epsilon} \leq KG_M^{1/2}/T^{1/2}$ (under Assumption 2.1(i,iib,iiib)).

5 Proof of Theorems 2.1 and 2.2

To do the analysis, we start by rewriting $Q_{T,M} - \mathbb{E}(Q_{T,M})$ as

$$Q_{T,M} - \mathbb{E}(Q_{T,M}) \;=\; \frac{1}{T}\sum_{t,\tau=1}^{T}G_{t,\tau,M}\big(h(X_{t,T},X_{\tau,T}) - \mathbb{E}(h(X_{t,T},X_{\tau,T})\big) = \sum_{t=1}^{T}Y_{t,T},$$

where $Y_{t,T}$ is defined in (20). To prove asymptotic normality we use a classical Bernstein blocking argument. Here we partition $\{Y_{t,T}; t = 1,\ldots,T\}$ into the sum of small and large blocks. Let $U_{i,T}$ and $V_{i,T}$ denote the big blocks and small blocks respectively, where

$$U_{i,T} = \sum_{t=ir_T+1}^{ir_T+p_T}Y_{t,T}, \qquad V_{i,T} = \sum_{t=ir_T+p_T+1}^{(i+1)r_T}Y_{t,T},$$

$p_T >> q_T >> M$ and $r_T = (p_T + q_T)$. Let $k_T = T/(p_T + q_T)$ and $q_T/(p_T + q_T) \to 0$ as $T \to \infty$. For the purpose of proving the results below we wil assume that $k_T = O((\log T)^{1/2})$. Using the above notation we let $Q_{T,M} - \mathbb{E}(Q_{T,M}) = \mathcal{S}_{k_T} + \mathcal{R}_{k_T}$, where

$$\mathcal{S}_{k_T} = \sum_{i=1}^{k_T} U_{i,T} \quad \text{and} \quad \mathcal{R}_{k_T} = \sum_{i=1}^{k_T} V_{i,T}.$$

Since $p_T >> q_T$, we will show that $\text{var}\big(\sqrt{\frac{T}{G_M}}\mathcal{R}_{k_T}\big) \to 0$. We first obtain moment bounds for $\{U_{i,T}\}$ and $\{V_{i,T}\}$. We note that under Assumption 2.1(i,iia,iiia), that $G_M := G \leq K\sum_{j=1}^{\infty} j^{-\delta} < \infty$.

Lemma 5.1 *Let us suppose Assumptions 2.1 holds. Then for some $\delta > 0$ we have*

$$\big\|U_{i,T}\big\|_{2+\delta} = O\big(\frac{p_T^{1/2}G_M^{1/2}}{T}\big) \qquad \big\|V_{i,T}\big\|_{2+\delta} = O\big(\frac{q_T^{1/2}G_M^{1/2}}{T}\big). \tag{27}$$

PROOF. We use Lemmas 4.2 and Lemma 4.3 to prove the result, with $p_T = S_T$ and $u = ir_T$. We first prove the result under Assumption 2.1(i,iia,iiia). By applying Lemma 4.2 for $q = 2 + \delta$ and $r > 2 + \delta$ we have

$$\big\|B_{T,S_T}^{(u)}\big\|_{2+\delta} \leq K \sup_{t,\tau,T} \|g(X_{t,T}, X_{\tau,T})\|_r T^{-1} S_T^{1/2} \sum_{j=1}^{\infty} \bigg(\frac{1}{j^{\delta-1}} + \frac{1}{j^{s(\frac{1}{2+\delta}-\frac{1}{r})}}\bigg).$$

Thus the above bound is finite for $r > s(2+\delta)/(s-2-\delta)$. In other words, if $r > 2s/(s-2)$, there exists a δ, such that $\big\|B_{T,S_T}^{(u)}\big\|_{2+\delta} = O(\frac{q_T^{1/2}G_M^{1/2}}{T})$.

To apply Lemma 4.3 for $q = 2 + \delta$, then for some $\tilde{r}/2 > r > 2 + \delta$ we have

$$\big\|U_{i,T}\big\|_{2+\delta} \leq KT^{-1}p_T^{1/2}\|X_{t,T}\|_{\tilde{r}}\bigg(G_M^{1/2}\big(\sum_{j=1}^{\infty}\frac{1}{j^{s(\frac{1}{2(2+\delta)}-\frac{1}{\tilde{r}})}}\big)^2 + \big(\sum_{j=1}^{\infty}\frac{1}{j^{s(\frac{1}{(2+\delta)}-\frac{1}{r})}}\big)\big(\sum_{j=1}^{\infty}\frac{1}{j^{s(\frac{1}{2r}-\frac{1}{\tilde{r}})}}\big)^2\bigg).$$

In order to ensure that the right hand side of the above is finite, $\tilde{r}$ should satisfy the conditions

$$\frac{1}{2(2+\delta)} - \frac{1}{\tilde{r}} > \frac{1}{s}, \quad \frac{1}{2+\delta} - \frac{1}{r} > \frac{1}{s} \quad \text{and} \quad \frac{1}{2r} - \frac{1}{\tilde{r}} > \frac{1}{s},$$

which implies

$$\tilde{r} > \frac{2(2+\delta)s}{(s-2(2+\delta)} \quad \text{and} \quad \tilde{r} > \frac{2s(2+\delta))}{(s-3(2+\delta)}. \tag{28}$$

Thus by Assumption 2.1(iiib) (we recall there exists an r such that $r > 4s/(s-6)$ and $\sup_{t,T}\|X_{t,T}\|_r < \infty$), there exists a $\tilde{r}$ and $\delta > 0$, such that (28) is satisfied.

Thus for both cases, (27) holds for some $\delta > 0$. The proof of $\big\|V_{i,T}\big\|_{2+\delta} = O\big(\frac{q_T^{1/2}G_M^{1/2}}{T}\big)$ is the same, hence we omit the details. $\square$

We now show that the contribution of the sum of small blocks, $\mathcal{R}_{k_T}$, is negligible with respect to the entire sum $Q_{T,M} - \mathbb{E}(Q_{T,M})$.

Lemma 5.2 *Suppose Assumption 2.1 holds and $q_T/(p_T+q_T) \to 0$ as $T \to \infty$. Then we have*

$$\big|cov\big(V_{i_1,T}, V_{i_2,T}\big)\big| \leq C\alpha\Big(|i_1 - i_2|p_T - M\Big)^{1-\frac{2}{2+\delta}}\big(\frac{G_M q_T}{T^2}\big) \tag{29}$$

and

$$var\Big(\sqrt{\frac{T}{G_M}}\mathcal{R}_{k_T}\Big) \leq C\frac{q_T}{(p_T+q_T)} \to 0, \tag{30}$$

as $T \to \infty$, where C is a finite constant.

PROOF. Define the sigma-algebras $\mathcal{G}_{i_2}^{\infty} = \sigma(Y_{i_2 r_T + p_T + 1,T}, Y_{i_2 r_T + p_T + 2,T}, \ldots)$ and
$\mathcal{G}_{-\infty}^{i_1} = \sigma(Y_{(i_1+1)r_T,T}, Y_{(i_1+1)r_T-1,T}, \ldots)$. To prove (29) for $i_2 > i_1$ we use Ibragimov's inequality to obtain

$$\begin{aligned}
\big|\mathrm{cov}\big(V_{i_1,T}, V_{i_2,T}\big)\big| &\leq C\Big\{\sup_{\substack{A \in \mathcal{G}_{i_2}^{\infty} \\ B \in \mathcal{G}_{-\infty}^{i_1}}} \big|P(A \cap B) - P(A)P(B)\big|\Big\}^{1-\frac{2}{2+\delta}}\big\|V_{i_1,T}\big\|_{2+\delta}^2 \\
&\leq C\big\{\alpha\big((i_2 - i_1 - 1)r_T + p_T + 1 - M\big)\big\}^{1-\frac{2}{2+\delta}}\big\|V_{i_1,T}\big\|_{2+\delta}^2 \\
&\leq C\alpha\big((i_2 - i_1)p_T - M\big)^{1-2/(2+\delta)}\big\|V_{i_1,T}\big\|_{2+\delta}^2.
\end{aligned} \tag{31}$$

This gives (29). To prove (30) we substitute (29) into $\mathrm{var}(\mathcal{R}_{k_T}) = \sum_{i_1,i_2=1}^{k_T}\mathrm{cov}\big(V_{i_1,T}, V_{i_2,T}\big)$ and use that $\|V_{i,T}\|_{2+\delta} = O(q_T^{1/2}G_M^{1/2}/T)$ to get

$$\mathrm{var}\big(\sqrt{\frac{T}{G_M}}\mathcal{R}_{k_T}\big) \;\leq\; C\frac{q_T}{T}\Big(\sum_{i=1}^{k_T} 1 + 2\sum_{i_1<i_2}^{k_T}\alpha\big(|i_1 - i_2|p_T - M\big)^{1-2/(2+\delta)}\Big).$$

Now by using that the mixing rate $\alpha(t) \leq Kt^{-s}$ and $k_T = T/(p_T+q_T)$ we have

$$\begin{aligned}
\mathrm{var}\big(\sqrt{\frac{T}{G_M}}\mathcal{R}_{k_T}\big) &\leq K\frac{q_T}{p_T+q_T}\Big(1 + \sum_{r=1}^{k_T}(rp_T - M)^{-s(1-\frac{2}{(2+\delta)})}\Big) \\
&\leq K\frac{q_T}{p_T+q_T}\Big(1 + (p_T - M)^{-s(1-\frac{2}{2+\delta})}k_T\Big).
\end{aligned}$$

Since $k_T = (\log T)^{1/2}$, we have $((p_T - M))^{-s(1-\frac{2}{2+\delta})}k_T < \infty$, which gives (30). $\square$

Using that $Q_{T,M} = \mathcal{S}_T + \mathcal{R}_T$ and $\mathrm{var}(Q_{T,M}) := V_T = O(\frac{G_M}{T})$, the above result implies that $\mathrm{var}{Q_{T,M}}^{-1/2}\mathcal{R}_T = o(1)$ and

$$V_T^{-1/2}(Q_{T,M} - \mathbb{E}(Q_{T,M})) = V_T^{-1/2}\mathcal{S}_{k_T} + o_p(1). \tag{32}$$

We now show normality of $\mathcal{S}_{k_T}$. We do this by replacing $\mathcal{S}_{k_T}$ with $\tilde{\mathcal{S}}_{k_T} = \sum_i \tilde{U}_{i,T}$, where $\tilde{U}_{i,T}$ and $U_{i,T}$ have identical distributions, but $\{\tilde{U}_{i,T}\}$ are independent random variables. Below we show that the distributions of $\mathcal{S}_{k_T}$ and $\tilde{\mathcal{S}}_{k_T}$ are asymptotically equivalent.

We require the following general theorem, which gives a bound on the differences of characteristic functions of sums mixing and independent random variables. A potentially useful aspect of this result, is that we allow for the mixing rate to change with T.

Theorem 5.1 *Suppose $\{Z_{t,T}\}$ is an α-mixing sequence which for $t < \tau + s_T$ satisfies*

$$\sup_{A\in\sigma(Z_{t,T},Z_{t-1,T},\ldots),B\in\sigma(Z_{\tau,T},Z_{\tau+1,T},\ldots)} \big|P(A\cap B) - P(A)P(B)\big| \leq a(|t-\tau| - s_T). \tag{33}$$

Let $W_{i,T} = \sum_{t=ir_T+1}^{ir_T+p_T} Z_{t,T}$, where $r_T = p_T + q_T$ and $\{\tilde{W}_{i,T}\}$ be independent random variables where the marginal distributions of $\tilde{W}_{i,T}$ and $W_{i,T}$ are the same. Then, for any $x\in\mathbb{R}$, we have

$$\left|\mathbb{E}\bigg(\exp(ix\sum_{j=1}^{k_T} W_{j,T})\bigg) - \prod_{j=1}^{k_T}\mathbb{E}\bigg(\exp(ix\tilde{W}_{j,T})\bigg)\right| \leq Ck_T a(q_T - s_T),$$

where C is a finite constant.

PROOF. By expanding $\mathbb{E}\bigg(\exp(ix\sum_{j=1}^{k_T} W_{j,T})\bigg) - \prod_{j=1}^{k_T}\mathbb{E}\bigg(\exp(ix\tilde{W}_{j,T})\bigg)$, we have

$$\begin{aligned} D_T &= \left|\mathbb{E}\bigg(\exp(ix\sum_{j=1}^{k_T} W_{j,T})\bigg) - \prod_{j=1}^{k_T}\mathbb{E}\bigg(\exp(ix\tilde{W}_{j,T})\bigg)\right| \\ &\leq \sum_{s=1}^{k_T-1}\Big|\prod_{r=1}^{s-1}\mathbb{E}\big(\exp(ixW_{r,T})\big)\Big|\left|\mathrm{cov}\bigg(\exp(ixW_s), \exp(ix\sum_{j=s+1}^{k_T} W_j)\bigg)\right|, \end{aligned}$$

(to simplify notation we denote $\prod_{r=1}^{0} A_r = 1$). From the definition of $W_{i,T}$ and by using Ibragimov's inequality (for bounded random variables) it is straightforward to show that

$$\leq \sum_{s=1}^{k_T-1} \sup_{\substack{A\in\sigma(Z_{(s+1)r_T+1,T},Z_{(s+1)r_T+2,T},\ldots)\\ B\in\sigma(Z_{sr_T+p_T,T},Z_{sr_T+p_T-1})}} |P(A\cap B) - P(A)P(B)|\big| \leq Ck_T a(q_T - s_T).$$

The above gives the required result. □

Lemma 5.3 *Suppose that Assumption 2.1 holds, and we choose* p_T *and* q_T *such that* $p_T >> q_T >> M$ *and* $k_T = (\log T)^{1/2}$*, where* $k_T = T/(p_T + q_T)$*, then the asymptotic distributions of* $V_T^{-1/2}(Q_{T,M} - \mathbb{E}(Q_{T,M}))$ *and* $V_T^{-1/2}\tilde{\mathcal{S}}_{k_T}$ *are equivalent.*

PROOF. From (32) we have $V_T^{-1/2}(Q_{T,M} - \mathbb{E}(Q_{T,M})) = V_T^{-1/2}\mathcal{S}_{k_T} + o_p(1)$. By using Theorem 5.1 with $Z_{t,T} := Y_{t,T} = T^{-1}\sum_{\tau=\max(t-M,1)}^{t} F_{t,\tau,M}(X_{t,T}X_{\tau,T} - \mathbb{E}(X_{t,T}X_{\tau,T}))$ and $W_{i,T} := U_{i,T}$ we have

$$|\Phi_{k_T}(x) - \tilde{\Phi}_{k_T}(x)| \leq k_T\alpha(q_T - M),$$

where $\Phi_{k_T}(\cdot)$ and $\tilde{\Phi}_{k_T}(\cdot)$ are the characteristic functions of $\mathcal{S}_{k_T}$ and $\tilde{\mathcal{S}}_{k_T}$. Since $p_T >> q_T >> M$ and $k_T = (\log T)^{1/2}$, and under Assumption 2.1(i) we have that $|\Phi_{k_T}(x) - \tilde{\Phi}_{k_T}(x)| \to 0$. Since the characteristic functions converge, we obtain the required result. □

We now show asymptotic normality of $V_T^{-1/2}\tilde{\mathcal{S}}_{k_T}$, this result together with the above lemma will give Theorems 2.1 and 2.2.

Lemma 5.4 *Suppose Assumption 2.1 is satisfied. Then we have*

$$V_T^{-1/2}\tilde{\mathcal{S}}_{k_T} \xrightarrow{D} \mathcal{N}(0,1).$$

PROOF. We will use the central limit theorem for independent random variables. Due to the independence of $\tilde{U}_{i,T}$ it is straightforward to show $\frac{1}{T}\sum_{i=1}^{k_T}\mathbb{E}(\tilde{U}_{i,T}^2) \to V_T$, hence it remains to verify Lindeberg's condition. By using (27) we have $\sum_{i=1}^{k_T}\mathbb{E}\big[(V_T^{-1/2}|\tilde{U}_{i,T}|)^{2+\delta}\big] \leq K\big(\frac{p_T}{T}\big)^{\delta/2} \to 0$, as $T \to \infty$. Thus Lindeberg's condition is fulfilled and we have asymptotic normality of $\tilde{\mathcal{S}}_{k_T}$. □.

PROOF of Theorem 2.1 To prove the result we show that Q_T we use that $Q_T = Q_{T,T^{1/2+\gamma}} + O_p(T^{-1/2-\gamma})$, where

$$Q_{T,T^{1/2+\gamma}} = \frac{1}{T}\sum_{t,\tau=1}^{T} I(\frac{t-\tau}{T^{1/2+\gamma}})G_{t,\tau,T}X_{t,T}X_{\tau,T}.$$

Thus by (32) we have

$$\mathrm{var}(Q_T)^{-1/2}\big(Q_T - \mathbb{E}(Q_T)\big) = \mathrm{var}(Q_T)^{-1/2}\mathcal{S}_{T,T^{1/2+\gamma}} + o_p(1), \qquad (34)$$

and $\mathrm{var}(\sqrt{T}Q_T) = \mathrm{var}(\sqrt{T}Q_{T,T^{1/2+\gamma}}) + O(T^{-\gamma})$. We observe that $Q_{T,T^{1/2+\gamma}}$ satisfies representation (3) and Assumption 2.1, thus by applying Lemma 5.4, then $V_T^{-1/2}(Q_{T,T^{1/2+\gamma}} - \mathbb{E}(Q_T)) \xrightarrow{D} \mathcal{N}(0,1)$. Therefore from (34) we have $V_T^{-1/2}(Q_T - \mathbb{E}(Q_T)) \xrightarrow{D} \mathcal{N}(0,1)$, which gives the desired result. □

PROOF of Theorem 2.2 By using Lemma 5.3, it is straightforward to show that $V_T^{-1/2}(Q_{T,M} - \mathbb{E}(Q_{T,M}))$ and $V_T^{-1/2}\tilde{\mathcal{S}}_{k_T}$ have asymptotically the same distribution. Now by using the same arguments as in the proof of Theorem 2.1 we obtain the result. □

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A Appendix

A.1 Proofs of results in Section 4

PROOF of Lemma 4.1 To prove the lemma we apply a result from Statulevicius and Jakimavicius (1988), Theorem 3, part (2), which states that if $t_1 \leq t_2 \leq \ldots \leq t_k$, then for all $2 \leq i \leq k$ we have $\big|\mathrm{cum}(X_{t_1,T}, X_{t_2,T}, \ldots, X_{t_k,T})\big| \leq 3(k-1)!2^{k-1}\alpha(t_i - t_{i-1})^{1-\frac{k}{r}} \sup_{t,T} \|X_{t,T}\|_r^k$.

To prove the result the first part of the lemma, we use a method similar to the proof of Neumann (1996), Remark 3.1. By taking the $(k-1)$th root of the above for all $2 \leq i \leq k$ we have

$$\big|\mathrm{cum}(X_{t_1,T}, X_{t_2,T}, \ldots, X_{t_k,T})\big|^{\frac{1}{k-1}} \leq C_k^{1/(k-1)}\alpha(t_i - t_{i-1})^{\frac{1-k/r}{k-1}} \sup_{t,T} \|X_{t,T}\|_r^{\frac{k}{k-1}},$$

where $C_k = 3(k-1)!2^{k-1}$. Since the above bound holds for all i, multiplying the above over i gives

$$\big|\mathrm{cum}(X_{t_1,T}, X_{t_2,T}, \ldots, X_{t_k,T})\big| \leq C_k \sup_{t,T} \|X_{t,T}\|_r^k \prod_{i=2}^{k} \alpha(t_i - t_{i-1})^{\frac{1-k/r}{k-1}}, \qquad (35)$$

thus proving the first part of the lemma.

To prove (i), we rewrite $\sum_{t_2,\ldots,t_k=1}^{\infty}$ as the sum of orderings, that is $\sum_{t_2,\ldots,t_k=1}^{\infty} = k! \sum_{1=t_2 \leq \ldots \leq t_k}^{\infty}$. Now since the number of orderings is finite, we can use (i) to obtain

$$\sum_{t_2,\ldots,t_k=1}^{\infty} \big|\mathrm{cum}(X_{t_1,T}, X_{t_2,T}, \ldots, X_{t_k,T})\big| \leq C_k \sup_{t,T} \|X_{t,T}\|_r^k \big\{\sum_r \alpha(r)^{\frac{(1-k/r)}{(k-1)}}\big\}^{k-1} < \infty,$$

which gives (16). To prove (ii) we use a similar argument to obtain

$$\begin{aligned}\sum_{t_2,\ldots,t_k=1}^{\infty}(1+|t_j|)\big|\mathrm{cum}(X_{t_1,T},X_{t_2,T},\ldots,X_{t_k,T})\big| &\leq \sum_{1\leq t_2<\ldots<t_k<\infty}(1+|t_j|)\big|\mathrm{cum}(X_{t_1,T},X_{t_2,T},\ldots,X_{t_k,T})\big| \\ &= k!\sum_{r_2,\ldots,r_k=1}^{\infty}(1+\sum_{i=2}^{j}|r_i|)\big|\mathrm{cum}(X_{t_1,T},X_{t_2,T},\ldots,X_{t_k,T})\big|,\end{aligned}$$

substituting (35) into the above gives the result. □

PROOF of Lemma 4.2 To prove the result we use the notion of Near Epoch Dependence. This requires bounding

$$\big\|Y_{t,T}-\mathbb{E}(Y_{t,T}|\mathcal{F}_{t-j})\|_q = A_1 + A_2$$

where

$$\begin{aligned}A_1 &= \Big\|\sum_{\tau<t}G_{t,\tau,M}h(X_{t,T},X_{\tau,T})-\mathbb{E}(\sum_{\tau<t}G_{t,\tau,M}h(X_{t,T},X_{\tau,T})|\mathcal{F}_{t-j})\Big\|_q \\ A_2 &= \Big\|\sum_{\tau\leq t}G_{\tau,t,M}h(X_{\tau,T},X_{t,T})-\mathbb{E}(\sum_{\tau\leq t}G_{\tau,t,M}h(X_{\tau,T},X_{t,T})|\mathcal{F}_{t-j})\Big\|_q.\end{aligned}$$

As the derivation of bounds on A_1 and A_2 are identical, we shall focus on A_1. We first observe that by using the Minkowski inequality we have

$$A_1 = \Big\|\sum_{\tau=1}^{t-1}G_{t,\tau,M}h(X_{t,T},X_{\tau,T})-\mathbb{E}(\sum_{\tau<t}G_{t,\tau,M}h(X_{t,T},X_{\tau,T})|\mathcal{F}_{t-j})\Big\|_q \leq I+II,$$

where

$$\begin{aligned}I &= \|\sum_{\tau=1}^{t-1}G_{t,\tau,M}h(X_{t,T},X_{\tau,T})-\mathbb{E}(\sum_{\tau<t}G_{t,\tau,M}h(X_{t,T},X_{\tau,T})|\mathcal{F}_{t-j/2}^{t})\big\|_q \\ II &= \big\|\mathbb{E}\big(\mathbb{E}(\sum_{\tau=1}^{t-1}G_{t,\tau,M}h(X_{t,T},X_{\tau,T})|\mathcal{F}_{t-j/2}^{t})|\mathcal{F}_{t-j}\big)-\mathbb{E}\big(\sum_{\tau=1}^{t-1}G_{t,\tau,M}h(X_{t,T},X_{\tau,T})\big)\big\|_q.\end{aligned}$$

and $\mathcal{F}^t_{t-j/2} = \sigma(X_{t,T}, X_{t-1,T}, \ldots, X_{t-j/2,T})$. To bound I we note that for $t > \tau$ and all j we have

$$\begin{aligned}
& \big\|\sum_{\tau=1}^{t-1} G_{t,\tau,M} h(X_{t,T}, X_{\tau,T}) - \mathbb{E}(\sum_{\tau=1}^{t-1} G_{t,\tau,M} h(X_{t,T}, X_{\tau,T}) | \mathcal{F}^t_{t-j})\big\|_q \\
= \;& \Big\| \sum_{k=j} G_{t,t-k} \big\{ h(X_{t,T}, X_{t-k,T}) - \mathbb{E}(h(X_{t,T}, X_{t-k,T}) | \mathcal{F}^t_{t-j}) \big\} \Big\|_q \\
\leq \;& \sum_{k=j} |G_{t,t-k}| \Big\| \big\{ h(X_{t,T}, X_{t-k,T}) - \mathbb{E}(h(X_{t,T}, X_{t-k,T}) | \mathcal{F}^t_{t-j}) \big\} \Big\|_q \\
\leq \;& K \sup_{\tau,T} \|h(X_{t,T}, X_{\tau,T})\|_q \sum_{k=j}^{\infty} k^{-\delta} = K \sup_{\tau,T} \|h(X_{t,T}, X_{\tau,T})\|_q (j/2)^{-(\delta-1)}.
\end{aligned}$$

where we use that $|G_{t,t-k,M}| \leq K|t-\tau|^{\delta}$. Furthermore, to bound II we use that $\mathbb{E}(\sum_{\tau=1}^{t-1} G_{t,\tau,M} h(X_{t,T}, X_{\tau,T})$
$\mathcal{F}^t_{t-j/2}$ together with Ibragimov's inequality to obtain

$$II \;\; \leq \;\; K(j/2)^{-s(\frac{1}{q}-\frac{1}{r})} \sum_{\tau<t} |G_{t,\tau,M}| \|h(X_{t,T}, X_{\tau,T})\|_r$$

Thus altogether we have

$$A_1 \;\; \leq \;\; K\bigg(j^{-(\delta-1)} \sup_{t,\tau,T} \|h(X_{t,T}, X_{\tau,T})\|_q + \sup_{t,\tau,T} \|h(X_{t,T}, X_{\tau,T})\|_r j^{-s(\frac{1}{q}-\frac{1}{r})} \bigg).$$

A similar bound also applies to A_2, thus altogether this gives

$$\big\| Y_{t,T} - \mathbb{E}(Y_{t,T} | \mathcal{F}_{t-j}) \big\|_q \leq K\bigg(j^{-(\delta-1)} \|h(X_{t,T}, X_{t-j,T})\|_q + \|h(X_{t,T}, X_{t-j,T})\|_r j^{-s(\frac{1}{q}-\frac{1}{r})} \bigg),$$

and we have shown the first part of the required result.

To show the second part we note that since $\big\| Y_{t,T} - \mathbb{E}(Y_{t,T} | \mathcal{F}_{t-j}) \big\|_q \to 0$ as $T \to \infty$, thus we almost surely have the representation $Y_{t,T} - \mathbb{E}(Y_{t,T}) = \sum_j N_{j,T}(t-j)$, where

$$N_{j,T}(t-j) = \mathbb{E}(Y_{t,T} | \mathcal{F}_{t-j}) - \mathbb{E}(Y_{t,T} | \mathcal{F}_{t-j-1}).$$

Thus substituting the above into $\|B^{(u)}_{T,S_T}\|_q$ and using the Burkholder inequality we have

$$\begin{aligned}
B^{(u)}_{T,S_T} \;\; = \;\; & \| \sum_{t=u}^{S_T+u} \sum_{j=0}^{\infty} N_{j,T}(t-j) \|_q \leq \sum_{j=0}^{\infty} \| \sum_{t=u}^{S_T+u} N_{j,T}(t-j) \|_q \leq \sum_{j=0}^{\infty} \bigg(\sum_{t=u}^{S_T+u} \|N_{j,T}(t-j)\|_q^2 \bigg)^{1/2} \\
\leq \;\; & S_T^{1/2} \bigg(\sum_{j=1}^{\infty} (j^{-\delta+2} + j^{-s(\frac{1}{q}-\frac{1}{r})+1}) \bigg),
\end{aligned}$$

as required. $\square$

PROOF of Lemma 4.3 The proof of (24) follows immediately from Ibragimov's inequality (Ibragimov (1962)) (see also Davidson (1994), Theorem 14.2). Using this we note that since $X_{t,T} = \mathbb{E}(X_{t,T}|\mathcal{F}_{t,T})$ and $\mathbb{E}(X_{t,T}|\mathcal{F}_{t-j}) \to 0$ as $j \to \infty$, almost surely we have

$$X_{t,T} = \sum_{j=0}^{\infty} \big(\mathbb{E}_{t-j}(X_{t,T}) - \mathbb{E}_{t-j-1}(X_{t,T})\big). \tag{36}$$

To prove (25), we use Ibragimov's and Chebyshev's inequalities and (24) to obtain

$$\begin{aligned}
& \big\|\mathbb{E}(M_{j_1}(t-j_1)M_{j_2}(t-j_1)|\mathcal{F}_{t-j_1-i}) - \mathbb{E}(M_{j_1}(t-j_1)M_{j_2}(t-j_1)|\mathcal{F}_{t-j_1-i-1})\big\|_q \\
\leq\ & 2\big\|\mathbb{E}(M_{j_1}(t-j_1)M_{j_2}(t-j_1)|\mathcal{F}_{t-j_1-i}) - \mathbb{E}(M_{j_1}(t-j_1)M_{j_2}(t-j_1))\big\|_q \\
\leq\ & 4(2^{1/q}+1)\|M_{j_1}(t-j_1)M_{j_2}(t-j_1)\|_r \alpha(i)^{\frac{1}{q}-\frac{1}{r}} \\
\leq\ & 12\|M_{j_1}(t-j_1)\|_{2r}\|M_{j_2}(t-j_1)\|_{2r}\alpha(i)^{\frac{1}{q}-\frac{1}{r}} \\
\leq\ & 12^3 \sup_{t,T}\|X_{t,T}\|_{\tilde{r}}^2 \alpha(j_1)^{\frac{1}{2r}-\frac{1}{\tilde{r}}}\alpha(j_2)^{\frac{1}{2r}-\frac{1}{\tilde{r}}}\alpha(i)^{\frac{1}{q}-\frac{1}{r}},
\end{aligned}$$

where $\tilde{r}/2 > r > q$. Now we prove (26). By substituting (36) into $B_{T,S_T}^{(u)}$ and using the above notation for conditional expectations we have

$$\begin{aligned}
B_{T,S_T}^{(u)} &= T^{-1}\sum_{t=u+1}^{S_T+u}\sum_{\tau} F_{t,\tau,M}(X_{t,T}X_{\tau,T} - \mathbb{E}(X_{t,T}X_{\tau,T})) \\
&= T^{-1}\sum_{j_1,j_2=0}^{\infty}\sum_{t=u+1}^{S_T+u}\sum_{\tau=\max(t-M,1)}^{t} F_{t,\tau,M}\big(M_{j_1}(t-j_1)M_{j_2}(\tau-j_2) - \mathbb{E}(M_{j_1}(t-j_1)M_{j_2}(\tau-j_2))\big).
\end{aligned}$$

Partitioning the above sum into various cases and using Minkowski's inequality gives

$$\begin{aligned}
\big\|B_{T,S_T}^{(u)}\big\|_q &= T^{-1}\sum_{j_1,j_2=0}^{\infty}\Big\|\sum_{t=u+1}^{S_T+u}\sum_{\tau}F_{t,\tau,M}\big\{M_{j_1}(t-j_1)M_{j_2}(\tau-j_2) - \mathbb{E}(M_{j_1}(t-j_1)M_{j_2}(\tau-j_2))\big\}\Big\|_q \\
&\leq I + II + III,
\end{aligned}$$

where

$$\begin{aligned}
I &= T^{-1}\sum_{j_1,j_2=0}^{\infty}\Big\|\sum_{t=u+1}^{S_T+u}\sum_{\tau<t-j_1+j_2}F_{t,\tau,M}M_{j_1}(t-j_1)M_{j_2}(\tau-j_2)\Big\|_q\\
II &= T^{-1}\sum_{j_1,j_2=0}^{\infty}\Big\|\sum_{\tau}\sum_{t<\tau-j_2+j_1}F_{t,\tau,M}M_{j_1}(t-j_1)M_{j_2}(\tau-j_2)\Big\|_q\\
III &= T^{-1}\Big\|\sum_{j_1,j_2=0}^{\infty}\sum_{t=u+1}^{S_T+u}F_{t,t-j_1+j_2,M}\big(M_{j_1}(t-j_1)M_{j_2}(t-j_1)-\mathbb{E}(M_{j_1}(t-j_1)M_{t-j_1+j_2}(t-j_1))\big)\Big\|_q.
\end{aligned}$$

We observe that $\{\sum_{\tau<t-j_1+j_2}F_{t,\tau,M}M_{j_1}(t-j_1)M_{j_2}(\tau-j_2)\}_t$ and $\{\sum_{t<\tau-j_2+j_1}^{T}F_{t,\tau,M}M_{j_1}(t-j_1)M_{j_2}(\tau-j_2)\}_\tau$ are martingale differences. Therefore by using the Burkholder-Rosenthal inequality twice together with Cauchy-Schwarz, for $q\geq 2$ we have

$$\begin{aligned}
I &\leq T^{-1}\sum_{j_1,j_2=0}^{\infty}\Big(\sum_{t=u+1}^{S_T+u}\Big\|\sum_{\tau<t-j_1+j_2}F_{t,\tau,M}M_{j_1}(t-j_1)M_{j_2}(\tau-j_2)\Big\|_q^2\Big)^{1/2}\\
&\leq T^{-1}\sum_{j_1,j_2=0}^{\infty}\Big(\sum_{t=u+1}^{S_T+u}\big\|M_{j_1}(t-j_1)\big\|_{2q}^2\Big\|\sum_{\tau<t-j_1+j_2}F_{t,\tau,M}M_{j_2}(\tau-j_2)\Big\|_{2q}^2\Big)^{1/2}\\
&\leq T^{-1}\sum_{j_1,j_2=0}^{\infty}\Big(\sum_{t=u+1}^{S_T+u}\big\|M_{j_1}(t-j_1)\big\|_{2q}^2\sum_{\tau<t-j_1+j_2}|F_{t,\tau,M}|^2\big\|M_{j_2}(\tau-j_2)\big\|_{2q}^2\Big)^{1/2}.
\end{aligned}$$

Using (24) we have $\|M_j(t-j)\|_{2q}\leq C\alpha(j)^{\frac{1}{2q}-\frac{1}{r}}$. Substituting these bounds into I and under Assumption 2.1(i) we have

$$\begin{aligned}
I &\leq T^{-1}C\Big(\sum_{j=0}^{\infty}\alpha(j)^{\frac{1}{2q}-\frac{1}{r}}\Big)^2\Big(\sum_{t=u+1}^{S_T+u}\sum_{\tau}|F_{t,\tau,M}|^2\Big)^{1/2}\leq T^{-1}S_T^{1/2}K\Big(\sum_{j=0}^{\infty}\alpha(j)^{\frac{1}{2q}-\frac{1}{r}}\Big)^2\sup_t\Big(\sum_{\tau}|F_{t,\tau,M}|^2\Big)^{1/2}\\
&\leq KT^{-1}S_T^{1/2}G_M^{1/2}\Big(\sum_{j=0}^{\infty}\alpha(j)^{\frac{1}{2q}-\frac{1}{r}}\Big)^2. \qquad (37)
\end{aligned}$$

Using the same methods we have

$$II \leq KT^{-1}S_T^{1/2}G_M^{1/2}\Big(\sum_{j=0}^{\infty}\alpha(j)^{\frac{1}{2q}-\frac{1}{r}}\Big)^2. \qquad (38)$$

Finally we obtain a bound for III. This requires a more delicate analysis since $\{M_j(t-j)M_{t-(\tau-j)}(t-j)-\mathbb{E}(M_j(t-j)M_{t-(\tau-j)}(t-j))\}$ are not necessarily martingale differences over t. We first represent $M_j(t-j_1)M_{j_2}(t-j_1)-\mathbb{E}(M_{j_1}(t-j_1)M_{j_2}(t-j_1))$ as the sum of martingale differences. Since $\mathbb{E}(M_{j_1}(t-j_1)M_{j_2}(t-j_1)|\mathcal{F}_{t-j_1-i})\overset{a.s.}{\to}\mathbb{E}(M_{j_1}(t-j_1)M_{j_2}(t-j_1))$, as $i\to\infty$, we have

$$M_j(t-j_1)M_{\tau-(t-j)}(t-j) - \mathbb{E}(M_j(t-j)M_{\tau-(t-j)}(t-j)) = \sum_{i=0}^{\infty} A_{j_1,j_2;i}(t-j_1-i),$$

almost surely, where

$$A_{j_1,j_2;i}(t-j_1-i) = \mathbb{E}(M_{j_1}(t-j_1)M_{j_2}(t-j_1)|\mathcal{F}_{t-j_1-i}) - \mathbb{E}(M_{j_1}(t-j_1)M_{j_2}(t-j_1)|\mathcal{F}_{t-j_1-i-1}).$$

Substituting this into III and using Minkowski's inequality gives

$$\begin{aligned} III &= \big\|T^{-1}\sum_{j_1,j_2=0}^{\infty}\sum_{t=u+1}^{u+S_T} F_{t,t-j_2+j_2,M}\big(M_{j_1}(t-j_1)M_{j_2}(t-j_1) - \mathbb{E}(M_{j_1}(t-j_1)M_{j_2}(t-j_1))\big)\big\|_q \\ &\leq T^{-1}\sum_{j_1,j_2=0}^{\infty}\sum_{i=0}^{\infty}\big\|\sum_{t=u+1}^{u+S_T} F_{t,t-j_1+j_2,M}A_{j_1,j_2;i}(t-j_1-i)\big\|_q. \end{aligned}$$

We observe that since $A_{j_1,j_2;i}(t-j_1-i) \in \sigma(X_{t-j_1-i}, X_{t-j_1-i-1}, \ldots)$ and $\mathbb{E}(A_{j_1,j_2;i}(t-j_1-i)|\sigma(X_{t-j-i-1}, X_{t-j-i-2}, \ldots)) = 0$, then $\{A_{j_1,j_2;i}(t-j_1-i)\}_t$ are martingale differences. Therefore by using the Burkholder-Rosenthal and Hölder on the above yields

$$III \leq T^{-1}\sum_{j_1,j_2=0}^{\infty}\sum_{i=0}^{\infty}\big(\sum_{t=u+1}^{u+S_T}|F_{t,t-j_1+j_2,M}|^2\big\|A_{j_1,j_2;i}(t-j_1-i)\big\|_q^2\big)^{1/2}$$

Substituting (25) into III gives

$$\begin{aligned} III &\leq CT^{-1}\sum_{j_1,j_2=0}^{\infty}\sum_{i=0}^{\infty}\big\{\sum_{t=u+1}^{u+S_T}|F_{t,t-j_1+j_2,M}|^2\big(\|X_t\|_{\tilde{r}}^2\alpha(j_1)^{\frac{1}{2r}-\frac{1}{\tilde{r}}}\alpha(j_2)^{\frac{1}{2r}-\frac{1}{\tilde{r}}}\alpha(i)^{\frac{1}{q}-\frac{1}{r}}\big)^2\big\}^{1/2} \\ &\leq T^{-1}\sum_{j_1,j_2=0}^{\infty}\sum_{i=0}^{\infty}\alpha(i)^{\frac{1}{q}-\frac{1}{r}}\alpha(j_1)^{\frac{1}{2r}-\frac{1}{\tilde{r}}}\alpha(j_2)^{\frac{1}{2r}-\frac{1}{\tilde{r}}}\big\{\sum_{t=u+1}^{u+S_T}\sup_{\tau} F_{t,\tau,M}^2\big\}^{1/2} \\ &\leq CT^{-1}S_T^{1/2}\big(\sum_{i=0}^{\infty}\alpha(i)^{\frac{1}{q}-\frac{1}{r}}\big)\big(\sum_{i=0}^{\infty}\alpha(i)^{\frac{1}{2r}-\frac{1}{\tilde{r}}}\big)^2 \end{aligned} \qquad (39)$$

Finally, we substitute (37), (38) and (39) into $\|B_{T,S_T}^{(u)}\|_q$ to obtain (26). ☐

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