A note on quadratic forms of nonstationary stochastic processes

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Abstract

In this paper quadratic forms of nonstationary, $\alpha$-mixing time series are considered. Under relatively weak mixing and moment assumptions, asymptotically normality and moment bounds of the quadratic form are derived. An important example of quadratic forms involving Discrete Fourier Transforms of locally stationary processes is studied. In particular, limits of quadratic forms of DFTs of locally stationary processes is derived, and it is shown that there is a significant difference between the behaviour of quadratic forms of stationary processes compared to quadratic forms of nonstationary processes. Such a property can be used to discriminate between stationary and nonstationary processes.

Keywords and phrases $\alpha$-mixing, discrete Fourier transform, locally stationary, quadratic forms.

1 Introduction

The study of the asymptotic theory of statistics often involves quadratic forms of random variables of the type

$$W_T = \frac{1}{T} \sum_{t,\tau=1}^{T} G_{t,\tau} X_t X_\tau,$$

where $\{X_t\}$ is a stochastic process and $\{G_{t,\tau}\}$ are weights, which vary according to the application. Various statistical tests and construction of confidence intervals depend on the asymptotic sampling distribution of above statistic. In view of its importance, several authors have studied $W_T$ under various assumptions on the stochastic process $\{X_t\}$.

For example, Mikosch (1990), Götze and Tikhomirov (1999) and the references therein, analysis $W_T$ under the assumption that $\{X_t\}$ are iid random variables. Kokoszka and Taqqu (1997) and Bhansali, Giraitis, and Kokoszka (2007) relax the independence assumption and establish asymptotic normality of $W_T$ under the assumption that $\{X_t\}$ is a realisation from stationary, linear time series. Rosenblatt (1984) allows for nonlinear time series, by assuming that $\{X_t\}$ are $\alpha$-mixing. In particular, he shows asymptotic normality of $W_T$ under the assumption that $\{X_t\}$ is a strictly stationary $\alpha$-mixing time series and has absolutely summability eight order
cumulants. The generalising to mixing random variables, allow \( \{X_t\} \) to be a non-linear time series, but the cumulant assumptions are quite strong. Recently, Gao and Anh (2000) relax the moment assumptions by considering geometric mixing \( \{X_t\} \) and Lin (2009) considers the case \( \{X_t\} \) is the sum of stationary \( \alpha \)-mixing random variables. It should be mentioned, that there are other methods for measuring dependence. For example, Wu and Shao (2007) show asymptotic normality when \( \{X_t\} \) can be written as a function of the innovations and can be approximated by the infinite sum of martingale differences. However, the underlying assumption in all the above references is that the process \( \{X_t\} \) is strictly stationary.

In the analysis of nonstationary time series (which is possibly nonlinear), quadratic forms of the above type do occur, for example estimators of the time-varying spectral density involve quadratic forms (see, for example, Dahlhaus (2000)). In this paper, our objective is to study the asymptotic theory of quadratic forms in such situations.

In Section 2 we show asymptotic normality of the quadratic form under some moment assumptions and \( \alpha \)-mixing of the stochastic process (which includes both nonstationary and nonlinear processes). By using Ibragimov-type inequalities (see Statulevicius and Jakimavicius (1988)) which link cumulants to the mixing rate, we avoid direct assumptions on the summability of the cumulants. The assumptions allow the weights \( G_{t,\tau} \) to also depend on \( T \), thus including the case of spectral density estimators. To understand how quadratic forms of stationary and nonstationary processes may differ, in Section 3, we consider the sampling properties of quadratic forms of locally stationary processes, which are a subclass of nonstationary time series (see Dahlhaus (1997), Dahlhaus and Subba Rao (2006), Subba Rao (2006) and Zhou and Wu (2009) for examples of locally stationary time series). We show that sampling properties of these quadratic forms under stationarity an local stationarity are rather different. From the perspective of statistical inference, these results may be future interest, as they could be used as an alternative characterisation of nonstationary time series.

In Section 4 we derive some results on cumulants and moments of the quadratic form. To prove the central limit theorem we use a similar Bernstein blocking argument, and this proof can be found in Section 5. The technical details are given in the appendix.

## 2 The quadratic form

Let \( \{X_{t,T}; 1 \leq t \leq T\} \) be a time series which is not necessarily stationary. By allowing \( X_{t,T} \) to depend on \( T \), the results below cover the case of triangular arrays, and in particular allow for locally stationary time series. We will assume that for all \( t \), \( \mathbb{E}(X_{t,T}) = 0 \) and also suppose that \( \inf_{t,T} \var(X_{t,T}) > 0 \) and \( \sup_{t,T} \var(X_{t,T}) < \infty \). This condition excludes degenerate cases by ensuring that \( \{X_{t,T}\} \) does not converge to a deterministic sequence. In this paper we consider quadratic forms of the type

\[
Q_{T,M} = \frac{1}{T} \sum_{t,\tau=1}^{T} G_{t,\tau,M} X_{t,T} X_{\tau,T}
\]

where for some \( 0 < \alpha < 1 \), \( M := M(T) = T^\alpha \), and if \( |t - \tau| > M \), then \( G_{t,\tau,M} = 0 \) (we will relax this condition below).
We now state some conditions, which we use to prove asymptotic normality of $Q_{T,M}$.

**Assumption 2.1** (i) For all $T$, $c_1 G_M \leq \text{var}(Q_{T,M}) \leq c_2 G_M$ (for some $0 < c_1 \leq c_2 < \infty$), where $G_M = \sup_T \sum_\tau |G_{t,\tau,M}|^2$ and $\inf_M G_M > 0$.

(ii) Let us suppose that $\{X_{t,T}\}$ is an $\alpha$-mixing time series which satisfies

$$
\sup_k \sup_{A \in \sigma(X_{t+k,T})} \sup_{B \in \sigma(X_{k,T})} |P(A \cap B) - P(A)P(B)| \leq \alpha(t),
$$

where $\alpha(t)$ are the mixing coefficients which satisfy $\alpha(t) \leq K|t|^{-s}$ for some $s > 0$.

(iii) For some $r > 4s/(s - 6) > 0$ we have $\sup_{t,T} \mathbb{E}|X_{t,T}|^r < \infty$.

Assumption 2.1(i) together with $G_{t,\tau,M} = 0$ for $|t - \tau| > M$ prevent pathological cases such as $Q_{T,M} = X_1 \sum_t X_t$. Moreover, the condition $G_{t,\tau,M} = 0$ for $|t - \tau| > M$ together with the mixing assumptions imply that $\sum_\tau G_{t_1,\tau,M} X_{t_1,T} X_{\tau,T}$ and $\sum_\tau G_{t_2,\tau,M} X_{t_2,T} X_{\tau,T}$ become asymptotically independent as $|t_1 - t_2| \to \infty$ and $T \to \infty$ (which allows us to verify the Lindeberg condition when proving asymptotic normality). We use Assumption 2.1(iii) to show that the covariances and fourth order cumulants are absolutely summable and the Lindeberg condition to prove asymptotic normality. We require assumptions which link moments and the mixing rate, because we work under the relatively weak assumption of $\alpha$-mixing and apply the classical Ibragimov inequality. For example, if the time series $\{X_t\}$ is geometrically mixing, (i.e. $\alpha(t) = K\rho^t$ for some $0 < \rho < 1$), then Assumption 2.1(iii) is satisfied if $\sup_{t,T} \mathbb{E}|X_{t,T}|^{4+\delta} < \infty$ for some $\delta > 0$. On the other hand if the mixing rate is slow (close to 6), then we need a large number of moments, for Assumption 2.1(iii) to be satisfied.

Many time series satisfy the $\alpha$-mixing conditions given in Assumption 2.1(ii), see, for example, Doukhan (1994), Cline and Pu (1999) and Bradley (2007). Several nonstationary time series are also mixing, for example, Fryzlewicz and Subba Rao (2010) show that the nonstationary time-varying ARCH process is geometrically $\alpha$-mixing and give conditions for $\alpha$-mixing of other nonstationary processes.

**Remark 2.1** (i) The variance of $Q_{T,M}$ is

$$
\text{var}(Q_{T,M}) = \frac{1}{T^2} \sum_{t_1,T_1=1}^T \sum_{t_2,T_2=1}^T G_{t_1,\tau_1,M} G_{t_2,\tau_2,M} \left[ \text{cov}(X_{t_1,T}, X_{t_2,T}) \text{cov}(X_{\tau_1,T}, X_{\tau_2,T}) + \text{cov}(X_{t_1,T}, X_{\tau_2,T}) \text{cov}(X_{\tau_1,T}, X_{t_2,T}) + \text{cum}(X_{t_1,T}, X_{\tau_1,T}, X_{t_2,T}, X_{\tau_2,T}) \right]. \tag{2}
$$

(ii) If $G_M = O(T^\alpha)$ (where $0 < \alpha < 1$), it can be shown that under Assumption 2.1(ii) the fourth order cumulant term in (2) is asymptotically negligible with respect to the covariances terms.

We now derive the limiting distribution of $Q_{T,M}$.

**Theorem 2.1** Suppose Assumption 2.1(ii,iii) is satisfied. Let $\text{var}(Q_{T,M}) = V_T$, then we have $V_T^{-1/2}(Q_{T,M} - \mathbb{E}(Q_{T,M})) \xrightarrow{D} \mathcal{N}(0,1)$ as $T \to \infty$. 

3
We prove Theorem 2.1 in Section 5. We now consider some corollaries of Theorem 2.1. In the corollary below, we consider quadratic forms whose weights are not necessarily zero when $|t - \tau| > M$.

**Corollary 2.1** Let us suppose that $\{X_{t,T}\}$ satisfies Assumption 2.1(ii,iii). Let

$$Q_T = \frac{1}{T} \sum_{t,\tau=1}^{T} G_{t,\tau,T} X_{t,T} X_{\tau,T},$$

where $|G_{t,\tau,T}| \leq K |t - \tau|^{-2}$ and $V_T := \text{var}(Q_T) = O(T^{-1})$. Then we have $V_T^{-1/2}(Q_T - \mathbb{E}(Q_T)) \overset{D}{\to} \mathcal{N}(0,1)$.

**PROOF.** To prove the result we show that $Q_T$ can closely be approximated by a quadratic sum which satisfies (1), thus by applying Theorem 2.1 we obtain the result. Define the quadratic form

$$Q_{T,1/2+\gamma} = \frac{1}{T} \sum_{t,\tau=1}^{T} I(t - \tau \in [-1/2+\gamma]) G_{t,\tau,T} X_{t,T} X_{\tau,T},$$

where $0 < \gamma < 1/2$ and $I$ is the indicator variable with $I(x) = 1$ for $x \in [-1,1]$ and zero elsewhere. Since $|G_{t,\tau,T}| \leq K |t - \tau|^{-2}$ we have that

$$Q_T = Q_{T,1/2+\gamma} + O_p(T^{-1/2-\gamma}),$$

(3)

and $\text{var}(\sqrt{T}Q_T) = \text{var}(\sqrt{T}Q_{T,1/2+\gamma}) + O(T^{-\gamma})$. We observe that $Q_{T,1/2+\gamma}$ satisfies representation (1) and Assumption 2.1, hence by applying Theorem 2.1 we have $V_T^{-1/2}(Q_T - \mathbb{E}(Q_T)) \overset{D}{\to} \mathcal{N}(0,1)$. Therefore from (3) we have $V_T^{-1/2}(Q_T - \mathbb{E}(Q_T)) \overset{D}{\to} \mathcal{N}(0,1)$, which gives the desired result. $\square$

We now state an extension of the above results, which involves quadratic forms of a vector time series.

**Corollary 2.2** Let us suppose that $\{X_{t,T}\}$ is a $d$-dimensional vector time series, which satisfies the mixing rate

$$\sup_k \sup_{A \in \sigma(X_{t+k,T},X_{t+1+k,T},...)} \sup_{B \in \sigma(X_{k,T},X_{k-1,T},...)} \left| P(A \cap B) - P(A)P(B) \right| \leq \alpha(t),$$

where $\alpha(t)$ are the mixing coefficients and are such that $\alpha(t) \leq K |t|^{-s}$ where $s > 0$, and suppose there exists some $r > \frac{4s}{s-6}$, such that $\sup_{t,T} \mathbb{E}(\sum_{j=1}^{d} |X_{t,T,j}|^r)^{1/r} < \infty$ (the multivariate version of Assumption 2.1(iii)). Define the quadratic form

$$Q_T = \frac{1}{T} \sum_{t,\tau=1}^{T} X'_{t,T} G_{t,\tau,T} X_{\tau,T},$$

where $|G_{t,\tau,T}| \leq K |t - \tau|^{-2}$ and $V_T := \text{var}(Q_T) = O(T^{-1})$. Then we have $V_T^{-1/2}(Q_T - \mathbb{E}(Q_T)) \overset{D}{\to} \mathcal{N}(0,1)$. Therefore from (3) we have $V_T^{-1/2}(Q_T - \mathbb{E}(Q_T)) \overset{D}{\to} \mathcal{N}(0,1)$, which gives the desired result. $\square$
where \( \{G_{t,\tau}\} \) is a \( d \times d \) matrix which satisfies \(|G_{t,\tau}| < K|t-\tau|^{-2} \) (\( |\cdot| \) is the \( \ell_1 \) norm of a matrix). We assume there exists \( 0 < c_1 \leq c_2 < \infty \) such that \( c_1/T \leq \text{var}(Q_T) \leq c_2/T \). Then we have \( V_T^{-1/2}(Q_T - \mathbb{E}(Q_T)) \xrightarrow{D} \mathcal{N}(0,1) \), where \( V_T = \text{var}(Q_T) \).

**PROOF.** The proof is exactly the same as the proof of Theorem 2.1, hence we omit the details. \( \square \)

### 3 Quadratic forms of locally stationary time series

In time series, quadratic forms often appear in terms of the integrated discrete Fourier transform (DFT)

\[
H_T = \frac{1}{T} \sum_{k=1}^{T} H(\omega_k)J_T(\omega_k)\overline{J_T(\omega_{k+\tau})},
\]

where \( J_T(\omega_k) = \frac{1}{\sqrt{2\pi T}} \sum_{\tau=1}^{T} X_{t,\tau,T} \exp(\frac{2\pi i k}{T}) \) and \( \omega_k = \frac{2\pi k}{T} \). By expanding \( J_T(\omega_k)\overline{J_T(\omega_{k+\tau})} \), \( H_T \) can be rewritten as the quadratic form

\[
H_T = \frac{1}{T} \sum_{t,\tau} X_{t,\tau,T}X_{\tau,T} \exp(-i\omega_r\tau) \left( \frac{1}{2\pi T} \sum_{k=1}^{T} H(\omega_k) \exp(i\omega_k(t-\tau)) \right) = \frac{1}{2\pi T} \sum_{t,\tau} G_{t,\tau,T}X_{t,\tau,T},
\]

where \( h_{t-\tau,T} = \frac{1}{2\pi T} \sum_{k=1}^{T} H(\omega_k) \exp(i\omega_k(t-\tau)) \) and \( G_{t,\tau,T} = \exp(-i\omega_r)h_{t-\tau,M} \). To simplify notation, from now on we will suppose \( \sup_{\omega} |H''(\omega)| < \infty \), then \( H(\omega) = \sum_{s=-\infty}^{\infty} h(s) \exp(is\omega) \), where \( h(s) = \int H(\omega) \exp(-is\omega) d\omega \) and \( |h(s)| \leq \sup_{\omega} |H''(\omega)| \cdot |s|^{-2} \). Since \( h_{t-\tau,T} \) is an approximation of \( h(t-\tau) \) we have \( H_T = \tilde{H}_T + O_p(T^{-1}) \), where

\[
\tilde{H}_T = \frac{1}{T} \sum_{t=1}^{T} \sum_{\tau=1}^{T} X_{t,\tau,T}h(t-\tau) \exp(-i\omega_r\tau).
\]

Under the assumption that \( \{X_{t,T}\} \) is a stationary time series with \( \sum_{r} |r| \cdot |\text{cov}(X_0, X_r)| < \infty \), it can be shown that

\[
\mathbb{E}(H_T) = \begin{cases} O(\frac{1}{T}) & r \neq 0 \\ \int_0^{2\pi} H(\omega) f(\omega) d\omega + O(\frac{1}{T}) & r = 0 \end{cases}
\]

(5)

an explicit expression for the variance \( \text{var}(H_T) \) can be obtained and by using Corollary 2.1 we can prove asymptotic normality of \( H_T \). In the stationary case, \( r = 0 \), is usually the only case of interest. On the other hand, the case \( r \neq 0 \) can be very useful in the study of nonstationary processes; as \( H_T \) may contain interesting information about nonstationary properties of \( \{X_{t,T}\} \) (see Dwivedi and Subba Rao (2010)). In the remainder of this section we will focus on the sampling properties of \( H_T \), when \( \{X_{t,T}\} \) is nonstationary.

In order to derive an asymptotic expression for the expectation and variance of \( H_T \) we will place some structure on the nonstationarity, and assume the nonstationarity changes only slowly over time. More precisely, let us suppose that \( \{X_{t,T}\} \) is a nonstationary time series, which can
We will use the following lemma to derive the cumulants of the DFT of a locally stationary ARCH process considered in Dahlhaus and Subba Rao (2006).

Let \( X_{t,T} \) be a time-varying MA process. So long as the sufficient number of moments exist, the above assumptions are satisfied by the locally stationary process \( X_{t,T} \) if it satisfies

\[
|X_{t,T} - X_t(u)| \leq (|\frac{t}{T} - u| + \frac{1}{T})V_t,
\]

where \( \{V_t\} \) is a positive stationary time series and \( \mathbb{E}[V_t] < \infty \). The asymptotics in this case are done in the rescaled sense. A simple example of a time series \( \{X_{t,T}\} \) which satisfies the above is the time-varying AR process \( X_{t,T} = a(\frac{t}{T})X_{t-1,T} + \varepsilon_t \) where \( \sup_u |a(u)| < 1 \). In this case, \( X_t(u) = a(u)X_{t-1} + \varepsilon_t \) and \( V_t = aV_{t-1} + |\varepsilon_t| \), where \( a = \sup_u |a(u)| \) (see Dahlhaus (1997) and Subba Rao (1970)).

To obtain an expression for the mean and variance of \( H_T \) we require the following definitions and assumptions.

**Definition 3.1** Define the local covariance and \( n \)th order cumulant as \( c(u, r) = \text{cov}(X_t(u), X_{t+r}(u)) \) and \( \kappa_n(u; t_2 - t_1, \ldots, t_n - t_1) = \text{cum}(X_{t_1}(u), \ldots, X_{t_n}(u)) \) respectively. Using this notation we define the local spectral density \( f(u; \omega) = \frac{1}{(2\pi)^{n-1}} \sum_{r=-\infty}^{\infty} c(u; r) \exp(ir\omega) \), and the \( n \)th order local spectra

\[
f(u; \omega_1, \ldots, \omega_{n-1}) = \frac{1}{(2\pi)^{n-1}} \sum_{t_1, \ldots, t_{n-1} = -\infty}^{\infty} \kappa_n(u; t_1, \ldots, t_{n-1}) \exp(i \sum_{j=1}^{n-1} t_j \omega_j), \tag{6}
\]

where \( \omega_1, \ldots, \omega_{n-1} \in [0, 2\pi] \).

**Assumption 3.1** For a given \( n \), there exists a sequence \( \{\kappa_n(t_1, \ldots, t_{n-1})\} \) such that

\[
\sum_{t_1, \ldots, t_{n-1}} (1 + |t_j|)|\kappa_n(t_1, \ldots, t_{n-1})| < \infty \quad \text{(for all } 1 \leq j \leq (n-1)\text{)}, \quad \text{and the cumulants corresponding to } \{X_t(u)\} \text{ satisfy }
\]

(i) \( |\text{cum}(X_{t_1,T}, \ldots, X_{t_n,T}) - \text{cum}(X_{t_1}(\frac{t}{T}), \ldots, X_{t_n}(\frac{t}{T}))| \leq T^{-1} \kappa_n(t_2 - t_1, \ldots, t_n - t_1) \).

(ii) For all \( u_j \in [0, 1] \) and \( v_j \in [0, 1] \), we have

\[
|\text{cum}(X_{t_1}(u_1), \ldots, X_{t_{j-1}}(u_{j-1}), X_{t_j}(v_j), X_{t_{j+1}}(u_{j+1}), \ldots, X_{t_n}(u_n)) - \text{cum}(X_{t_1}(u_1), \ldots, X_{t_{j-1}}(u_{j-1}), X_{t_j}(u_j), X_{t_{j+1}}(u_{j+1}), \ldots, X_{t_n}(u_n))| \\
\leq |u_j - v_j| \kappa_n(t_2 - t_1, \ldots, t_n - t_1),
\]

(iii) For all \( 1 \leq j \leq n - 1 \), we have \( \sup_u \sum_{r_1, \ldots, r_{n-1}} (1 + |t_j|)|\kappa_n(u; t_1, \ldots, t_{n-1})| < \infty \).

(iv) The second derivative of \( f_n(u; \omega_1, \ldots, \omega_{n-1}) \) with respect to \( u \), is bounded over \( u, \omega_1, \ldots, \omega_{n-1} \).

So long as the sufficient number of moments exist, the above assumptions are satisfied by the time-varying MA(\( \infty \)) process proposed in Dahlhaus and Polonik (2006) and the time-varying ARCH process considered in Dahlhaus and Subba Rao (2006).

We will use the following lemma to derive the cumulants of the DFT of a locally stationary process.
Lemma 3.1 Suppose Assumption 3.1. Then we have
\[
\sup_{\omega_1, \ldots, \omega_{n-1}} |F_n(k; \omega_1, \ldots, \omega_{n-1})| \leq C \sup_{u, \omega_1, \ldots, \omega_{n-1}} \left| \frac{\partial f_n(u; \omega_1, \ldots, \omega_{n-1})}{\partial u} \right| \frac{1}{|k|^2}
\]  
and \( \sup_{\omega_1, \ldots, \omega_{n-1}} \sum_k |F_n(k; \omega_1, \ldots, \omega_{n-1})| < \infty \), where \( \{F_n(\cdot; \omega_1, \ldots, \omega_{n-1})\} \) are the Fourier coefficients of the local nth order spectra.

PROOF. The proof follows from Assumption 3.1(iv), and is a straightforward application of Briggs and Henson (1997), Theorem 6.2.

Using the following lemma, we derive a generalisation of Brillinger (1981), Theorem 4.3.2, which considers the cumulants of DFTs of stationary time series.

Lemma 3.2 Suppose Assumption 3.1 holds, then we have
\[
cum(J_{T}(\omega_{j_1}), \ldots, J_{T}(\omega_{j_n})) = \frac{(2\pi)^{(n/2)-1}}{T^{(n/2)-1}} F_n((j_1 + \ldots + j_n); \omega_{j_1}, \ldots, \omega_{j_{n-1}}) + O\left(\frac{1}{T^{n/2}}\right). \tag{9}
\]
where \( \omega_{j_k} = \frac{2\pi j_k}{T} \) and \( j_k \in \mathbb{Z} \).

PROOF. Expanding the cumulant term we have
\[
cum(J_{T}(\omega_{j_1}), \ldots, J_{T}(\omega_{j_n})) = \frac{1}{(2\pi T)^{n/2}} \sum_{t_1, \ldots, t_n=1}^{T} \cum(X_{t_1,T}, \ldots, X_{t_n,T}) \exp(it_1 \omega_{j_1} + \ldots + it_n \omega_{j_n}).
\]
We now relace \( \cum(X_{t_1,T}, \ldots, X_{t_n,T}) \) with \( \kappa_n \left(\frac{t}{T}, t_2 - t_1, \ldots, t_n - t_1\right) \) and use Assumption 3.1 to obtain
\[
cum(J_{T}(\omega_{j_1}), \ldots, J_{T}(\omega_{j_n})) = \frac{1}{(2\pi T)^{n/2}} \sum_{t_1, \ldots, t_n=1}^{T} \kappa_n \left(\frac{t}{T}, t_2 - t_1, \ldots, t_n - t_1\right) \exp(it_1 \omega_{j_1} + \ldots + it_n \omega_{j_n}) + O(T^{-n/2}).
\]
Replacing the above with \( f_n(t; \omega_{j_1}, \ldots, \omega_{j_n}) \) (see (6)) we have
\[
cum(J_{T}(\omega_{j_1}), \ldots, J_{T}(\omega_{j_n})) = \frac{(2\pi)^{(n/2)-1}}{T^{n/2}} \sum_{t=1}^{T} f_n(t; \omega_{j_1}, \ldots, \omega_{j_{n-1}}) \exp(it \sum_{s=1}^{n} \omega_{j_s}) + O\left(\frac{1}{T^{n/2}}\right)
\]
\[
= \frac{(2\pi)^{(n/2)-1}}{T^{n/2-1}} \int_{0}^{1} f_n(u; \omega_{j_1}, \ldots, \omega_{j_{n-1}}) \exp(iu \sum_{s=1}^{n} j_s) du + O\left(\frac{1}{T^{n/2}}\right)
\]
Finally, substituting (7) into the above gives (9). It is interesting to note that Paparoditis (2009) derived a similar result to (10) for time-varying MA(\( \infty \)) processes. \( \square \)
An interesting observation which follows from the lemma above are the differences between \( \text{cov}(J_T(\omega_k), J_T(\omega_{k'})) = \text{cum}(J_T(\omega_k), J_T(\omega_{k'})) \), for stationary and locally stationary time series. In the case that \( \{X_{t,T}\} \) is second order stationary we have \( \text{cov}(J_T(\omega_k), J_T(\omega_{k'})) = o(1) \), whereas if \( \{X_{t,T}\} \) were (second order) nonstationary there is an ‘ordering’ in correlation between the DFTs. More precisely, \(|\text{cov}(J_T(\omega_k), J_T(\omega_{k'}))| \leq C|k_1 - k_2|^{-2} \), where \( C \) is a finite constant. Hence the correlation between the DFTs decay the further apart the frequencies. It should be mentioned that in the case that \( \{X_{t,T}\} \) is nonstationary but not necessarily locally stationary it is not clear whether the correlation structure between the DFTs decay the further apart the frequencies. It is possible, that the differences in correlation of the DFT, could be a way of discriminating locally stationary from general nonstationary behaviour.

Using the above we can can now generalise the results in (5) to locally stationary processes.

**Lemma 3.3** Suppose Assumption 3.1 holds with \( n = 1, \ldots, 4 \), and \( H(\omega) \) has a bounded second derivative. Then we have

\[
\mathbb{E}(H_T) = \int_0^1 \int_0^{2\pi} H(\omega)f(u, \omega) \exp(-i2\pi ru) d\omega du + O(\frac{1}{T}) \quad \text{(11)}
\]

and

\[
\text{var}(\sqrt{T}H_T) = \frac{1}{T} \sum_{k_1,k_2=1}^{T} H(\omega_{k_1})H(\omega_{k_2}) \left[ F_2(k_1-k_2,\omega_{k_1})F_2(k_2-k_1,-\omega_{k_1+r}) + F_2(-(k_1+k_2+r),-\omega_{k_1+r})F_2(k_1+k_2+r,\omega_{k_1}) + \frac{(2\pi)^{1/2}}{T} F_4(0,\omega_{k_1},-\omega_{k_1+2},-\omega_{k_2}) \right]. \quad \text{(12)}
\]

If, in addition, Assumption 2.1 holds, then we have

\[
V_T^{-1/2} \left( H_T - \int_0^1 \int_0^{2\pi} H(\omega)f(u, \omega) \exp(-i2\pi ru) d\omega du \right) \overset{D}{\longrightarrow} \mathcal{N}(0,1), \quad \text{(13)}
\]

where \( V_T = \text{var}(H_T) \), and \( \text{var}(\sqrt{T}H_T) \) is given in (12).

**PROOF.** Now by using Lemma 3.2 we have

\[
\mathbb{E}(H_T) = \frac{1}{T} \sum_{k=1}^{T} H(\omega_k)\mathbb{E}(J_T(\omega_k)J_T(\omega_{k+r})) = \frac{1}{T} \sum_{k=1}^{T} H(\omega_k)F_n(-r,\omega_k) + O(\frac{1}{T}).
\]

Since \( H(\cdot) \) has a bounded first derivative, we can exchange summands with integrals we obtain (11). To prove (12) we expand \( \text{var}(H_T) \) using the expansion of \( \text{cov}(Y_1Y_2, Y_3Y_4) \) in terms of cumulants and use Lemma 3.2 to get the result. To prove (13) we use (11), (12) and apply Theorem 2.1. \( \square \)

\( H_T \) defined in (4) does not allow for the function \( H(\cdot) \) to depend on \( T \), thus it does not include spectral density estimators. We now consider a variation of \( H_T \), which allows for these type of
estimators. Let

$$H_T = \frac{1}{T} \sum_{k=1}^{T} H_M(\omega_k) J_T(\omega_k) J_T(\omega_{k+r}), \quad (14)$$

where the function $H_M(\omega_k)$ depends on $T$ and is not necessarily uniformly bounded over $T$, and the Fourier coefficients of $H_M$, $h_M(t - \tau) = \frac{1}{2\pi} \int_0^{2\pi} H_M(\omega) \exp(i\omega(t - \tau)) d\omega$, are such that $\sum_r |h_M(r)|^2 = O(M)$ and for $|r| > M$, we have $h_M(r) = 0$. Let us suppose that the second derivative of $H_M(\cdot)$ exists and is bounded for each $M$, then $H_T$ can be approximated by the quadratic form

$$H_T = \frac{1}{T} \sum_{t, \tau} X_{t,T}X_{\tau,T} \exp(-i\omega\tau) \left( \frac{1}{2\pi T} \sum_{k=1}^{T} H_M(\omega_k) \exp(i\omega_k(t - \tau)) \right)$$

$$= \frac{1}{2\pi T} \sum_{t=1}^{T} \sum_{\tau=1}^{T} X_{t,T}X_{\tau,T} h_M(t - \tau) \exp(-i\omega\tau) + O_p\left( \sup\omega |H'_M(\omega)| T \right).$$

Lemma 3.4 Let $H_T$ be defined as in (14). Suppose Assumption 2.1 and 3.1 (with $m = 1, \ldots, 4$) holds, and that for each $M$, the second derivative of $H_M(\omega)$ is bounded, then we have

$$V_{T}^{-1/2} \left( H_T - \int_0^1 \int_0^{2\pi} H_M(\omega) f(u, \omega) \exp(-i2\pi ru) d\omega du \right) \overset{D}{\to} \mathcal{N}(0, 1),$$

where

$$V_T = \frac{1}{T^2} \sum_{k_1, k_2=1}^{T} H_M(\omega - \omega_{k_1})H_M(\omega - \omega_{k_2}) \left[ F_2(k_1 - k_2, \omega_{k_1})F_2(k_2 - k_1, -\omega_{k_1+r}) + F_2(-(k_1 + k_2 + r), -\omega_{k_1+r})F_2(k_1 + k_2 + r, \omega_{k_1}) \right] + O\left( \frac{1}{M_H T^2} \right).$$

and $M_H = \frac{1}{T} \sum_{k=1}^{T} |H_M(\omega_k)|^2$.

PROOF. The proof is of the asymptotic normality is identical to the proof in Lemma 3.3, however the the variance is slightly different. In this case, because the weights of $\sum_r |h_M(r)|^2 = O(M) \to \infty$, as $T \to \infty$, then we observe that the covariance terms in the variance of $V_T$ is of order $O\left( \frac{M}{T} \right)$, and dominates the fourth order cumulant terms (which is of order $O(T^{-1})$, if the absolute sum of the fourth order cumulants is finite). □

4 Some bounds on sums of cumulants and moments

In this section we state some bounds on the sums of moments and cumulants. These results will be used to prove Theorem 2.1, but they may also be of independent interest.

We first state a bound for the sum of cumulants based on the mixing rate. The result below gives bounds on cumulants of a time series and can be considered as a generalisation of Ibragimov’s
inequality, which is a bound on the covariance of two mixing random variables. Let \( \|X\|_p = (\mathbb{E}|X|^p)^{1/p} \).

**Lemma 4.1** Let us suppose that \( \{X_{t,T}\} \) is a \( \alpha \)-mixing time series with rate \{\( \alpha(t) \)\}. If \( t_1 \leq t_2 \leq \ldots \leq t_k \), then we have \( |\text{cum}(X_{t_1,T}, \ldots, X_{t_k,T})| \leq C_k \sup_t \|X_{t,T}\|_r \prod_{i=2}^{k} \alpha(t_i - t_{i-1})^{\frac{1-k/r}{k-1}}, \)

\[
(i) \sup_{t_1} \sum_{t_2,\ldots,t_k=1}^{\infty} |\text{cum}(X_{t_1,T}, \ldots, X_{t_k,T})| \leq C_k \sup_{t,T} \|X_{t,T}\|_r \left( \sum_{t} \alpha(t) \right)^{\frac{1-k/r}{k-1}} < \infty, \tag{15}
\]

and for all \( 2 \leq j \leq k \) we have

\[
(ii) \sup_{t_1} \sum_{t_2,\ldots,t_k=1}^{\infty} (1 + |t_j|) |\text{cum}(X_{t_1,T}, \ldots, X_{t_k,T})| \leq C_k \sup_{t,T} \|X_{t,T}\|_r \left( \sum_{t} \alpha(t) \right)^{\frac{1-k/r}{k-1}} < \infty, \tag{16}
\]

where \( C_k \) is a finite constant which depends only on \( k \).

**PROOF.** See Appendix. \( \square \)

Using the lemma above, the following corollary on the absolute summability of the fourth order cumulants immediately follows. This corollary gives sufficient conditions Assumption 3.1 to hold.

**Corollary 4.1** Suppose that \( \{X_{t,T}\} \) is a \( \alpha \)-mixing time series which satisfies Assumption 2.1(ii), where \( \alpha(t) \leq K \cdot |t|^{−s} \).

\( \text{(i) Let us suppose that } r > 4s/(s-3) \text{ and } \sup_t \mathbb{E}|X_{t,T}|^r < \infty, \text{ then we have } |\text{cov}(X_{t,T}, X_{\tau,T})| \leq C|t - \tau|^{-\frac{s+3}{2}} \text{ and } \sup_t \sum_{t_2,t_3,t_4=-\infty}^{\infty} |\text{cum}(X_{t_1,T}, X_{t_2,T}, X_{t_3,T}, X_{t_4,T})| < \infty. \)

\( \text{(ii) Let us suppose that } r > 4s/(s-6) \text{ and } \sup_t \mathbb{E}|X_{t,T}|^r < \infty, \text{ then we have } |\text{cov}(X_{t,T}, X_{\tau,T})| \leq C|t - \tau|^{-\frac{s+6}{2}} \text{ and for all } 2 \leq j \leq 4, \)

\( \sup_t \sum_{t_2,t_3,t_4=-\infty}^{\infty} (1 + |t_j|) |\text{cum}(X_{t_1,T}, X_{t_2,T}, X_{t_3,T}, X_{t_4,T})| < \infty. \)

To simplify notation we rewrite the quadratic form in (1) as

\[ Q_{T,M} = \frac{1}{T} \sum_{t=1}^{T} \sum_{\tau=1}^{T} F_{t,\tau,M} X_{t,T} X_{\tau,T} \quad \text{where } F_{t,\tau,M} = \begin{cases} G_{t,\tau,M} & \text{if } t = \tau \\ (G_{t,\tau,M} + G_{\tau,t,M}) & \text{if } t \neq \tau. \end{cases} \]

We now define sub-blocks of \( Q_{T,M} \), which will be used to prove Theorem 2.1. For \( r > S_T \), define the ‘partial’ shift summand

\[ B^{(u)}_{T,S_T} = \frac{1}{T} \sum_{t=u+1}^{T} \sum_{\tau=\max(t-M,1)}^{T} F_{t,\tau,M} \left( X_{t,T}X_{\tau,T} - \mathbb{E}(X_{t,T}X_{\tau,T}) \right). \tag{17} \]

Lemma 4.1 can be used to obtain bounds for \( \text{var}(Q_{T,M}) \) and other integer moments of \( Q_{T,M} \). However, in order prove asymptotic normality under relatively weak assumptions we will require
Lemma 4.2 Suppose Assumption 2.1(i,ii) holds. Let $F_{t,T}$ = $\sigma(X_{t,T}, X_{t-1,T}, \ldots)$ and denote $E(Z|F_{j,T}) = E_j(Z)$. If for some $r > q$ we have $\sup_{t,T} ||X_{t,T}||_r < \infty$, then we obtain the bound
\begin{equation}
||E_{t-j}(X_{t,T}) - E_{t-j-1}(X_{t,T})||_q \leq 4(2^{1/q} + 1)\alpha(j)^{1/2} \frac{1}{r} ||X_{t,T}||_r, \tag{18}
\end{equation}
and $X_{t,T}$ almost surely admits the representation $X_{t,T} = \sum_{j=0}^{\infty} (E_{t-j}(X_{t,T}) - E_{t-j-1}(X_{t,T})).$
Let $M_j(t-j) = E(X_{t,T}|F_{t-j}) - E(X_{t,T}|F_{t-j-1}).$ If for some $\tilde{r}/2 > r > q$ we have $\sup_{t,T} ||X_{t,T}||_{\tilde{r}} < \infty$, then
\begin{align*}
||E(M_j(t-j)M_{t-j}(t-\tau)|F_{t-j-2}) - E(M_j(t-j)M_{\tau-(t-j)}(t-j)|F_{t-j-2})||_q & \leq K\|X_{t,T}\|_{r}^2 \alpha(j)\frac{1}{2} \alpha(\tau - (t-j))\frac{1}{2} \alpha(i)\frac{1}{2} \frac{1}{r} \frac{1}{\tilde{r}}. \tag{19}
\end{align*}
Let $q \geq 2$ and $B_{T,S_T}^{(u)}$ be defined as in (17). If there exists, a $\tilde{r}$, such that $\sup_{t,T} ||X_{t,T}||_{\tilde{r}} < \infty$, where $\tilde{r}/2 > r > q$, then we have
\begin{equation}
\|B_{T,S_T}^{(u)}\|_q \leq KT^{-1}G_M^{1/2}S_T^{1/2}L\left(\frac{s}{2q} - \frac{s}{r}\right)^2 + L\left(\frac{s}{q} - \frac{s}{r}\right)L\left(\frac{s}{2r} - \frac{s}{\tilde{r}}\right)^2, \tag{20}
\end{equation}
where $L(sp) = \sum_{i=1}^{\infty} i^{-sp}$.

PROOF. See the appendix.

A simple application of the lemma above is to derive bounds for the moments of the quadratic form $Q_{T,M}$ (since $Q_{T,M}$ is a special case of $B_{T,S_T}^{(u)}$, with $u = 0$ and $S_T = T$). By using the arguments in Lemma 5.1, below, it can be shown that under Assumption 2.1, that for some $\delta > 0$, we have $\|Q_{T,M}\|_{2+\delta} = G_M^{1/2}/T^{1/2}$ (where we set $p_T = T$).

Remark 4.1 It is worth mentioning that the above result also applies to quadratic forms of linear time series, which do not necessarily have to be $\alpha$-mixing (this is rare but it can happen if the innovations of the linear process follow a binomial distribution, see, for example, Andrews (1984)). In this case, the mixing assumptions can be replaced by assumptions on the MA($\infty$) coefficients. More precisely, let $X_{t,T} = \sum_{j=0}^{\infty} a_t(\cdot)\varepsilon_{t-j}$, and assume that $|a_t(\cdot)| \leq (j(\log j)^{1+\varepsilon})^{-1}$, and $\|\varepsilon_t\|_q < \infty$. By following the same proof as that of Lemma 4.2 we can show that
\begin{equation}
\|B_{T,S_T}^{(u)}\|_q \leq KT^{-1}G_M^{1/2}(S_T^{1/2} + 1).
\end{equation}

5 Proof of Theorem 2.1

To do the analysis, we start by rewriting $(Q_{T,M} - E(Q_{T,M}))$ as
\begin{equation}
Q_{T,M} - E(Q_{T,M}) = \frac{1}{T} \sum_{t=1}^{T} G_{t,T}(X_{t,T}X_{t,T} - E(X_{t,T}X_{t,T})) = \sum_{t=1}^{T} Y_{t,T}
\end{equation}
where $Y_{t,T} = T^{-1} \sum_{t=\max(t-M,1)}^t F_{t,T} (X_{t,T} - \mathbb{E}(X_{t,T}))$. To prove asymptotic normality we use a classical Bernstein blocking argument. Here we partition $\{Y_{t,T}; t = 1, \ldots, T\}$ into the sum of small and large blocks. Let $U_{i,T}$ and $V_{i,T}$ denote the big blocks and small blocks respectively, where

$$U_{i,T} = \sum_{t=ir_T+p_T}^{(i+1)r_T} Y_{t,T}, \quad V_{i,T} = \sum_{t=ir_T+p_T+1}^{(i+1)r_T} Y_{t,T},$$

$p_T \gg q_T \gg M$ and $r_T = (p_T + q_T)$. Let $k_T = T / (p_T + q_T)$ and $q_T / (p_T + q_T) \to 0$ as $T \to \infty$. For the purpose of proving the results below we will assume that $k_T = O((\log T)^{1/2})$. Using the above notation we decompose $(Q_{T,M} - \mathbb{E}(Q_{T,M})) = S_{k_T} + R_{k_T}$, where

$$S_{k_T} = \sum_{i=1}^{k_T} U_{i,T} \quad \text{and} \quad R_{k_T} = \sum_{i=1}^{k_T} V_{i,T}.$$

Since $p_T \gg q_T$, we will show that $\text{var}(\sqrt{\frac{T}{G_M}} R_{k_T}) \to 0$. We first obtain moment bounds for $\{U_{i,T}\}$ and $\{V_{i,T}\}$.

**Lemma 5.1** Let us suppose Assumptions 2.1 holds. Then for some $\delta > 0$ we have

$$\|U_{i,T}\|_{2+\delta} = O\left(\frac{p_T^{1/2} G_M^{1/2}}{T}\right) \quad \|V_{i,T}\|_{2+\delta} = O\left(\frac{q_T^{1/2} G_M^{1/2}}{T}\right). \quad (21)$$

**PROOF.** We use Lemma 4.2 to prove the result, with $p_T = S_T$ and $u = ir_T$. To apply Lemma 4.2 for $q = 2+\delta$, we require that there exists an $\tilde{r}$ such that $\sup_{t,T} \|X_{t,T}\|_{r} < \infty$, where $\tilde{r}/2 > r > 2+\delta$. Under these conditions we have

$$\|U_{i,T}\|_{2+\delta} \leq KT^{-1} G_M^{1/2} \left( p_T^{1/2} L\left(\frac{s}{2(2+\delta)} - \frac{s}{\tilde{r}}\right)^2 + L\left(\frac{s}{2+\delta} - \frac{s}{r}\right) L\left(\frac{s}{2} - \frac{s}{\tilde{r}}\right)^2 \right).$$

In order to ensure that the right hand side of the above is finite, $\tilde{r}$ should satisfy the conditions

$$\frac{1}{2(2+\delta)} - \frac{1}{r} > \frac{1}{s}, \quad \frac{1}{2+\delta} - \frac{1}{r} > \frac{1}{s} \quad \text{and} \quad \frac{1}{2r} - \frac{1}{\tilde{r}} > \frac{1}{s},$$

then we have $L\left(\frac{s}{2(2+\delta)} - \frac{s}{\tilde{r}}\right) < \infty$, $L\left(\frac{s}{2+\delta} - \frac{s}{r}\right) < \infty$ and $L\left(\frac{s}{2} - \frac{s}{\tilde{r}}\right) < \infty$. The above conditions imply that $\tilde{r}$ should satisfy

$$\tilde{r} > \frac{2(2+\delta)s}{s - 2(2+\delta)} \quad \text{and} \quad \tilde{r} > \frac{2s(2+\delta)}{(s-3)(2+\delta)}. \quad (22)$$

Assumption 2.1(iii) (we recall there exists an $r$ such that $r > 4s / (s - 6)$ and $\sup_{t,T} \|X_{t,T}\|_{r} < \infty$), implies that there exists a $\tilde{r}$ and $\delta > 0$, such that (22) is satisfied. Thus, (21) holds for some $\delta > 0$. The proof of $\|V_{i,T}\|_{2+\delta} = O\left(\frac{q_T^{1/2} G_M^{1/2}}{T}\right)$ is the same, hence we omit the details. \qed

We now show that the contribution of the sum of small blocks, $R_{k_T}$, is negligible with respect to the entire sum $Q_{T,M} - \mathbb{E}(Q_{T,M})$. 

12
Lemma 5.2 Suppose Assumption 2.1 holds and $q_T/(p_T + q_T) \to 0$ as $T \to \infty$. Then we have

$$|\text{cov}(V_{i,T}, V_{i,T})| \leq C_\alpha \left(|i_1 - i_2| p_T - M\right)^{1 - \frac{2}{p + q}} \left(\frac{G_M q_T}{T^2}\right)$$

(23) and

$$\text{var} \left(\frac{T}{G_M} \mathcal{R}_{k_T}\right) \leq C \frac{q_T}{(p_T + q_T)} \to 0,$$

(24)
as $T \to \infty$, where $C$ is a finite constant.

PROOF. Define the sigma-algebras $\mathcal{F}_{i_2}^\infty = \sigma(Y_{i_2r_T+1,T}, Y_{i_2r_T+2,T}, \ldots)$ and $\mathcal{F}_{i_1}^\infty = \sigma(Y_{(i_1+1)r_T,T}, Y_{(i_1+1)r_T-1,T}, \ldots)$. To prove (23) we use Ibragimov’s inequality (assuming $i_2 > i_1$) to obtain

$$|\text{cov}(V_{i_1,T}, V_{i_2,T})| \leq C\left\{ \sup_{A \in \mathcal{F}_{i_1}^\infty} \mathbb{P}(A \cap B) - \mathbb{P}(A) \mathbb{P}(B) \right\}^{1 - \frac{2}{p + q}} \|V_{i_1,T}\|_{2 + \delta}^2$$

$$\leq C\left\{ \alpha((i_2 - i_1) \rho_T + p_T + 1 - M) \right\}^{1 - \frac{2}{p + q}} \|V_{i_1,T}\|_{2 + \delta}^2$$

$$\leq C\alpha((i_2 - i_1) p_T - M)^{1 - \frac{2}{p + q}} \|V_{i_1,T}\|_{2 + \delta}^2.$$  
(25)

This gives (23). To prove (24) we substitute (23) into $\text{var}(\mathcal{R}_{k_T}) = \sum_{i_1, i_2} \text{cov}(V_{i_1,T}, V_{i_2,T})$ and use that $\|V_{i,T}\|_{2 + \delta} = O(q_T^{1/2} G_M^{1/2}/T)$ to obtain

$$\text{var} \left(\frac{T}{G_M} \mathcal{R}_{k_T}\right) \leq K \left(\frac{T}{G_M} \sum_{i_1} \frac{q_T G_M}{T^2} + \frac{2T}{G_M} \sum_{i_1, i_2} C \alpha \left(|i_1 - i_2| p_T - M\right)^{1 - \frac{2}{p + q}} \frac{q_T G_M}{T^2}\right).$$

Now by using that $\alpha(r) \leq Kr^{-2}$ and $k_T = T/(p_T + q_T)$ we have

$$\text{var} \left(\frac{T}{G_M} \mathcal{R}_{k_T}\right) \leq K \frac{q_T}{p_T + q_T} \left(1 + \sum_{r=1}^{k_T} (rp_T - M)^{-s(1 - \frac{2}{p + q})}\right)$$

$$\leq K \frac{q_T}{p_T + q_T} \left(1 + (p_T - M)^{-s(1 - \frac{2}{p + q})} k_T\right).$$

Since $k_T = (\log T)^{1/2}$ and $p_T = (p_T - M)^\eta$ (for some $0 < \eta < 1$), for all $T$ we have $((p_T - M)^{-s(1 - \frac{2}{p + q})} k_T < \infty$, which gives (24). 

Since $\text{var}(Q_{T,M}) = V_T = O(\frac{G_M}{T})$ (by Assumption 2.1(ii), the above result implies that

$$V_T^{-1/2}(Q_{T,M} - \mathbb{E}(Q_{T,M})) = V_T^{-1/2} S_{k_T} + o_p(1).$$

(26)

We now show normality of $S_{k_T}$. We do this by replacing $S_{k_T}$ with $\tilde{S}_{k_T} = \sum_i U_{i,T}$, where $U_{i,T}$ and $U_{i,T}$ have identical distributions, but $\{U_{i,T}\}$ are independent random variables. Below we show that the distributions of $S_{k_T}$ and $\tilde{S}_{k_T}$ are asymptotically equivalent.
We require the following general theorem, which gives a bound on the differences of characteristic functions of sums mixing and independent random variables. A potentially useful aspect of this result, is that we allow for the mixing rate to change with $T$.

**Theorem 5.1** Suppose $\{Z_{t,T}\}$ is an $\alpha$-mixing sequence which for $t < \tau + s_T$ satisfies

$$
sup_{A \in \sigma(Z_{t,T},Z_{t-1,T},...), B \in \sigma(Z_{T},Z_{T+1,T},...)} |P(A \cap B) - P(A)P(B)| \leq a(|t - \tau| - s_T). \tag{27}
$$

Let $W_{i,T} = \sum_{t=r_T+1}^{r_T+p_T} Z_{t,T}$, where $r_T = p_T + q_T$ and $\{\tilde{W}_{i,T}\}$ be independent random variables where the marginal distributions of $\tilde{W}_{i,T}$ and $W_{i,T}$ are the same. Then, for any $x \in \mathbb{R}$, we have

$$
\left| E\left( \exp(ix \sum_{j=1}^{k_T} W_{j,T}) \right) - \prod_{j=1}^{k_T} E\left( \exp(ix \tilde{W}_{j,T}) \right) \right| \leq Ck_Ta(q_T - s_T),
$$

where $C$ is a finite constant.

**PROOF.** By expanding $E\left( \exp(ix \sum_{j=1}^{k_T} W_{j,T}) \right) - \prod_{j=1}^{k_T} E\left( \exp(ix \tilde{W}_{j,T}) \right)$, we have

$$
D_T = \left| \prod_{j=1}^{k_T} E\left( \exp(ix W_{j,T}) \right) - \prod_{j=1}^{k_T} E\left( \exp(ix \tilde{W}_{j,T}) \right) \right|
$$

$$
\leq \sum_{s=1}^{k_T-1} \left| \prod_{r=1}^{s-1} E\left( \exp(ix W_{r,T}) \right) \right| \left| \text{cov}\left( \exp(ix W_s), \exp(ix \sum_{j=s+1}^{k_T} W_j) \right) \right|,
$$

(to simplify notation we denote $\prod_{r=1}^{0} A_r = 1$). From the definition of $W_{i,T}$ and by using Ibragimov’s inequality (for bounded random variables) it is straightforward to show that

$$
\leq \sum_{s=1}^{k_T-1} \sup_{A \in \sigma(Z_{s+1,r_T+1,T},Z_{s+1,r_T+2,T},...)} B \in \sigma(Z_{s},Z_{s+p_T},Z_{s+p_T+1}) |P(A \cap B) - P(A)P(B)| \leq Ck_Ta(q_T - s_T).
$$

The above gives the required result. \qed

**Lemma 5.3** Suppose that Assumption 2.1 holds, and we choose $p_T$ and $q_T$ such that $p_T \gg q_T \gg M$ and $k_T = (\log T)^{1/2}$, where $k_T = T/(p_T + q_T)$, then the asymptotic distributions of $V^{-1/2}_T(Q_{T,M} - E(Q_{T,M}))$ and $V^{-1/2}_T \tilde{S}_{k_T}$ are equivalent.

**PROOF.** From (26) we have $V^{-1/2}_T(Q_{T,M} - E(Q_{T,M})) = V^{-1/2}_T \tilde{S}_{k_T} + o_p(1)$. By using Theorem 5.1 with $Z_{i,T} := Y_{i,T} = T^{-1} \sum_{t=\max(t-M,1)}^{t} F_{t,T,M}(X_{i,T}X_{r,T} - E(X_{i,T}X_{r,T}))$ and $W_{i,T} := U_{i,T}$ we have

$$
|\Phi_{k_T}(x) - \tilde{\Phi}_{k_T}(x)| \leq k_Ta(q_T - M),
$$

where $\Phi_{k_T}(\cdot)$ and $\tilde{\Phi}_{k_T}(\cdot)$ are the characteristic functions of $S_{k_T}$ and $\tilde{S}_{k_T}$. Since $p_T \gg q_T \gg M$ and $k_T = (\log T)^{1/2}$, and using Assumption 2.1(ii) we have that $|\Phi_{k_T}(x) - \tilde{\Phi}_{k_T}(x)| \to 0$. Since
the characteristic functions converge, we obtain the required result.

We now show asymptotic normality of $V_T^{-1/2} \tilde{S}_{k_T}$, this result together with the above lemma will give Theorem 2.1.

**Lemma 5.4** Suppose Assumption 2.1 is satisfied. Then we have

$$V_T^{-1/2} \tilde{S}_{k_T} \xrightarrow{D} \mathcal{N}(0,1).$$

**Proof.** We will use the central limit theorem for independent random variables. Due to the independence of $\tilde{U}_{i,T}$ it is straightforward to show

$$\frac{1}{T} \sum_{i=1}^{k_T} \mathbb{E}(\tilde{U}_{i,T}^2) \rightarrow V_T,$$

hence it remains to verify Lindeberg’s condition. By using (21) we have

$$\sum_{i=1}^{k_T} \mathbb{E}\left[(V_T^{-1/2}|\tilde{U}_{i,T}|)^2 + \delta\right] \leq K(\frac{p_T}{T})^\delta \rightarrow 0,$$

as $T \rightarrow \infty$. Thus Lindeberg’s condition is fulfilled and we have asymptotic normality of $\tilde{S}_{k_T}$. □

**Proof of Theorem 2.1** By using Lemma 5.3, it is straightforward to show that $V_T^{-1/2}(Q_{T,M} - \mathbb{E}(Q_{T,M}))$ and $V_T^{-1/2} \tilde{S}_{k_T}$ have asymptotically the same distribution. In Lemma 5.4 we show that $V_T^{-1/2} \tilde{S}_{k_T} \xrightarrow{D} \mathcal{N}(0,1)$. Therefore, we have that $V_T^{-1/2}(Q_{T,M} - \mathbb{E}(Q_{T,M})) \xrightarrow{D} \mathcal{N}(0,1)$ as $T \rightarrow \infty$, which gives the desired result. □

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**A Appendix**

**A.1 Proofs of results in Section 4**

**Proof of Lemma 4.1** To prove the lemma we apply a result from Statulevicius and Jakimavicius (1988), Theorem 3, part (2), which states that if $t_1 \leq t_2 \leq \ldots \leq t_k$, then for all $2 \leq i \leq k$ we have

$$|\text{cum}(X_{t_1,T}, X_{t_2,T}, \ldots, X_{t_k,T})| \leq 3(k - 1)!2^{k-1} \alpha(t_i - t_{i-1})^{1 - \frac{k}{r}} \sup_{t,T} \|X_{t,T}\|^r.$$

To prove the result the first part of the lemma, we use a method similar to the proof of Neumann (1996), Remark 3.1. By taking the $(k-1)$th root of the above for all $2 \leq i \leq k$ we have

$$\left|\text{cum}(X_{t_1,T}, X_{t_2,T}, \ldots, X_{t_k,T})\right|^\frac{1}{k-1} \leq C_k^{1/(k-1)} \alpha(t_i - t_{i-1})^{\frac{1-k/r}{k-1}} \sup_{t,T} \|X_{t,T}\|^\frac{k}{r},$$

where $C_k = 3(k - 1)!2^{k-1}$. Since the above bound holds for all $i$, multiplying the above over $i$ gives

$$\left|\text{cum}(X_{t_1,T}, X_{t_2,T}, \ldots, X_{t_k,T})\right| \leq C_k \sup_{t,T} \|X_{t,T}\|^k \prod_{i=2}^{k} \alpha(t_i - t_{i-1})^{\frac{1-k/r}{k-1}}, \quad (28)$$
thus proving the first part of the lemma.

To prove (i), we rewrite $\sum_{t_2,\ldots,t_k=1}^{\infty} \alpha(r) \frac{(1-k/r)^{k-1}}{(k-1)!}$ as the sum of orderings, that is $\sum_{t_2,\ldots,t_k=1}^{\infty} k! \sum_{1=t_2 \leq \cdots \leq t_k}^{\infty} \alpha(r) \frac{(1-k/r)^{k-1}}{(k-1)!} < \infty$.

Now since the number of orderings is finite, we can use (i) to obtain

$$\sum_{t_2,\ldots,t_k=1}^{\infty} \left| \text{cum}(X_{t_1,T}, X_{t_2,T}, \ldots, X_{t_k,T}) \right| \leq C_k \sup_{t,T} \|X_{t,T}\|^k \left\{ \sum_r \alpha(r) \left( \frac{1}{(k-1)!} r - \frac{1}{(k-1)!} \right)^{k-1} \right\} < \infty,$$

which gives (15). To prove (ii) we use a similar argument to obtain

$$\sum_{t_2,\ldots,t_k=1}^{\infty} (1 + |t_j|) \left| \text{cum}(X_{t_1,T}, X_{t_2,T}, \ldots, X_{t_k,T}) \right| \leq \sum_{1 \leq t_2 < \cdots < t_k < \infty} (1 + |t_j|) \left| \text{cum}(X_{t_1,T}, X_{t_2,T}, \ldots, X_{t_k,T}) \right|$$

$$= k! \sum_{r_2,\ldots,r_k=1}^{\infty} (1 + \sum_{i=2}^{j} |r_i|) \left| \text{cum}(X_{t_1,T}, X_{t_2,T}, \ldots, X_{t_k,T}) \right|,$$

substituting (28) into the above gives the result.

**PROOF of Lemma 4.2** The proof of (18) follows immediately from Ibragimov’s inequality (Ibragimov (1962)) (see also Davidson (1994), Theorem 14.2). Using this we note that since $X_{t,T} = \mathbb{E}(X_{t,T}|\mathcal{F}_T)$ and $\mathbb{E}(X_{t,T}|\mathcal{F}_{t-j}) \to 0$ as $j \to \infty$, we have almost surely have

$$X_{t,T} = \sum_{j=0}^{\infty} (\mathbb{E}_{t-j}(X_{t,T}) - \mathbb{E}_{t-j-1}(X_{t,T})). \quad (29)$$

To prove (19) we use Ibragimov’s and Chebyshev’s inequalities and (18) to obtain

$$\left\| \mathbb{E}(M_j(t-j)M_{r-(t-j)}(t-j)|\mathcal{F}_{t-j-1}) - \mathbb{E}(M_j(t-j)M_{r-(t-j)}(t-j)|\mathcal{F}_{t-j-1}) \right\|_q \leq 2\left\| \mathbb{E}(M_j(t-j)M_{r-(t-j)}(t-j)|\mathcal{F}_{t-j-1}) - \mathbb{E}(M_j(t-j)M_{r-(t-j)}(t-j)) \right\|_q \leq 4(2^{1/q} + 1)\|M_j(t-j)M_{r-(t-j)}(t-j)\|_2 \alpha(i) \frac{1}{2} - \frac{1}{2} \leq 12\|M_j(t-j)\|_2 \|M_{r-(t-j)}(t-j)\|_2 \alpha(i) \frac{1}{2} - \frac{1}{2} \leq 12^2\|X_{t,T}\|^2 \alpha(j) \frac{1}{2} - \frac{1}{2} \alpha(\tau - (t-j))^2 \frac{1}{2} - \frac{1}{2} \alpha(i) \frac{1}{2} - \frac{1}{2},$$

where $\tilde{r}/2 > r > q$ and $M_j(t-j) = \mathbb{E}_{t-j}(X_t) - \mathbb{E}_{t-j-1}(X_t)$. Now we prove (20). By substituting (29) into $B_{T,S_T}^{(u)}$ and using the above notation for conditional expectations we have

$$B_{T,S_T}^{(u)} = T^{-1} \sum_{t=u+1}^{S_T+u} \sum_{\tau} F_{t,\tau,M}(X_{t,T}X_{\tau,T} - \mathbb{E}(X_{t,T}X_{\tau,T}))$$

$$= T^{-1} \sum_{j_1,j_2=0}^{\infty} \sum_{t=u+1}^{S_T+u} \sum_{\tau=\max(t-M,1)}^{\infty} F_{t,\tau,M}(M_{j_1}(t-j_1)M_{j_2}(\tau-j_2) - \mathbb{E}(M_{j_1}(t-j_1)M_{j_2}(\tau-j_2))).$$
Now partitioning the above sum into various cases and using Minkowski's inequality we have

\[
\|B_{T,S_T}^{(u)}\|_q = T^{-1} \sum_{j_1,j_2=0}^{\infty} \left\| \sum_{t=u+1}^{\infty} \sum_j F_{t,\tau,M} M_{j_1}(t-j_1)M_{j_2}(\tau-j_2) - \mathbb{E}(M_{j_1}(t-j_1)M_{j_2}(\tau-j_2)) \right\|_q \\
\leq I + II + III,
\]

where

\[
I = T^{-1} \sum_{j_1,j_2=0}^{\infty} \left\| \sum_{t=u+1}^{\infty} \sum_{\tau<t-j_1+j_2} F_{t,\tau,M} M_{j_1}(t-j_1)M_{j_2}(\tau-j_2) \right\|_q
\]

\[
II = T^{-1} \sum_{j_1,j_2=0}^{\infty} \left\| \sum_{t<\tau-j_1+j_2} F_{t,\tau,M} M_{j_1}(t-j_1)M_{j_2}(\tau-j_2) \right\|_q
\]

\[
III = T^{-1} \left\| \sum_{j=0}^{\infty} \sum_{t,\tau=u+1} F_{t,\tau,M} (M_j(t-j)M_{\tau-(t-j)}(t-j) - \mathbb{E}(M_j(t-j)M_{\tau-(t-j)}(t-\tau))) \right\|_q.
\]

We observe that \(\{\sum_{t<\tau-j_1+j_2} F_{t,\tau,M} M_{j_1}(t-j_1)M_{j_2}(\tau-j_2)\}_t\) and \(\{\sum_{t<\tau-j_1+j_2} F_{t,\tau,M} M_{j_1}(t-j_1)M_{j_2}(\tau-j_2)\}_\tau\) are martingale differences. Therefore by using the Burkholder-Rosenthal inequality twice together with Cauchy-Schwarz, for \(q \geq 2\) we have

\[
I \leq T^{-1} \sum_{j_1,j_2=0}^{\infty} \left( \sum_{t=u+1}^{\infty} \left\| \sum_{\tau<t-j_1+j_2} F_{t,\tau,M} M_{j_1}(t-j_1)M_{j_2}(\tau-j_2) \right\|^2 \right)^{1/2} \leq T^{-1} \sum_{j_1,j_2=0}^{\infty} \left( \sum_{t=u+1}^{\infty} \left\| M_{j_1}(t-j_1) \right\|_{2q}^2 \sum_{\tau<t-j_1+j_2} \left\| F_{t,\tau,M} M_{j_2}(\tau-j_2) \right\|_{2q}^2 \right)^{1/2} \leq T^{-1} \sum_{j_1,j_2=0}^{\infty} \left( \sum_{t=u+1}^{\infty} \left\| M_{j_1}(t-j_1) \right\|_{2q}^2 \sum_{\tau<t-j_1+j_2} |F_{t,\tau,M}|^2 \left\| M_{j_2}(\tau-j_2) \right\|_{2q}^2 \right)^{1/2}
\]

and

\[
II \leq T^{-1} \sum_{j_1,j_2=0}^{\infty} \left( \sum_{\tau=1}^{T} \left\| M_{j_2}(\tau-j_2) \right\|_{2q}^2 \sum_{t<\tau-j_2+j_1} |F_{t,\tau,M}|^2 \left\| M_{j_1}(t-j_1) \right\|_{2q}^2 \right)^{1/2}.
\]

Using (18) we have \(\|M_j(t-j)\|_{2q} \leq C\alpha(j)^{\frac{1}{2q}-\frac{1}{2}}\). Substituting these bounds into \(I\) and under Assumption 2.1(i) we have

\[
I \leq T^{-1} C \left( \sum_{j=0}^{\infty} \alpha(j)^{\frac{1}{2q}-\frac{1}{2}} \right)^2 \left( \sum_{t=1}^{T} \sum_{\tau} |F_{t,\tau,M}|^2 \right)^{1/2} \leq T^{-1} S_{T}^{1/2} C \left( \sum_{j=0}^{\infty} \alpha(j)^{\frac{1}{2q}-\frac{1}{2}} \right)^2 \sup_{\tau} \left( \sum_{t} |F_{t,\tau,M}|^2 \right)^{1/2} \leq KT^{-1} S_{T}^{1/2} G_{M}^{1/2} L \left( \frac{s}{2q} - \frac{s}{r} \right)^2.
\]
Using the same methods we have

\[ \| I I \|_q \leq T^{-1} S_T^{1/2} C \left( \sum_{j=0}^{\infty} \alpha(j) \frac{1}{2q} \right)^2 \text{sup}_{t} \left( \sum_{\tau} |F_{t,\tau,M}|^2 \right)^{1/2} \leq KT^{-1} S_T^{1/2} G_M^{1/2} L \left( \frac{s}{2q} - \frac{s}{r} \right)^2. \]  

(31)

Finally we obtain a bound for III. This requires a more delicate analysis since \{M_j(t - j)M_{t-(\tau-j)}(t - j) - E(M_j(t - j)M_{t-(\tau-j)}(t - j))\} are not necessarily martingale differences over \(t\).

We first represent \(M_j(t - j)M_{t-(\tau-j)}(t - j) - E(M_j(t - j)M_{t-(\tau-j)}(t - j))\) as the sum of martingale differences. Since \(E(M_j(t - j)M_{t-(\tau-j)}(t - j)|\mathcal{F}_{t-j-1-i}) \overset{a.s.}{\to} E(M_j(t - j)M_{t-(\tau-j)}(t - j))\), as \(i \to \infty\), we have

\[ M_j(t - j)M_{t-(\tau-j)}(t - j) - E(M_j(t - j)M_{t-(\tau-j)}(t - j)) = \sum_{i=0}^{\infty} a_{i,j}(t, \tau), \]

almost surely, where

\[ a_{i,j}(t, \tau) = E(M_j(t - j)M_{t-(\tau-j)}(t - j)|\mathcal{F}_{t-j-1-i}) - E(M_j(t - j)M_{t-(\tau-j)}(t - j)|\mathcal{F}_{t-j-1}). \]

Substituting this into III and using Minkowski’s inequality gives

\[ III = \left\| T^{-1} \sum_{\tau} \sum_{j=\tau}^{u+S_T} \sum_{t=\tau}^{\infty} F_{t,\tau,M} \left( M_j(t - j)M_{t-(\tau-j)}(t - j) - E(M_j(t - j)M_{t-(\tau-j)}(t - j)) \right) \right\|_q \]

\[ \leq T^{-1} \sum_{\tau=0}^{M} \sum_{j=\tau}^{\infty} \sum_{t=\tau}^{\infty} \sum_{i=0}^{u+S_T} F_{t,\tau,M} a_{i,j}(t, \tau) \right\|_q, \]

We observe that \{a_{i,j}(t, \tau)\} are martingale differences over \(t\), hence by using the Burkholder-Rosenthal inequality we have

\[ III \leq T^{-1} \sum_{\tau=0}^{M} \sum_{j=\tau}^{\infty} \sum_{i=0}^{u+S_T} \left\| \sum_{t=\tau}^{\infty} F_{t,\tau,M} a_{i,j}(t, \tau) \right\|_q \leq T^{-1} \sum_{\tau=0}^{M} \sum_{j=\tau}^{\infty} \left( \sum_{i=0}^{u+S_T} \|a_{i,j}(t, \tau)\|^2 \right)^{1/2}. \]

Substituting (19) into III gives

\[ III \leq C T^{-1} \sum_{i=0}^{\infty} \alpha(i)^{1/2} \sum_{\tau=0}^{\infty} \sum_{j=\tau}^{\infty} \alpha(j)^{1/2} \alpha(j - \tau)^{1/2} \left( \sum_{t=\tau}^{\infty} |F_{t,\tau,M}|^2 \right)^{1/2} \]

\[ \leq KT^{-1} G_M^{1/2} L \left( \frac{s}{q} - \frac{s}{r} \right) L \left( \frac{s}{2r} - \frac{s}{r} \right)^2. \]

(32)

Finally, we substitute (30), (31) and (32) into \(\| B_{T,S_T}^{(u)} \|_q\) to obtain (20).
References


