Fourier based statistics for irregular spaced spatial data

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Abstract

A class of Fourier based statistics for irregular spaced spatial data is introduced, examples include, the Whittle likelihood, a parametric estimator of the covariance function based on the \(L_2\)-contrast function and a simple nonparametric estimator of the spatial autocovariance which is a non-negative function. The Fourier based statistic is a quadratic form of a discrete Fourier-type transform of the spatial data. Evaluation of the statistic is computationally tractable, requiring \(O(nb)\) operations, where \(b\) are the number Fourier frequencies used in the definition of the statistic (which varies according to the application) and \(n\) is the sample size. The asymptotic sampling properties of the statistic are derived using mixed spatial asymptotics, where the number of locations grows at a faster rate than the size of the spatial domain and under the assumption that the spatial random field is stationary and the irregular design of the locations are independent, identically distributed random variables. The asymptotic analysis allows for the case that the frequency grid over which the estimator is defined is unbounded. These results are used to obtain the asymptotic sampling properties of certain estimators and also in variance estimation.

Keywords and phrases: Fixed and increasing frequency domain asymptotics, mixed spatial asymptotics, random locations, spectral density function, stationary spatial random fields.

1 Introduction

In recent years irregular spaced spatial data has become ubiquitous in several disciplines as varied as the geosciences to econometrics. The analysis of such data poses several challenges which do not arise in data which is sampled on a regular lattice. One important problem is the computational costs when dealing with large irregular sampled data sets. If spatial data are sampled on a regular lattice then algorithms such as the Fast Fourier transform can be employed to reduce the computational burden (see, for example, Chen, Hurvich, and Lu (2006)). Unfortunately, such algorithms have little benefit if the spatial data are irregularly sampled. To address this issue, within the spatial domain, several authors, including, Vecchia (1988), Cressie and Huang (1999),
Stein, Chi, and Welty (2004), have proposed estimation methods which are designed to reduce the computational burden.

In contrast to the above references, Fuentes (2007) argues that working within the frequency domain can simplify the problem. Fuentes assumes that the irregular spaced data can be embedded on a grid and the missing mechanism is deterministic and ‘locally smooth’. Based on these assumptions Fuentes proposes a tapered Whittle estimator to estimate the parameters of a spatial covariance function. However, a possible drawback is that, if the locations are extremely irregular, the local smooth assumption will not hold. Therefore, in order to work within the frequency domain, methodology and inference devised specifically for irregular sampled data is required. Matsuda and Yajima (2009) and Bandyopadhyay and Lahiri (2009) have pioneered this approach, by assuming that the irregular locations are independent, identically distributed random variables (thus allowing the data to be extremely irregular) and define the irregular sampled discrete Fourier transform (DFT) as

\[ J_n(\omega) = \frac{\lambda^{d/2}}{n} \sum_{j=1}^{n} Z(s_j) \exp(i s_j \cdot \omega), \]  

where \( s_j \in [-\lambda/2, \lambda/2]^d \) denotes the spatial locations observed in the space \([-\lambda/2, \lambda/2]^d\) and \( \{Z(s_j)\} \) denotes the spatial random field at these locations. It’s worth mentioning a similar transformation on irregular sampled data goes back to Masry (1978), who defines the discrete Fourier transform of Poisson sampled continuous time series. Using the definition of the DFT given in (1) Matsuda and Yajima (2009) define the Whittle likelihood for the spatial data as

\[ \int_{\Omega} \left( \log f_\theta(\omega) + \frac{|J_n(\omega)|^2}{f_\theta(\omega)} \right) d\omega, \]  

where \( \Omega \) is a compact set in \( \mathbb{R}^d \), here we will assume that \( \Omega = 2\pi[-C, C]^d \) and \( f_\theta(\omega) \) is the candidate parametric spectral density function. A clear advantage of this approach is that it avoids the inversion of a large matrix. However, one still needs to evaluate the integral, which can be computationally quite difficult, especially if the aim is to minimise over the entire parameter space of \( \theta \).

For frequency domain methods to be computationally tractable and to be used as a viable alternative to ‘spatial domain’ methods, the frequency domain needs to be gridified in such a way that tangible estimators are defined on the grid. The problem is how to choose an appropriate lattice on \( \mathbb{R}^d \). A possible solution can be found in Bandyopadhyay and Subba Rao (2015), Theorem 2.1, who show that on the lattice \( \{\omega_k = (\frac{2\pi k_1}{\lambda}, \ldots, \frac{2\pi k_d}{\lambda}); k = (k_1, \ldots, k_d) \in \mathbb{Z}^d\} \) the DFT \( \{J_n(\omega_k)\} \) is ‘close to uncorrelated’ if the random field is second order stationary and the locations \( \{s_j\} \) are independent, identical, uniformly distributed random variables. Heuristically, this result suggests that \( \{J_n(\omega_k)\} \) contains all the information about the random field, such that one can perform the analysis on the transformed data \( \{J_n(\omega_k)\} \). For example, rather than use (2) one can use the discretized likelihood

\[ \frac{1}{\lambda^d} \sum_{k_1, \ldots, k_d = -C\lambda}^{C\lambda} \left( \log f_\theta(\omega_k) + \frac{|J_n(\omega_k)|^2}{f_\theta(\omega_k)} \right), \]  

(3)
to estimate the parameters. Indeed Matsuda and Yajima (2009), Remark 2, mention that in practice one should use the discretized likelihood to estimate the parameters, but, unfortunately, they did not derive any results for the discretized likelihood.

Integrated periodogram statistics such as the discretized Whittle likelihood defined above, are widely used in time series and their properties well understood. However, with the exception of Bandopadhyay, Lahiri, and Norman (2015), as far as we are aware, there does not exist any results for integrated periodograms of irregularly sampled spatial data. Therefore one of the objectives of this paper is to study the class of integrated statistics which have the form

$$Q_{a,\lambda}(g_\theta; r) = \frac{1}{\lambda^d} \sum_{k_1,\ldots,k_d=-a}^{a} g_\theta(\omega_k)J_n(\omega_k)J_n(\omega_{k+r}) \quad r \in \mathbb{Z}^d,$$  \tag{4}

where $\theta$ is a finite dimension parameter, $J_n(\omega_k)$ is defined in (1) and $g_\theta : \mathbb{R}^d \to \mathbb{R}$ is a weight function which depends on the application. Note that for $r = 0$, $a = C\lambda$ and $g_\theta(\omega) = f_\theta(\omega)^{-1}$ we have the discretized Whittle likelihood (up to a non-random constant). We have chosen to consider the case $r \neq 0$ because in the case that the locations are uniformly sampled, $Q_{a,\lambda}(g_\theta;0)$ and $\{Q_{a,\lambda}(g_\theta; r): r \neq 0\}$ asymptotically have the same variance, which we exploit to estimate the variance of $Q_{a,\lambda}(g_\theta;0)$ in Section 5. In terms of computation, evaluation of $\{J_n(\omega_k); k = (k_1, \ldots, k_d), k_j = -a, \ldots, a\}$ requires $O(a^d n)$ operations (as far as are aware the FFT cannot be used to reduce the number of operations for irregular spaced data). However, once $\{J_n(\omega_k); k = (k_1, \ldots, k_d), k_j = -a, \ldots, a\}$ has been evaluated the evaluation of $Q_{a,\lambda}(g_\theta; r)$ only requires $O(a^d)$ operations.

To understand why such a class of statistics may be of interest in spatial statistics, consider the case $r = 0$ and replace $|J_n(\omega_k)|^2$ in $Q_{a,\lambda}(g_\theta;0)$ with the spectral density, $f : \mathbb{R}^d \to \mathbb{R}$ of the spatial process $\{Z(s); s \in \mathbb{R}^d\}$. This replacement suggests $Q_{a,\lambda}(g_\theta;0)$ is estimating the functional $I(g_\theta; \frac{a}{\lambda})$ where

$$I \left( g_\theta; \frac{a}{\lambda} \right) = \frac{1}{(2\pi)^d} \int_{[-a/\lambda,a/\lambda]^d} g_\theta(\omega)f(\omega)d\omega. \tag{5}$$

In Section 3 we show that this conjecture is true up to a multiplicative constant. Before giving examples of functionals $I \left( g_\theta; \frac{a}{\lambda} \right)$ of interest we first compare $Q_{a,\lambda}(g_\theta;0)$ to the integrated periodogram estimators which are often used in time series (see for example, Walker (1964), Hannan (1971), Dunsmuir (1979), Dahlhaus and Janas (1996), Can, Mikosch, and Samorodnitsky (2010) and Niebuhr and Kreiss (2014)). However, there are some fundamental differences, which mean the analysis of (4) is very different to those in classical time series. We observe that the user-chosen frequency grid $\{\omega_k; k = (k_1, \ldots, k_d), k_j \in [-a,a]\}$ used to define $Q_{a,\lambda}(g_\theta; r)$ influences the limits of the functional $I(g_\theta; \frac{a}{\lambda})$. This is different to regular sampled locations where frequency grid is defined over $[0,2\pi]^d$ (outside this domain aliasing occurs). Returning to irregular sampled spatial data, in certain situations, such as the Whittle likelihood described in (3) or the spectral density estimator (described in Section 2), bounding the frequency grid to $a = C\lambda$ is necessary. In the case of the Whittle likelihood this is important, because $|f_\theta(\omega)| \to 0$ as $\|\omega\| \to \infty$ (for any norm $\| \cdot \|$), thus the discretized Whittle likelihood is only well defined over a bounded frequency grid.
The choice of $C$ is tied to how fast the tails in the parametric class of spectral density functions $\{f_\theta; \theta \in \Theta\}$ (where $\Theta$ is a compactly supported parameter space) decay to zero. In contrast, in most situations the aim is to estimate the functional $I(g_\theta; \infty)$. For example, the spatial covariation can be written as $c(v) = I(e^{i\theta}; \infty)$ or a loss function defined over all frequencies. In this case, the only plausible method for estimating $I(g_\theta; r)$ without asymptotic bias is to allow the frequency grid $\{\omega_k; k = (k_1, \ldots, k_d), -a \leq k_i \leq a\}$ to be unbounded, in other words let $a/\lambda \to \infty$ as $\lambda \to \infty$. In which case $I(g_\theta; \frac{a}{\lambda}) \to I(g; \infty)$ as the domain $\lambda \to \infty$. To understand whether this is feasible, we recall that if the spatial process were observed only on a grid then aliasing means that it is impossible to estimate $I(g_\theta; \infty)$. On the other hand, unlike regularly spaced or near regularly spaced data, ‘truly’ irregular sampling means that the DFT can estimate high frequencies, without the curse of aliasing (a phenomena which was noticed as early as Shapiro and Silverman (1960) and Beutler (1970)). In this case, in the definition of $Q_{a, \lambda}(g_\theta; r)$ there’s no need for the frequency grid to be bounded, and $a$ can be magnitudes larger than $\lambda$. One contribution of this paper is to understand how an unbounded frequency grid influences inference.

As alluded to in the previous paragraph, in this paper we derive the asymptotic sampling properties of $Q_{a, \lambda}(g_\theta; r)$ under the asymptotic set-up that the spatial domain $\lambda \to \infty$. The main focus will be the mixed asymptotic framework, introduced in Hall and Patil (1994) (see also Hall, Fisher, and Hoffman (1994) and used in, for example, Lahiri (2003), Matsuda and Yajima (2009), Bandyopadhyay and Lahiri (2009), Bandyopadhyay et al. (2015) and Bandyopadhyay and Subba Rao (2015)). This is where $\lambda \to \infty$ (the size of the spatial domain grows) and the number of observations $n \to \infty$ in such a way that $\lambda^d/n \to 0$. In other words, the sampling density gets more dense as the spatial domain grows. The results differ slightly under the pure increasing domain framework (see Bandyopadhyay et al. (2015) who compares the mixed and pure increasing domain framework), where $\lambda^d/n \to c$ ($0 < c < \infty$) as $n \to \infty$ and $\lambda \to \infty$. In Section 3.3 we outline some of these differences. We should be mention that under the fixed-domain asymptotic framework (where $\lambda$ is kept fixed but the number of locations, $n$ grows) considered in Stein (1994, 1999), $Q_{a, \lambda}(g_\theta; r)$ cannot consistently estimate $I(g_\theta; \frac{a}{\lambda})$. There is a non-trivial bias (which can be calculated using the results in Section 3.1) and a variance that does not asymptotically converge to zero as $n \to \infty$.

Obtaining the moments of $Q_{a, \lambda}(g_\theta; r)$ in the case that $a = O(\lambda)$ is analogous to those for regular sampled time series and spatial processes and follows from the DFT analysis given in Section 2.1. However, when the frequency grid is unbounded, the number of terms within the summand of $Q_{a, \lambda}(g_\theta; r)$ is $O(a^d)$ which is increasing at a rate faster than the standardisation $\lambda^d$. This turns the problem into one that is technically rather challenging; bounds on the cumulant of DFTs cannot be applied to estimators defined on an increasing frequency grid. One of the aims in this paper is to develop the machinery for the analysis of estimators defined with an unbounded frequency grid. We show that under Gaussianity of the spatial process and some mild conditions on the spectral density function, the choice of $a$ does not play a significant role in the sampling properties of $Q_{a, \lambda}(g_\theta; r)$. In particular, we show that $Q_{a, \lambda}(g_\theta; r)$ is a mean squared consistent estimator of $I(g_\theta; \infty)$ (up to a finite constant) for any $a$ so long as $a/\lambda \to \infty$ as $\lambda \to \infty$. Furthermore, by constraining the rate of growth of the frequency grid to $a = O(\lambda^k)$ for some $k$ such that $1 \leq k < \infty$ (both
the bounded and unbounded frequency grid come under this canopy), the asymptotic sampling properties (mean, variance and asymptotic normality) of \( Q_{a,\lambda}(g_{\theta}; r) \) can be derived. On the other hand, we show that if the spatial random process is non-Gaussian then the additional constraint that \((\lambda a)^d/n^2 \to c < \infty \) is required as \( \lambda, a, n \to \infty \). This bound is quite strong and highlights stark differences between the statistical properties of a Gaussian random field to those of a non-Gaussian random field. The results in this paper (at least in the case of Gaussianity) can be used to verify condition C.4(ii) in Bandyopadhyay et al. (2015).

We now summarize the paper. In Section 2 we obtain the properties of the DFT \( J_n(\omega_k) \) and give examples of estimators which can be written as \( Q_{a,\lambda}(g_{\theta}; r) \). In Section 3 we derive the asymptotic sampling properties of \( Q_{a,\lambda}(g_{\theta}; r) \). In Section 4 we apply the results derived in Section 3 to some of the statistics proposed in Section 2. The expressions for the variance \( Q_{a,\lambda}(g_{\theta}; 0) \) are complex and difficult to estimate in practice. In Section 5, we exploit the results in Section 3.4, where we show that in the case that the locations are uniformly distributed \( \{Q_{a,\lambda}(g_{\theta}; r)\} \) forms a ‘near uncorrelated’ sequence over \( r \) which asymptotically has the same variance when \( r/\lambda \to 0 \) as \( \lambda \to \infty \). Using these results we define an ‘orthogonal’ sample which can be used to obtain a simple estimator of the variance of \( Q_{a,\lambda}(g_{\theta}; 0) \) which is computationally fast to evaluate and has useful theoretical properties.

An outline of the proofs in the case of uniform sampling can be found in Appendix A. The proofs for non-uniform sampling and technical proofs can be found in the supplementary material, Subba Rao (2015b); many of these results build on the work of Kawata (1959) and may be of independent interest.

## 2 Preliminary results and assumptions

We observe the spatial random field \( \{Z(s); s \in \mathbb{R}^d\} \) at the locations \( \{s_j\}_{j=1}^n \). Throughout this paper we will use the following assumptions on the spatial random field.

**Assumption 2.1 (Spatial random field)**

(i) \( \{Z(s); s \in \mathbb{R}^d\} \) is a second order stationary random field with mean zero and covariance function \( c(s_1 - s_2) = \text{cov}(Z(s_1), Z(s_2)|s_1, s_2) \). We define the spectral density function as \( f(\omega) = \int_{\mathbb{R}^d} c(s) \exp(-is'\omega)ds \).

(ii) \( \{Z(s); s \in \mathbb{R}^d\} \) is a stationary Gaussian random field.

In the following section we require the following definitions. For some finite \( 0 < C < \infty \) and \( \delta > 0 \), let

\[
\beta_\delta(s) = \begin{cases} 
C & |s| \in [-1, 1] \\
C|s|^{-(1+\delta)} & |s| > 1 
\end{cases}.
\]

Let \( \beta_\delta(s) = \prod_{j=1}^d \beta_\delta(s_j) \). To minimise notation we will often use \( \sum_{k=-a}^a \) to denote the multiple sum \( \sum_{k_1=-a}^a \cdots \sum_{k_d=-a}^a \). Let \( \| \cdot \|_1 \) denote the \( \ell_1 \)-norm of a vector and \( \| \cdot \|_2 \) denote the \( \ell_2 \)-norm.
2.1 Properties of the DFTs

We first consider the behaviour of the DFTs under both uniform and non-uniform sampling of the locations.

Below we summarize Theorem 2.1, Bandyopadhyay and Subba Rao (2015), which defines frequencies where the Fourier transform is ‘close to uncorrelated’, this result requires the following additional assumption on the distribution of the spatial locations.

Assumption 2.2 (Uniform sampling) The locations \( \{s_j\} \) are independent uniformly distributed random variables on \([-\lambda/2, \lambda/2]^d\).

Theorem 2.1 Let us suppose that \( \{Z(s); s \in \mathbb{R}^d\} \) is a stationary spatial random field whose covariance function (defined in Assumption 2.1(i)) satisfies \( |c(s)| \leq \beta_{1+\delta}(s) \) for some \( \delta > 0 \). Furthermore, the locations \( \{s_j\} \) satisfy Assumption 2.2. Then we have

\[
\text{cov} \left[ J_n(\omega_{k_1}), J_n(\omega_{k_2}) \right] = \left\{ \begin{array}{ll}
f(\omega_k) + O\left(\frac{1}{\lambda^d} + \frac{\lambda^d}{n}\right) & k_1 = k_2(= k) \\
O\left(\frac{1}{\lambda^{2-d}}\right) & k_1 - k_2 \neq 0 \end{array} \right.
\]

where \( b = b(k_1 - k_2) \) denotes the number of zero elements in the vector \( k_1 - k_2, \omega_k = (\frac{2\pi k_1}{\lambda}, \ldots, \frac{2\pi k_d}{\lambda}) \), \( k \in \mathbb{Z}^d \) and the bounds are uniform in \( k, k_1, k_2 \in \mathbb{Z}^d \).

PROOF See Theorem 2.1, Bandyopadhyay and Subba Rao (2015). \( \square \)

To understand what happens in the case that the locations are not uniformly sampled, we adopt the assumptions of Hall and Patil (1994), Matsuda and Yajima (2009) and Bandyopadhyay and Lahiri (2009) and assume that \( \{s_j\} \) are iid random variables with density \( \frac{1}{\lambda^d}h(\frac{s}{\lambda}) \), where \( h: [-1/2, -1/2]^d \rightarrow \mathbb{R} \). We use the following assumptions on the sampling density \( h \).

Assumption 2.3 (Non-uniform sampling) The locations \( \{s_j\} \) are independent distributed random variables on \([-\lambda/2, \lambda/2]^d\), where the density of \( \{s_j\} \) is \( \frac{1}{\lambda^d}h(\frac{s}{\lambda}) \), and \( h(\cdot) \) admits the Fourier representation

\[
h(u) = \sum_{j \in \mathbb{Z}^d} \gamma_j \exp(i2\pi j'u),
\]

where \( \sum_{j \in \mathbb{Z}^d} \|\gamma_j\|_1 < \infty \) such that \( |\gamma_j| \leq C \prod_{i=1}^d |j_i|^{-(1+\delta)}I(j_i \neq 0) \) (for some \( \delta > 0 \)). This assumption is satisfied if the second derivative of \( h \) is bounded on the \( d \)-dimensional torus \([-1/2, 1/2]^d\).

Note that if \( h \) is such that \( \sup_{s \in [-1/2, 1/2]^d} \left| \frac{\partial^{m_1+\ldots+m_d}h(s_1, \ldots, s_d)}{\partial s_1^{m_1} \ldots \partial s_d^{m_d}} \right| < \infty \) \( (0 \leq m_i \leq 2) \) but \( h \) is not continuous on the \( d \)-dimensional torus \([-1/2, 1/2]^d\) then \( |\gamma_j| \leq C \prod_{i=1}^d |j_i|^{-(1+\delta)}I(j_i \neq 0) \) and the above condition will not be satisfied. However, this assumption can be induced by tapering the observations such that \( X(s_j) \) is replaced with \( \tilde{X}(s_j) \), where \( \tilde{X}(s_j) = t(s_j)X(s_j), t(s) = \prod_{i=1}^d t(s_i) \) and \( t \) is a weight function which has a bounded second derivative, \( t(-1/2) = t(1/2) = 0 \) and \( t'(1/2) = t'(-1/2) = 0 \). By using \( \tilde{X}(s_j) \) instead of \( X(s_j) \), in all the derivations we can replace the density \( h(s) \) with \( t(s)h(s) \). This means the results now rely on the Fourier coefficients of \( t(s)h(s) \),
which decay at the rate $|\int_{[-1/2,1/2]^d} t(s)h(s)\exp(i2\pi j's)ds| \leq C \prod_{i=1}^d |ji|^{-2}I(ji \neq 0)$, and thus the above condition is satisfied. Note that Matsuda and Yajima (2009), Definition 2, use a similar data-tapering scheme to induce a similar condition.

We now show that under this general sampling scheme the near ‘uncorrelated’ property of the DFT given in Theorem 2.1 does not hold.

**Theorem 2.2** Let us suppose that $\{Z(s); s \in \mathbb{R}^d\}$ is a stationary spatial random field whose covariance function (defined in Assumption 2.1(i)) satisfies $|c(s)| \leq \beta_{1+\delta}(s)$ for some $\delta > 0$. Furthermore, the locations $\{s_j\}$ satisfy Assumption 2.3. Then we have

$$\text{cov}[J_n(\omega_{k_1}), J_n(\omega_{k_2})] = \langle \gamma, \gamma_{(k_2 - k_1)} \rangle f(\omega_{k_1}) + \frac{c(0)\gamma_{k_2 - k_1} \lambda^d}{n} + O\left(\frac{1}{\lambda}\right),$$

where the bounds are uniform in $k_1, k_2 \in \mathbb{Z}^d$ and $\langle \gamma, \gamma_r \rangle = \sum_{j \in \mathbb{Z}^d} \gamma_j \gamma_{r-j}$.

**PROOF** See Appendix B.

In the case of uniform sampling $\gamma_j = 0$ for all $j \neq 0$ and $\gamma_0 = 1$, in this case $\text{cov}[J_n(\omega_{k_1}), J_n(\omega_{k_2})] = \left[ f(\omega_{k_1}) + c(0)\lambda^d/n \right] I(k_1 = k_2) + O(\lambda^{-1})$. Thus we observe that Theorem 2.2 includes Theorem 2.1 as a special case (up to the number of zeros in the vector $k_1 - k_2$; if $b(k_1 - k_2) \leq (d-2)$ Theorem 2.1 gives a faster rate). On the other hand, we note that in the case the locations are not sampled from a uniform distribution the DFTs are not (asymptotically) uncorrelated. However, they do satisfy the property

$$\text{cov}[J_n(\omega_{k_1}), J_n(\omega_{k_2})] = O\left(\left[1 + \frac{I(k_1 \neq k_2)}{||k_1 - k_2||_1}\right] \left[1 + \frac{\lambda^d}{n}\right] + 1/\lambda\right).$$

In other words, the correlations between the DFTs decay the further apart the frequencies. A similar result was derived in Bandyopadhyay and Lahiri (2009) who show that the correlation between $J_n(\omega_1)$ and $J_n(\omega_2)$ are asymptotic uncorrelated if their frequencies are ‘asymptotically distant’ such that $\lambda^d||\omega_1 - \omega_2||_1 \to \infty$.

**Remark 2.1** In both Theorems 2.1 and 2.2 we use the assumption that $|c(s)| \leq \beta_{1+\delta}(s_j)$. This assumption is satisfied by a wide range of covariance functions. Examples include:

- The Wendland covariance, since its covariance is bounded and has a compact support.
- The Matern covariance, which for $\nu > 0$ is defined as $c_\nu(\|s\|_2) = \frac{1}{\Gamma(\nu)2^{\nu-1}(\sqrt{2\nu}\|s\|_2)^\nu}K_\nu(\sqrt{2\nu}\|s\|_2)$ ($K_\nu$ is the modified Bessel function of the second kind). To see why, we note that if $\nu > 0$ then $c_\nu(s)$ is a bounded function. Furthermore, for large $\|s\|_2$ we note that $c_\nu(\|s\|_2) \sim C_\nu\|s\|_2^{-0.5}\exp(\sqrt{2\nu}\|s\|_2)$ as $\|s\|_2 \to \infty$ (where $C_\nu$ is a finite constant). Thus by using the inequality

$$d^{-1/2}(|s_1| + |s_2| + \ldots + |s_d|) \leq \sqrt{s_1^2 + s_2^2 + \ldots + s_d^2} \leq (|s_1| + |s_2| + \ldots + |s_d|),$$

we can show $|c_\nu(s)| \leq \beta_{1+\delta}(s)$ for any $\delta > 0$. 


Remark 2.2 (Random vs Fixed design of locations) Formally, the methods in this paper are developed under the assumption that the locations are random.

Of course, as a referee pointed out, it is unclear whether a design is random or not. However, in the context of the current paper, ‘random design’ really refers to the locations being ‘irregular’ in the sense of not being defined on or in the proximity of a lattice (such a design would be considered ‘near regular’). A simple way to check for this is to evaluate the Fourier transform of the locations

$$\hat{\gamma}_r = \frac{1}{n} \sum_{j=1}^{n} \exp(i s'_j \omega_r).$$

In the case that the design is random $\hat{\gamma}_r$ is an estimator of $\gamma_r$ (the Fourier coefficients of $h(\cdot)$) and $\hat{\gamma}_r \approx 0$ for $\|r\|_1$ large. On the other hand, if the design is ‘near regular’ $\{\hat{\gamma}_r\}$ would be periodic on the grid $\mathbb{Z}^d$. Therefore, by making a plot of $\{\hat{\gamma}_r; r \in [-Kn^{1/d}, Kn^{1/d}]^d\}$ (for some $K \in \mathbb{Z}^+$), we can search for repetitions at shifts of $n^{1/d}$, i.e., $\hat{\gamma}_r \approx \hat{\gamma}_{r+n^{1/d}}$ where $n^{1/d} = (n^{1/d}, \ldots, n^{1/d})$. If $\hat{\gamma}_r \approx 0$ for large $\|r\|_1$, then the locations are sufficiently irregular such that we can treat them as random and the results in this paper are valid. On the other hand repetitions suggest the locations lie are ‘near regular’ and the results in this paper do not hold in this case.

In the case that spatial locations are defined on a regular lattice, then it is straightforward to transfer the results for discrete time time series to spatial data on the grid (in this case $a = \lambda = n^{1/d}$).

2.2 Assumptions for the analysis of $Q_{a,\lambda}(\cdot)$

In order to asymptotically analyze the statistic, $Q_{a,\lambda}(g_\theta; r)$, we will assume that the spatial domain $[-\lambda/2, \lambda/2]^d$ grows as the number of observations $n \to \infty$ and we will mainly work under the mixed domain framework described in the introduction, where $\lambda^d/n \to 0$ as $\lambda \to \infty$ and $n \to \infty$.

Assumption 2.4 (Assumptions on $g_\theta(\cdot)$)

(i) If the frequency grid is bounded to $[-C, C]^d$ (thus $a = C\lambda$), then $\sup_{\omega \in [-C, C]^d} |g_\theta(\omega)| < \infty$ and for all $1 \leq j \leq d$, $\sup_{\omega \in [-C, C]^d} \left| \frac{\partial g_\theta(\omega)}{\partial \omega_j} \right| < \infty$.

(ii) If the frequency grid is unbounded, then $\sup_{\omega \in \mathbb{R}^d} |g_\theta(\omega)| < \infty$ and for all $1 \leq j \leq d$, $\sup_{\omega \in \mathbb{R}^d} \left| \frac{\partial g_\theta(\omega)}{\partial \omega_j} \right| < \infty$.

In the case that $a = C\lambda$ (where $|C| < \infty$), the asymptotic properties of $Q_{a,\lambda}(g_\theta; r)$ can be mostly derived by using either Theorems 2.1 or 2.2. For example, in the case that the locations are uniformly sampled, it follows from Theorem 2.1 that

$$E[Q_{a,\lambda}(g_\theta; r)] = \begin{cases} I(g_\theta; C) + O(\frac{\lambda^d}{n} + \frac{1}{\lambda}) & r = 0 \\ O(\frac{1}{\lambda}) & \text{otherwise} \end{cases}$$

where $I(g_\theta; \cdot)$ is defined in (5), with $\lambda^d/n \to 0$ as $\lambda \to \infty$ and $n \to \infty$ (the full details will be given in Theorem 3.1). However, this analysis only holds in the case that $a = O(\lambda)$, in other words the frequency grid is bounded. Situations where this statistic is of interest are rare, notable exceptions include the Whittle likelihood and the spectral density estimator. But usually our aim is to estimate
the functional $I(g_\theta; \infty)$, therefore the frequency grid $\{\omega_k; k = (k_1, \ldots, k_d), -a \leq k_i \leq a\}$ should be unbounded. In this case, using Theorems 2.1 or 2.2 to analyse $E[Q_{a,\lambda}(g_\theta; r)]$ will lead to errors of the order $O(\frac{\sigma}{\lambda^2})$, which cannot be controlled without placing severe restrictions on the expansion of the frequency grid.

Below we state the assumptions required on the spatial process. Below each assumption we explain where it is used. Note that Assumption 2.5(a) is used for the bounded frequency grid and 2.5(b) is (mainly) used for the nonbounded frequency grid (however the same assumption can also be used for the bounded frequency, see the comments below 2.5(b)). It is unclear which Assumption 2.5(a) or (b) is stronger, since neither assumption implies the other.

**Assumption 2.5 (Conditions on the spatial process)**

(a) $|c(s)| \leq \beta_1 + \delta(s)$

Required for the bounded frequency grid, to obtain the DFT calculations.

(b) For some $\delta > 0$, $f(\omega) \leq \beta_\delta(\omega)$

Required for the unbounded frequency grid - using this assumption instead of (a) in the case of an bounded frequency grid leads to slightly larger errors bounds in the derivation of the first and second order moments. This assumption is also used to obtain the CLT result for both the bounded and unbounded frequency grids.

(c) For all $1 \leq j \leq d$ and some $\delta > 0$, the partial derivatives satisfy $\left| \frac{\partial f(\omega)}{\partial \omega_j} \right| \leq \beta_\delta(\omega)$.

We use this condition to approximate sums with integral for both the bounded and unbounded frequency grids. It is also used to make a series of approximations to derive the limiting variance in the case that the frequency grid is unbounded.

(d) For some $\delta > 0$, $\left| \frac{\partial^d f(\omega)}{\partial \omega_1 \cdots \partial \omega_d} \right| \leq \beta_\delta(\omega)$.

Required only in the proof of Theorem 3.1(ii)(b).

**Remark 2.3** Assumption 2.5(b,c,d) appears quite technical, but it is satisfied by a wide range of spatial covariance functions. For example, the spectral density of the Matern covariance defined in Remark 2.1 is $f_\nu(\omega) = \sigma(1 + \|\omega\|^2)^{-\nu+d/2}$. It is straightforward to show that this spectral density satisfies Assumption 2.5(b,c,d), noting that the $\delta$ used to define $\beta_\delta$ will vary with $\nu$, dimension $d$ and the order of the derivative $f_\nu(\cdot)$.

We show in Appendix F, Subba Rao (2015b), that

$$E[Q_{a,\lambda}(g_\theta; r)] \rightarrow \langle \gamma, \gamma_r \rangle I \left(g_\theta; \frac{a}{\lambda} \right) + E \left[ Z(s_j)^2 \exp(-is'\omega_r) \right] \frac{1}{n} \sum_{k=-a}^{a} g(\omega_k),$$

where $I(g_\theta; \frac{a}{\lambda})$ is defined in (5). We observe that in the case that the frequency grid is bounded then the second term of the above is asymptotically negligible. However, in the case that the frequency grid is unbounded then the second term above plays a role, indeed it can induce a bias unless the
rate of growth of the frequency grid is constrained such that $a^d/n \to 0$. Therefore, the main focus of this paper will be on a slight variant of the statistic $Q_{a,\lambda}(g_0; r)$, defined as

$$
\tilde{Q}_{a,\lambda}(g_0; r) = Q_{a,\lambda}(g_0; r) - \frac{1}{n} \sum_{k=-a}^{a} g_0(\omega_k) \frac{1}{n} \sum_{j=1}^{n} Z(s_j)^2 \exp(-is_j^t\omega_r).
$$

(8)

$\tilde{Q}_{a,\lambda}(g_0; r)$ avoids some of the bias issues when the frequency grid is unbounded. For the rest of this paper our focus will be on $\tilde{Q}_{a,\lambda}(g_0; r)$. Note that Matsuda and Yajima (2009) and Bandyopadhyay et al. (2015) use a similar bias correction method.

The results for the non-bias corrected version ($Q_{a,\lambda}(g_0; r)$) can be found in Appendix F, Subba Rao (2015b).

Below we give some examples of statistics of the form (4) and (8) which satisfy Assumption 2.4.

**Example 2.1 (Fixed frequency grid)** (i) The discretized Whittle likelihood given in equation (3) can be written in terms of $Q_{a,\lambda}(g_0; 0)$ where

$$
\frac{1}{\lambda^d} \sum_{k_1,\ldots,k_d=-C\lambda}^{C\lambda} \left( \log f_0(\omega_k) + \frac{|J_n(\omega_k)|^2}{f_0(\omega_k)} \right) = Q_{a,\lambda}(f_0^{-1}; 0) + \frac{1}{\lambda^d} \sum_{k_1,\ldots,k_d=-C\lambda}^{C\lambda} \log f_0(\omega_k)
$$

and $\{f_0(\cdot); \theta \in \Theta\}$ (where $\Theta$ has compact support) is a parametric family of spectral density functions. The choice of $a = C\lambda$ depends on the rate of decay of $f_0(\omega)$ to zero.

(ii) A nonparametric estimator of the spectral density function $f(\omega)$ is

$$
\hat{f}_{\lambda,n}(\omega) = \frac{1}{b^d} \prod_{j=1}^{d} W(\omega_j/b) \text{ and } W : [-1/2, -1/2] \to \mathbb{R} \text{ is a spectral window. In this case we set } a = \lambda/2.
$$

**Example 2.2 (Unbounded frequency grid)** (i) In Bandyopadhyay and Subba Rao (2015) a test statistic of the type

$$
\tilde{a}_n(g; r) = \tilde{Q}_{a,\lambda}(g; r) \text{ where } r \in \mathbb{Z}^d/\{0\},
$$

is proposed. We show in Section 3.4 under the assumptions that the locations are uniformly distributed and $r \neq 0$, that $E[\tilde{Q}_{a,\lambda}(g; r)] \approx 0$. On the other hand, if the spatial process is nonstationary, this property does not hold (see Bandyopadhyay and Subba Rao (2015) for the details). Using this dichotomy, Bandyopadhyay and Subba Rao (2015) test for stationarity of a spatial process.

(ii) A nonparametric estimator of the spatial (stationary) covariance function is

$$
\hat{c}_n(v) = T \left( \frac{2v}{\lambda} \right) \left( \frac{1}{\lambda^d} \sum_{k=-a}^{a} |J_n(\omega_k)|^2 \exp(iv^t\omega_k) \right) = T \left( \frac{2v}{\lambda} \right) Q_{a,\lambda}(e^{iv^t}; 0)
$$

(9)
where \( T(u) = \prod_{j=1}^{d} T(u_j) \) and \( T : \mathbb{R} \rightarrow \mathbb{R} \) is the triangular kernel defined as \( T(u) = (1 - |u|) \) for \( |u| \leq 1 \) and \( T(u) = 0 \) for \( |u| > 1 \). It can be shown that the Fourier transform of \( \hat{c}_n(\nu) \) is
\[
\hat{f}_\lambda(\omega) = \int_{\mathbb{R}^d} \hat{c}_n(\nu) \exp(-i\omega' \nu) d\nu = \sum_{k=-a}^{a} |J_n(\omega_k)|^2 \text{Sinc}^2[\lambda(\omega_k - \omega)]
\]
where \( \text{Sinc}(\lambda \omega/2) = \prod_{j=1}^{d} \text{sinc}(\lambda \omega_j/2) \) and \( \text{sinc}(\omega) = \sin(\omega)/\omega \). Clearly, \( \hat{f}_\lambda(\omega) \) is non-negative, therefore, the sample covariance function \( \{\hat{c}_n(\nu)\} \) is a non-negative definite function and thus a valid covariance function.

Hall et al. (1994) also propose a nonparametric estimator of the spatial covariance function which has a valid covariance function. Their scheme is a three-stage scheme; (i) a nonparametric pre-estimator of the covariance function is evaluated (ii) the Fourier transform of the pre-estimator covariance function is evaluated over \( \mathbb{R}^d \) (iii) negative values of the Fourier transform are placed to zero and the inverse Fourier transform of the result forms an estimator of the spatial covariance function. The estimator (9) avoids evaluating Fourier transforms over \( \mathbb{R}^d \) and is simpler to compute.

(iii) We recall that Whittle likelihood can only be defined on a bounded frequency grid. This can be an issue if the observed locations are dense on the spatial domain and thus contain a large amount of high frequency information which would be missed by the Whittle likelihood. An alternative method for parameter estimation of a spatial process is to use a different loss function. Motivated by Rice (1979), consider the quadratic loss function
\[
L_n(\theta) = \frac{1}{\lambda^d} \sum_{k_1,\ldots,k_d=-a}^{a} (|J_n(\omega_k)| - f_\theta(\omega_k))^2,
\]
and let \( \hat{\theta}_n = \arg\min_{\theta \in \Theta} L_n(\theta) \) or equivalently solve \( \nabla_\theta L_n(\theta) = 0 \), where
\[
\nabla_\theta L_n(\theta) = -\frac{2}{\lambda^d} \sum_{k_1,\ldots,k_d=-a}^{a} \Re \nabla_\theta f_\theta(\omega_k) \{ |J_n(\omega_k)|^2 - f_\theta(\omega_k) \} + Q_{a,\lambda}(-2\Re f_\theta(\cdot)\theta;0) + \frac{2}{\lambda^d} \sum_{k=-a}^{a} f_\theta(\omega_k) \nabla_\theta f_\theta(\omega_k),
\]
and \( \Re X \) denotes the real part of the random variable \( X \). It is well known that the distributional properties of a quadratic loss function are determined by its first derivative. In particular, the asymptotic sampling properties of \( \hat{\theta}_n \) are determined by \( Q_{a,\lambda}(-2\Re f_\theta(\cdot)\theta;0) \).

To minimize notation, from now onwards we will suppress the \( \theta \) in \( \tilde{Q}_{a,\lambda}(g;\theta) \), using instead \( \tilde{Q}_{a,\lambda}(g;r) \) (and only give the full form when necessary).

### 3 Sampling properties of \( \tilde{Q}_{a,\lambda}(g;r) \)

In this section we show that under certain conditions \( \tilde{Q}_{a,\lambda}(g;r) \) is a consistent estimator of \( I(g;\frac{a}{\lambda}) \) (or some multiple of it), where \( I(g;\frac{a}{\lambda}) \) is defined in (5). We note that \( I\left( g;\frac{a}{\lambda}\right) \rightarrow I(g;\infty) \) if \( a/\lambda \rightarrow \infty \) as \( a \rightarrow \infty \) and \( \lambda \rightarrow \infty \).
3.1 The mean of $\tilde{Q}_{a,\lambda}(g; r)$

We start with the expectation of $\tilde{Q}_{a,\lambda}(g; r)$. As we mentioned earlier, it is unclear how to choose the number of frequencies $a$ in the definition of $\tilde{Q}_{a,\lambda}(g; r)$. We show if $\sup_{\omega \in \mathbb{R}^d} |g(\omega)| < \infty$, the choice of $a$ does not play a significant role in the asymptotic properties of $\tilde{Q}_{a,\lambda}(g; r)$. To show this result, we note that by not constraining $a$, such that $a = O(\lambda)$, then the number of terms in the sum $Q_{a,\lambda}(g; r)$ grows at a faster rate than the standardisation $\lambda^d$. In this case, the analysis of $\tilde{Q}_{a,\lambda}(g; r)$ requires more delicate techniques than those used to prove Theorems 2.1 and 2.2. We start by stating some pertinent features in the analysis of $\tilde{Q}_{a,\lambda}(g; r)$, which gives a flavour of our approach. By writing $\tilde{Q}_{a,\lambda}(g; r)$ as a quadratic form it is straightforward to show that

$$E[Q_{a,\lambda}(g; r)] = c_2 \sum_{k=-a}^{a} g(\omega_k) \frac{1}{\lambda^d} \int_{[-\lambda/2,\lambda/2]^d} c(s_1 - s_2) \exp(i\omega'_k(s_1 - s_2) - is'_2 \omega_r) h\left(\frac{s_1}{\lambda}\right) h\left(\frac{s_2}{\lambda}\right) ds_1 ds_2,$$

where $c_2 = n(n-1)/n^2$. The proof of Theorems 2.1 and 2.2 are based on making a change of variables $v = s_1 - s_2$ and then systematically changing the limits of the integral. This method works if the frequency grid $[-a/\lambda, a/\lambda]^d$ is fixed for all $\lambda$. However, if the frequency grid $[-a/\lambda, a/\lambda]^d$ is allowed to grow with $\lambda$, applying this brute force method to $E[\tilde{Q}_{a,\lambda}(g; r)]$ has the disadvantage that it aggregates the errors within the sum of $E[\tilde{Q}_{a,\lambda}(g; r)]$. Instead, to further the analysis, we replace $c(s_1 - s_2)$ by its Fourier transform $c(s_1 - s_2) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\omega) \exp(i\omega'(s_1 - s_2)) d\omega$ and focus on the case that the sampling design is uniform; $h(s/\lambda) = \lambda^{-d} \mathbb{1}_{[-\lambda/2,\lambda/2]^d}(s)$ (later we consider general densities). This reduces the first term in $E[\tilde{Q}_{a,\lambda}(g; r)]$ to the Fourier transforms of step functions, which is the product of sinc functions, $\text{Sinc}(\frac{x}{\lambda})$ (noting that the Sinc function is defined in Example 2.2). Specifically, we obtain

$$E\left[\tilde{Q}_{a,\lambda}(g; r)\right] = \frac{c_2}{(2\pi)^d} \sum_{k=-a}^{a} g(\omega_k) \int_{\mathbb{R}^d} f(\omega) \text{Sinc}\left(\frac{\lambda \omega}{2} + k\pi\right) \text{Sinc}\left(\frac{\lambda \omega}{2} + (k + r)\pi\right) d\omega$$

$$= \frac{c_2}{\lambda^d} \int_{\mathbb{R}^d} \text{Sinc}(y) \text{Sinc}(y + r\pi) \left[\frac{1}{\lambda^d} \sum_{k=-a}^{a} g(\omega_k) f\left(\frac{2y}{\lambda} - \omega_k\right)\right] dy,$$

where the last line above is due to a change of variables $y = \frac{-\omega}{\lambda} + k\pi$. Since the spectral density function is absolutely integrable it is clear that $\left[\frac{1}{\lambda^d} \sum_{k=-a}^{a} g(\omega_k) f\left(\frac{2y}{\lambda} - \omega_k\right)\right]$ is uniformly bounded over $y$ and that $E\left[\tilde{Q}_{a,\lambda}(g; r)\right]$ is finite for all $\lambda$. Furthermore, if $f\left(\frac{2y}{\lambda} - \omega_k\right)$ were replaced with $f(-\omega_k)$, then what remains in the integral are two shifted sinc functions, which is zero if $r \in \mathbb{Z}^d/\{0\}$, i.e.

$$E\left[\tilde{Q}_{a,\lambda}(g; r)\right] = \frac{c_2}{\lambda^d} \int_{\mathbb{R}^d} \text{Sinc}(y) \text{Sinc}(y + r\pi) \left[\frac{1}{\lambda^d} \sum_{k=-a}^{a} g(\omega_k) f\left(\frac{2y}{\lambda} - \omega_k\right)\right] dy + R,$$

where

$$R = \frac{c_2}{\lambda^d} \int_{\mathbb{R}^d} \text{Sinc}(y) \text{Sinc}(y + r\pi) \left[\frac{1}{\lambda^d} \sum_{k=-a}^{a} g(\omega_k) \left(f\left(\frac{2y}{\lambda} - \omega_k\right) - f(-\omega_k)\right)\right] dy.$$
In the following theorem we show that under certain conditions on \( f, R \) is asymptotically negligible. Let \( b = b(r) \) denote the number of zero elements in the vector \( r \in \mathbb{Z}^d \).

**Theorem 3.1** Let \( I(g; \cdot) \) be defined as in (5). Suppose Assumptions 2.1(i) and 2.2 hold.

(i) If Assumptions 2.4(i) and 2.5(a,c) hold, then we have

\[
E[\tilde{Q}_{a,\lambda}(g; r)] = \begin{cases} 
O\left(\frac{1}{\lambda \varepsilon} \right) & r \in \mathbb{Z}^d / \{0\} \\
I(g; C) + O\left(\frac{1}{\lambda}\right) & r = 0
\end{cases}
\]  

(ii) Suppose Assumptions 2.4(ii) holds and

(a) Assumption 2.5(b) holds, then \( \sup_{r} \left| E[\tilde{Q}_{a,\lambda}(g; r)] \right| < \infty \).

(b) Assumption 2.5(b,c) holds and \( \{m_1, \ldots, m_{d-b}\} \) is the subset of non-zero values in \( r = (r_1, \ldots, r_d) \), then we have

\[
E[\tilde{Q}_{a,\lambda}(g; r)] = \begin{cases} 
O\left(\frac{1}{\lambda \varepsilon} \prod_{j=1}^{d-b} (\log \lambda + \log |m_j|)\right) & r \in \mathbb{Z}^d / \{0\} \\
I(g; \frac{\sigma}{\lambda}) + O\left(\frac{\log \lambda}{\lambda} + \frac{1}{n}\right) & r = 0
\end{cases}
\]  

(c) If only Assumption 2.5(b,c) holds, then the \( O\left(\frac{1}{\lambda \varepsilon} \prod_{j=1}^{d-b} (\log \lambda + \log |m_j|)\right) \) term in (b) is replaced with the slower rate \( O\left(\frac{1}{\lambda} (\log \lambda + \log \|r\|_1)\right) \).

Note that the above bounds for (b) and (c) are uniform in \( a \).

**PROOF** See Appendix A.1. \( \square \)

We observe that if \( r \neq 0 \), then \( \tilde{Q}_{a,\lambda}(g; r) \) is estimating zero. It would appear that these terms don’t contain any useful information, however in Section 5 we show how these terms can be used to estimate nuisance parameters.

In order to analyze \( E[\tilde{Q}_{a,\lambda}(g; r)] \) in the case that the locations are not from a uniform distribution we return to (12) and replace \( c(s_1 - s_2) \) by its Fourier representation and also replace the sampling density \( h(s/\lambda) \) by its Fourier representation \( h(s/\lambda) = \sum_{j \in \mathbb{Z}^d} \gamma_j \exp(i2\pi j' s/\lambda) \) to give

\[
E[\tilde{Q}_{a,\lambda}(g; r)] = \frac{c_2}{\pi^d} \sum_{j_1, j_2 \in \mathbb{Z}} \gamma_{j_1} \gamma_{j_2} \frac{1}{\lambda^d} \sum_{k=-a}^{a} g(\omega_k) \int_{-\infty}^{\infty} f\left(\frac{2y}{\lambda} - \omega_k\right) \text{Sinc}(y) \text{Sinc}(y + (r - j_1 - j_2)\pi) dy.
\]

This representation allows us to use similar techniques to those used in the uniform sampling case to prove the following result.

**Theorem 3.2** Let \( I(g; \cdot) \) be defined as in (5). Suppose Assumptions 2.1(i) and 2.3 hold.

(i) If in addition Assumptions 2.4(i) and 2.5(a,c) hold, then we have

\[
E[\tilde{Q}_{a,\lambda}(g; r)] = \langle \gamma, \gamma r \rangle I\left( g; \frac{\alpha}{\lambda} \right) + O(\lambda^{-1}),
\]

\( O(\lambda^{-1}) \) is uniform over \( r \in \mathbb{Z}^d \).
Theorem 3.3 We observe that by applying Theorem 3.2 to the case that $h$ is uniform (using that $\gamma_0 = 1$ else $\gamma_j = 0$) gives $E[\bar{Q}_{a,\lambda}(g; r)] = O(\lambda^{-1})$ for $r \neq 0$. Hence, in the case that the sampling is uniform, Theorems 3.1 and 3.2 give similar results, though the bounds in Theorem 3.1 are sharper.

Remark 3.1 (Estimation of $\sum_{j \in \mathbb{Z}^d} |\gamma_j|^2$) The above lemma implies that $E[\bar{Q}_{a,\lambda}(g; 0)] = \langle \gamma, \gamma_0 \rangle I \left( g; \frac{a}{\lambda} \right)$. Therefore, to estimate $I \left( g; \frac{a}{\lambda} \right)$ we require an estimator of $\langle \gamma, \gamma_0 \rangle$. To do this, we recall that

$$
\langle \gamma, \gamma_0 \rangle = \sum_{j \in \mathbb{Z}^d} |\gamma_j|^2 = \frac{1}{\lambda^d} \int_{[-\lambda/2, \lambda/2]^d} h_\lambda(\omega)^2 d\omega.
$$

Therefore one method for estimating the above integral is to define a grid on $[-\lambda/2, \lambda/2]^d$ and estimate $h_\lambda$ at each point, then to take the average squared over the grid (see Remark 1, Matsuda and Yajima (2009)). An alternative, computationally simpler method, is to use the method proposed in Gine and Nickl (2008), that is

$$
\hat{\langle \gamma, \gamma_0 \rangle} = \frac{2}{n(n-1)b} \sum_{1 \leq j_1 < j_2 \leq n} K \left( \frac{s_{j_1} - s_{j_2}}{b} \right)^2,
$$

as an estimator of $\langle \gamma, \gamma_0 \rangle$, where $K : [-1/2, 1/2]^d \to \mathbb{R}$ is a kernel function. Note that multiplying the above kernel with $\exp(-i\omega' s_{j_2})$ results in an estimator of $\langle \gamma, \gamma_r \rangle$. In the case $d = 1$ and under certain regularity conditions, Gine and Nickl (2008) show if the bandwidth $b$ is selected in an appropriate way then $\langle \gamma, \gamma_0 \rangle$ attains the classical $O(n^{-1/2})$ rate under suitable regularity conditions (see, also, Bickel and Ritov (1988) and Laurent (1996)). It seems plausible a similar result holds for $d > 1$ (though we do not prove it here). Therefore, an estimator of $I \left( g; \frac{a}{\lambda} \right)$ is $\bar{Q}_{a,\lambda}(g; r) / \hat{\langle \gamma, \gamma_0 \rangle}$.

3.2 The covariance and asymptotic normality

In the previous section we showed that the expectation of $\bar{Q}_{a,\lambda}(g; r)$ depends only the number of frequencies $a$ through the limit of the integral $I(g; \frac{a}{\lambda})$ (if $\sup_{\omega \in \mathbb{R}^d} |g(\omega)| < \infty$). In this section, we show that $a$ plays a mild role in the higher order properties of $\bar{Q}_{a,\lambda}(g; r)$. We focus on the case that the random field is Gaussian and later describe how the results differ in the case that the random field is non-Gaussian.

Theorem 3.3 Suppose Assumptions 2.1, 2.3 and

(ii) If in addition Assumptions 2.4(ii) and 2.5(b,c) hold, then we have

$$
E[\bar{Q}_{a,\lambda}(g; r)] = \langle \gamma, \gamma_r \rangle I \left( g; \frac{a}{\lambda} \right) + O \left( \frac{\log \lambda + I(r \neq 0) \log \| r \|_1}{\lambda} \right).
$$

PROOF See Appendix B in (Subba Rao, 2015b).

$$
\lambda^d \text{cov} \left[ \bar{Q}_{a,\lambda}(g; r_1), \bar{Q}_{a,\lambda}(g; r_2) \right] = U_1(r_1, r_2; \omega_{r_1}, \omega_{r_2}) + O \left( \frac{1}{\lambda^4} \right).
$$

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and
\[
\lambda^d \text{cov} \left[ \overline{Q_{a, \lambda}(g; r_1)}, \overline{Q_{a, \lambda}(g; r_2)} \right] = U_2(r_1, r_2; \omega_{r_1}, \omega_{r_2}) + O \left( \frac{1}{\lambda} + \frac{\lambda^d}{n} \right)
\]

(ii) Assumption 2.4(ii) and Assumption 2.5(b) hold. Then we have
\[
\lambda^d \sup_{a, r_1, r_2} \left| \text{cov} \left[ \overline{Q_{a, \lambda}(g; r_1)}, \overline{Q_{a, \lambda}(g; r_2)} \right] \right| < \infty \quad \text{and} \quad \lambda^d \sup_{a, r_1, r_2} \left| \text{cov} \left[ \overline{Q_{a, \lambda}(g; r_1)}, \overline{Q_{a, \lambda}(g; r_2)} \right] \right| < \infty,
\]
if \( \lambda^d / n \to c \) (where \( 0 \leq c < \infty \)) as \( \lambda \to \infty \) and \( n \to \infty \).

(iii) Assumption 2.4(ii) and 2.5(b,c) hold. Then uniformly for all \( 0 \leq ||r_1||_1, ||r_2||_1 \leq C|a| \) (for some finite constant \( C \)) we have
\[
\lambda^d \text{cov} \left[ \overline{Q_{a, \lambda}(g; r_1)}, \overline{Q_{a, \lambda}(g; r_2)} \right] = U_1(r_1, r_2; \omega_{r_1}, \omega_{r_2}) + O(\ell_{\lambda, a, n})
\]
\[
\lambda^d \text{cov} \left[ \overline{Q_{a, \lambda}(g; r_1)}, \overline{Q_{a, \lambda}(g; r_2)} \right] = U_2(r_1, r_2; \omega_{r_1}, \omega_{r_2}) + O(\ell_{\lambda, a, n}),
\]
where
\[
\ell_{\lambda, a, n} = \log^2(a) \left[ \frac{\log a + \log \lambda}{\lambda} \right] + \frac{\lambda^d}{n}.
\]
Note, if we drop the restriction on \( r_1 \) and \( r_2 \) and simply let \( r_1, r_2 \in \mathbb{Z}^d \) then the bound for \( \ell_{\lambda, a, n} \) needs to include the additional term \( \log^2(a)(\log ||r_1||_1 + \log ||r_2||_1)/\lambda \).

\( U_1(\cdot) \) and \( U_2(\cdot) \) are defined as
\[
U_1(r_1, r_2; \omega_{r_1}, \omega_{r_2}) = U_{1,1}(r_1, r_2; \omega_{r_1}, \omega_{r_2}) + U_{1,2}(r_1, r_2; \omega_{r_1}, \omega_{r_2})
\]
\[
U_2(r_1, r_2; \omega_{r_1}, \omega_{r_2}) = U_{2,1}(r_1, r_2; \omega_{r_1}, \omega_{r_2}) + U_{2,2}(r_1, r_2; \omega_{r_1}, \omega_{r_2})
\]
with
\[
U_{1,1}(r_1, r_2; \omega_{r_1}, \omega_{r_2}) = \frac{1}{2\pi} \sum_{j_1 + \ldots + j_4 = r_1 - r_2} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \int_{D_{j_1 + j_3}} g(\omega) g(\omega + \omega_{j_1 + j_3}) f(\omega + \omega_{j_1}) f(\omega + \omega_{r_1 - j_2}) d\omega
\]
\[
U_{1,2}(r_1, r_2; \omega_{r_1}, \omega_{r_2}) = \frac{1}{2\pi} \sum_{j_1 + j_2 + j_3 + j_4 = r_1 - r_2} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \int_{D_{j_1 + j_3}} g(\omega) g(-\omega - \omega_{j_2 - j_3 - j_4}) f(\omega + \omega_{j_1}) f(\omega + \omega_{r_1 - j_2}) d\omega
\]
\[
U_{2,1}(r_1, r_2; \omega_{r_1}, \omega_{r_2}) = \frac{1}{2\pi^3} \sum_{j_1 + j_2 + j_3 + j_4 = r_1 + r_2} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \int_{D_{j_1 + j_3}} g(\omega) g(-\omega_{j_1 - j_3} - \omega) f(\omega + \omega_{j_1}) f(\omega + \omega_{r_1 - j_2}) d\omega
\]
\[
U_{2,2}(r_1, r_2; \omega_{r_1}, \omega_{r_2}) = \frac{1}{2\pi} \sum_{j_1 + j_2 + j_3 + j_4 = r_1 + r_2} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \int_{D_{j_1 + j_3}} g(\omega) g(\omega - \omega_{j_2 + j_3 - r_1}) f(\omega + \omega_{j_1}) f(\omega + \omega_{r_1 - j_2}) d\omega,
\]
where the integral is defined as
\[
\int_{D_r} = \int_{2\pi \min(a, a-r_1)/\lambda}^{2\pi \min(a, a-r_1)/\lambda} \cdots \int_{2\pi \min(a, a-r_4)/\lambda}^{2\pi \min(a, a-r_4)/\lambda}.
\]

We now briefly discuss the above results. From Theorem 3.3(iii) we see that \( \tilde{Q}_{a,\lambda}(g; r) \) is a mean squared consistent estimator of \( (\gamma, \gamma_r)I(g; a/\lambda) \), i.e. \( \text{E}[\tilde{Q}_{a,\lambda}(g; r) - (\gamma, \gamma_r)I(g; \frac{a}{\lambda})]^2 = O(\lambda^{-d} + (\log \lambda + \frac{1}{\lambda})^2) \) as \( a \to \infty \) and \( \lambda \to \infty \). Indeed, if \( \sup_{\omega \in \mathbb{R}^d} |g(\omega)| < \infty \), then the rate that \( a/\lambda \to \infty \) plays no role.

However, in order to obtain an explicit expression for the variance additional conditions are required. In the case that the frequency grid is bounded we obtain an expression for the variance of \( \tilde{Q}_{a,\lambda}(g; r) \). On the other hand, we observe (see Theorem 3.3(iii)) that if the frequency grid is unbounded we require some additional conditions on the spectral density function and some mild constraints on the rate of grow of the frequency domain \( a \). More precisely, \( a \) should be such that \( a = O(\lambda^k) \) for some \( 1 \leq k < \infty \). If these conditions are fulfilled, then the asymptotic variance of \( \tilde{Q}_{a,\lambda}(g; r) \) (up to the limits of an integral) is the equivalent for both the bounded and unbounded frequency grid.

By comparing Theorem 3.3(i) and (ii) we observe that in the case the frequency grid is bounded, the same result can be proved Assumption 2.5(b,c) rather than 2.5(a,c), however the error bounds would be slightly larger.

The expressions above are rather unwieldy, however, some simplifications can be made if \( \|r_1\|_1 << \lambda \) and \( \|r_2\|_1 << \lambda \).

Corollary 3.1 Suppose Assumptions 2.3, 2.4 and 2.5(a,c) or 2.5(b,c) hold. Then we have

\[
U_1(r_1, r_2; \omega_{r_1}, \omega_{r_2}) = \Gamma_{r_1-r_2} C_1 + O\left(\frac{\|r_1\|_1 + \|r_2\|_1 + 1}{\lambda}\right)
\]

\[
U_2(r_1, r_2; \omega_{r_1}, \omega_{r_2}) = \Gamma_{r_1+r_2} C_2 + O\left(\frac{\|r_1\|_1 + \|r_2\|_1 + 1}{\lambda}\right)
\]

where \( \Gamma_r = \sum_{j_1 + j_2 + j_3 + j_4 = r} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \) and

\[
C_1 = \frac{1}{(2\pi)^d} \int_{2\pi[-\lambda/\lambda,\lambda/\lambda]^d} f(\omega)^2 \left[ |g(\omega)|^2 + g(\omega)g(-\omega) \right] d\omega
\]

\[
C_2 = \frac{1}{(2\pi)^d} \int_{2\pi[-\lambda/\lambda,\lambda/\lambda]^d} f(\omega)^2 \left[ |g(\omega)g(-\omega) + g(\omega)g(\omega)| \right] d\omega.
\]

Using the above expressions for \( \lambda^d \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] \) and \( \lambda^d \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] \) the variance of the real and imaginary parts of \( \tilde{Q}_{a,\lambda}(g; r) \) can easily be deduced.

In the following theorem we derive bounds for the cumulants of \( \tilde{Q}_{a,\lambda}(g; r) \), which are subsequently used to show asymptotical normality of \( \tilde{Q}_{a,\lambda}(g; r) \).

Theorem 3.4 Suppose Assumptions 2.1, 2.3, 2.4 and 2.5(b) hold. Then for all \( q \geq 3 \) and uniform in \( r_1, \ldots, r_q \in \mathbb{Z}^d \) we have

\[
\text{cum}_q\left[ \tilde{Q}_{a,\lambda}(g, r_1), \ldots, \tilde{Q}_{a,\lambda}(g, r_q) \right] = O\left( \frac{\log^{2d(q-2)}(a)}{\lambda^{d(q-1)}} \right)
\]
if \( \frac{\lambda^d}{n \log^2(a)} \rightarrow 0 \) as \( n \rightarrow \infty \), \( a \rightarrow \infty \) and \( \lambda \rightarrow \infty \).

**PROOF** See Appendix B in Subba Rao (2015b).

From the above theorem we see that if \( \frac{\lambda^d}{n \log^2(a)} \rightarrow 0 \) and \( \log^2(a)/\lambda^{1/2} \rightarrow 0 \) as \( \lambda \rightarrow \infty \), \( n \rightarrow \infty \) and \( a \rightarrow \infty \), then we have \( \lambda^{d/2} \text{cum}_{q}(Q_{a,\lambda}(g, r_1), \ldots, Q_{a,\lambda}(g, r_q)) \rightarrow 0 \) for all \( q \geq 3 \). Using this result we show asymptotic Gaussianity of \( Q_{a,\lambda}(g, r) \).

**Theorem 3.5** Suppose Assumptions 2.1, 2.3, 2.4 and 2.5(b,c) hold. Let \( C_1 \) and \( C_2 \), be defined as in Corollary 3.1. Under these conditions we have

\[
\lambda^{d/2} \Delta^{-1/2} \begin{pmatrix}
\Re \left( \tilde{Q}_{a,\lambda}(g, r_1) - \langle \gamma; \gamma-r \rangle I(g; \frac{a}{\lambda}) \right) \\
\Im \left( \tilde{Q}_{a,\lambda}(g, r_1) - \langle \gamma; \gamma-r \rangle I(g; \frac{a}{\lambda}) \right)
\end{pmatrix} \overset{d}{\rightarrow} \mathcal{N}(0, I_2),
\]

where \( \Re X \) and \( \Im X \) denote the real and imaginary parts of the random variable \( X \), and

\[
\Delta = \frac{1}{2} \begin{pmatrix}
\Re (\Gamma_0 C_1 + \Gamma_2 r C_2) & -\Im (\Gamma_0 C_1 + \Gamma_2 r C_2) \\
-\Im (\Gamma_0 C_1 + \Gamma_2 r C_2) & \Re (\Gamma_0 C_1 - \Gamma_2 r C_2)
\end{pmatrix}
\]

with \( \frac{\log^2(a)}{\lambda^{1/2}} \rightarrow 0 \) as \( \lambda \rightarrow \infty \), \( n \rightarrow \infty \) and \( a \rightarrow \infty \).

**PROOF** See Appendix D, Subba Rao (2015b).

It is likely that the above result also holds when the assumption of Gaussianity of the spatial random field is relaxed and replaced with the conditions stated in Theorem 3.6 (below) together with some mixing-type assumptions. We leave this for future work. However, in the following theorem, we obtain an expression for the variance of \( \tilde{Q}_{a,\lambda}(g, r) \) for non-Gaussian random fields. We make the assumption that \( \{Z(s); s \in \mathbb{R}^d\} \) is fourth order stationary, in the sense that \( E[Z(s)] = \mu \), \( \text{cov}[Z(s_1), Z(s_2)] = c(s_1 - s_2) \), \( \text{cum}[Z(s_1), Z(s_2), Z(s_3)] = \kappa_2(s_1 - s_2, s_1 - s_3) \) and \( \text{cum}[Z(s_1), Z(s_2), Z(s_3), Z(s_4)] = \kappa_4(s_1 - s_2, s_1 - s_3, s_1 - s_4) \), for some functions \( \kappa_2(\cdot) \) and \( \kappa_4(\cdot) \) and all \( s_1, \ldots, s_4 \in \mathbb{R}^d \).

**Theorem 3.6** Let us suppose that \( \{Z(s); s \in \mathbb{R}^d\} \) is a fourth order stationary spatial random field that satisfies Assumption 2.1(i). Furthermore, we suppose that Assumptions 2.3, 2.4 and 2.5(a,c) or 2.5(b,c) are satisfied. Let \( f_4(\omega_1, \omega_2, \omega_3) = \int_{\mathbb{R}^d} \kappa_4(s_1, s_2, s_3) \exp(-i \sum_{j=1}^3 s_j^4 \omega_j) d\omega_1 d\omega_2 d\omega_3 \). We assume that for some \( \delta > 0 \) the spatial tri-spectral density function is such that \( |f_4(\omega_1, \omega_2, \omega_3)| \leq \beta_3(\omega_1) \beta_3(\omega_2) \beta_3(\omega_3) \) and \( \frac{\partial f_4(\omega_1, \omega_2, \omega_3)}{\partial \omega_j} \leq \beta_3(\omega_1) \beta_3(\omega_2) \beta_3(\omega_3) \) (where \( \beta_3(\cdot) \) is defined in (6)).

If \( \|r_1\|, \|r_2\| < < \lambda \), then we have

\[
\lambda^d \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] = \Gamma_{r_1-r_2}(C_1 + D_1) + O \left( \ell_{a,n} + \frac{(a \lambda)^d}{n^2} + \frac{\|r_1\| + \|r_2\|}{\lambda} \right),
\]

and

\[
\lambda^d \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] = \Gamma_{r_1+r_2}(C_2 + D_2) + O \left( \ell_{a,n} + \frac{(a \lambda)^d}{n^2} + \frac{\|r_1\| + \|r_1\|}{\lambda} \right),
\]
where $C_1$ and $C_2$ are defined as in Corollary 3.1 and

$$D_1 = \frac{1}{(2\pi)^{2d}} \int_{2\pi[-a/\lambda, a/\lambda]^{2d}} g(\omega_1) g(\omega_2) f_4(-\omega_1, -\omega_2, \omega_2) d\omega_1 d\omega_2$$

$$D_2 = \frac{1}{(2\pi)^{2d}} \int_{2\pi[-a/\lambda, a/\lambda]^{2d}} g(\omega_1) g(\omega_2) f_4(-\omega_1, -\omega_2, \omega_2) d\omega_1 d\omega_2.$$

Note that we can drop the restriction that $\|r\|_1, \|r\|_2 <\lambda$ and let $r_1, r_2 \in \mathbb{Z}^d$, under this more general assumption we obtain expressions that are similar to those for $U_1$ and $U_2$ (given in Theorem 3.3) and we need to include $\log^2(a)\|r\|_1 + \|r_2\|_1/\lambda$ in the bounds.

**PROOF** See Appendix B in Subba Rao (2015b).

We observe that to ensure the term $\frac{(a\lambda)^d}{n^2} \to 0$ we need to choose $a$ such that $a^d = o(n^2/\lambda^d)$. We recall that for Gaussian random fields $a = O(\lambda^k)$ for some $1 \leq k < \infty$ was sufficient for obtaining an expression for the variance and asymptotic normality. However, in contrast, in the case that the spatial random field is non-Gaussian stronger assumptions are required on $a$ (indeed if $\frac{(a\lambda)^d}{n^2} \to \infty$ the variance $\lambda^d \text{var}[\tilde{Q}_{a,\lambda}(g; r)]$ is not bounded).

### 3.3 Alternative frequency grids and asymptotics

In this subsection we discuss two issues related to the estimator $\tilde{Q}_{a,\lambda}(g; r)$.

#### 3.3.1 Frequency grids

In a recent paper, Bandyopadhyay et al. (2015) define the spatial empirical likelihood within the frequency domain. They use the spatial ‘periodogram’ $\{|J_n(\omega_{a,k})|^2; k \in \mathbb{Z}^d \text{ and } k \in C[-\lambda^\beta, \lambda^\beta]^d\}$ with $\omega_{a,k} = \frac{2\pi \lambda^\beta k}{\lambda}$ as the building blocks of the empirical likelihood. The way in which they define their frequency grid is different to the frequency grid defined in this paper. More precisely, whereas $\{\omega_k = \frac{2\pi \lambda^\beta k}{\lambda}; k = (k_1, \ldots, k_d), -a \leq k_j \leq a\}$ is used to construct $\tilde{Q}_{a,\lambda}(g; r)$, Bandyopadhyay et al. (2015) use the frequency grid $\{\omega_{a,k} = \frac{2\pi \lambda^\beta k}{\lambda}; k = (k_1, \ldots, k_d), -\lambda^\beta \leq k_j \leq \lambda^\beta\}$ where $0 < (1-\eta) < \alpha < 1$. In other words, by using $\omega_{a,k}$ to define their frequency grid they ensure that the frequencies are sufficiently far apart to induce the near uncorrelatedness (indeed near independence) of the $\{|J_n(\omega_{a,k})|^2; k \in \mathbb{Z}^d \text{ and } k \in C[-\lambda^\beta, \lambda^\beta]^d\}$ even in the case that the distribution of the locations in not uniform (see the discussion below Theorem 2.2). This construction is crucial in ensuring that the statistics used to define the empirical likelihood are asymptotically pivotal.

In this section we consider the sampling properties of $\tilde{Q}_{a,\lambda}(g; r)$ under the $\{\omega_{a,k} = \frac{2\pi \lambda^\beta k}{\lambda}; k = (k_1, \ldots, k_d), -a \leq k_j \leq a\}$ frequency grid. To do this, we note that under this frequency grid $\tilde{Q}$ is defined as

$$\tilde{Q}_{a,\lambda}(g; r) = \frac{1}{\lambda^d(1-\alpha)} \sum_{k_1, \ldots, k_d = -a}^a g(\omega_{a,k}) J_n(\omega_{a,k}) J_n(\omega_{a,k+r}) - \frac{\lambda^d \alpha}{n} \sum_{k = -a}^a g(\omega_{a,k}) \frac{1}{n} \sum_{j=1}^n Z(s_j)^2 \exp(-i s_j^T \omega_{a,r}). \quad (19)$$

where we assume $\lambda^\alpha \in \mathbb{Z}^+$. Comparing $\tilde{Q}_{a,\lambda,\alpha}(g; r)$ with $\tilde{Q}_{a,\lambda}(g; r)$ we see that they are the same when we set $\alpha = 0$. The methods developed in this paper can easily be extended to the analysis of
\[ \bar{Q}_{a, \lambda, \alpha}(g; \mathbf{r}). \] More precisely, using the same method discussed in Section 3.1 it can be shown that

\[
E \left[ \bar{Q}_{a, \lambda, \alpha}(g; \mathbf{r}) \right] = \frac{c_2}{\pi^d} \sum_{j_1, j_2 \in \mathbb{Z}} \gamma_{j_1} \gamma_{j_2} \frac{1}{\lambda^{d(1-\alpha)}} \sum_{k=-a}^{a} g(\omega_{a,k}) \int_{-\infty}^{\infty} f \left( \frac{2y}{\lambda} - \omega_{a,k} \right) \text{Sinc}(y) \text{Sinc}(y + \lambda^\alpha (r - j_1 - j_2)\pi) dy.
\]

And by using the same method used to prove Lemma 3.2 we have

\[
E[\bar{Q}_{\lambda, a, \alpha}(g; \mathbf{r})] = \langle \gamma, \gamma - \lambda^\alpha \mathbf{r} \rangle I \left( g; \frac{a}{\lambda^{1-\alpha}} \right) + O \left( \frac{\log \lambda + \log \| \mathbf{r} \|_1}{\lambda} + \frac{1}{\lambda^{1-\alpha}} \right). \tag{20}
\]

Similarly it can be shown that

\[
\lambda^{d(1-\alpha)} \text{cov} \left[ \bar{Q}_{a, \lambda, \alpha}(g; \mathbf{r}_1), \bar{Q}_{a, \lambda, \alpha}(g; \mathbf{r}_2) \right] = \Gamma \lambda^{\alpha(r_1-r_2)} \tilde{C}_1 + O \left( \ell_{\lambda, a, n} + \left[ \frac{(a\lambda)^d}{n^2} + \frac{1}{\lambda^\alpha} \right] I_{S=NG} + \frac{\| \mathbf{r}_1 \|_1 + \| \mathbf{r}_2 \|_1 + 1}{\lambda^{1-\alpha}} \right) \tag{21}
\]

where \( I_{S=NG} \) denotes the indicator variable, such that \( I_{S=NG} = 1 \) if the spatial process \( \{Z(s); s \in \mathbb{R}^d\} \) is non-Gaussian else it is zero, and \( \tilde{C}_1 \) is defined in exactly the same way as \( C_1 \), but the integral \( \int_{2\pi[a/\lambda^{1-\alpha}/\lambda]} f \) replaces \( \int_{2\pi[a/\lambda_{\lambda, a}/\lambda]} f \). Note, when using this grid, for non-Gaussian random fields the the term involving the fourth order cumulant is asymptotically negligible (however, this is not true for the case \( \alpha = 0 \)). Bandyopadhyay et al. (2015), Lemma 7.4, prove the same result for \( \lambda^{d(1-\alpha)} \text{var} \left[ \bar{Q}_{a, \lambda, \alpha}(g; \mathbf{r}) \right] \).

In many respects, the results above are analogous to those in the case that the locations come from the uniform distribution (see Section 3.4, below). More precisely, by using a frequency grid where the frequencies are defined sufficiently far apart \( \bar{Q}_{a, \lambda, \alpha}(g; 0) \) is a consistent estimator of

\[
\langle \gamma, \gamma - 0 \rangle I \left( g; \frac{a}{\lambda^{1-\alpha}} \right)
\]

whereas for \( \mathbf{r} \neq 0, \bar{Q}_{a, \lambda, \alpha}(g; \mathbf{r}) \) is a consistent estimator of zero. Furthermore, \( \{\bar{Q}_{a, \lambda, \alpha}(g; \mathbf{r}); \mathbf{r} \in \mathbb{Z}^d\} \) is a near uncorrelated sequence. However, unlike the case that the locations are uniformly distributed it is unclear how \( \{\bar{Q}_{a, \lambda, \alpha}(g; \mathbf{r}); \mathbf{r} \neq 0\} \) can be exploited to estimate the variance of \( \bar{Q}_{a, \lambda, \alpha}(g; 0) \).

### 3.3.2 Mixed Domain verses Pure Increasing Domain asymptotics

The asymptotics in this paper are done using mixed domain asymptotics, that is, as the domain \( \lambda \to \infty \), the number of locations observed grows at a faster rate than \( \lambda \), in other words \( \lambda^d/n \to 0 \) as \( n \to \infty \). However, as rightly pointed out by a referee, for a given application it may be difficult to disambiguate Mixed Domain (MD) from the Pure Increasing Domain (PID), where \( \lambda^d/n \to c \) \((0 < c < \infty)\) set-up. We briefly discuss how the results change under PID asymptotics and the implications of this. We find that the results point to a rather intriguing difference for spatial processes that are Gaussian and non-Gaussian.
As mentioned above, in the case of PID asymptotic $\lambda^d/n$ is not asymptotically negligible. This means that the covariance $\lambda^d \text{cov}\left[\overline{Q}_{a,\lambda}(g; r_1), \overline{Q}_{a,\lambda}(g; r_2)\right]$ will involve extra terms. These additional terms can be deduced from the proof, but are long so we do not state them here. Interestingly, in the case that the locations are uniformly distributed the procedure described in Section 5 can still be used to estimate the variance without any adjustments to the method (in other words, this method is robust to the asymptotics used).

This brings us to the most curious difference between the MID and PD case. Comparing Theorems 3.3 and 3.6 we observe the following:

- In the case that spatial process is Gaussian, using both MD and PID asymptotics we have $\lambda^d \text{var}[\overline{Q}_{a,\lambda}(g; r)] = O(1)$ (see Theorem 3.3(i)). Furthermore, an asymptotic expression for the variance is

$$\lambda^d \text{var}[\overline{Q}_{a,\lambda}(g; r)] = \Gamma_0[C_1 + E_1] + O \left(\log^2(a) \frac{\log a + \log \lambda}{\lambda} + \frac{\|r\|_1 + 1}{\lambda}\right)$$

where $E_1$ is the additional term that is asymptotically not negligible under PID asymptotics; $E = O(\lambda^d/n)$. From the above we see that if we choose $a$ such that $a = O(\lambda^k)$ for some $1 < k < \infty$ then the frequency grid is unbounded and similar results as those stated in Sections 3.1 and 3.2 hold under PID asymptotics.

- In the case that the process is non-Gaussian, using Theorem 3.6 we have

$$\lambda^d \text{var}[\overline{Q}_{a,\lambda}(g; r)] = \Gamma_0[C_1 + D_1 + E_1 + F_1] + O \left(\log^2(a) \frac{\log a + \log \lambda}{\lambda} + \frac{(a\lambda)^d}{n^2} + \frac{\|r\|_1 + 1}{\lambda}\right),$$

where $E_1 + F_1$ are asymptotically negligible under MD asymptotics but not negligible under PID asymptotics; $E_1 + F_1 = O(\lambda^d/n)$. Now $a$ plays a vital role in the properties of the variance. Indeed from the proof of Theorem 3.6, we can show that if $\frac{(a\lambda)^d}{n^2} \to \infty$ as $a, \lambda, n \to \infty$, then $\lambda^d \text{var}[\overline{Q}_{a,\lambda}(g; r)]$ is not bounded.

In the case of MD asymptotics we choose $a$ such that $(a\lambda)^d/n^2 \to 0$ and $\log^2(a)/\lambda \to \infty$. Under these two conditions the frequency grid can be unbounded and grow at the rate $a/\lambda$ as $\lambda \to \infty$. However, under PID asymptotics (where $\lambda = O(n^{1/d})$) in order to ensure that $(a\lambda)^d/n^2 = O(1)$ we require $a = O(n^{1/d}) = O(\lambda)$. This constrains the frequency grid to be bounded. In other words, in the case that the spatial process is non-Gaussian, in order that $\text{var}[\overline{Q}_{a,\lambda}(g; r)] = O(\lambda^{-d})$, the frequency grid must be bounded.

However, it is interesting to note that it is still possible to have an unbounded frequency grid for non-Gaussian random processes under PID asymptotics if we use the estimator $\overline{Q}_{a,\lambda,\alpha}(g; r)$ (defined in (19) with $0 < \alpha < 1$). In this case let $a = O(\lambda)$ (thus the condition $(a\lambda)^d/n^2 = O(1)$ is satisfied), which gives rise to the frequency grid $\{\omega_{a, k} = \frac{2\pi \lambda^a}{\lambda} k; k = (k_1, \ldots, k_d), -\lambda \leq k_j \leq \lambda\}$, thus the frequency grid grows at the rate $a/\lambda^{1-\alpha} = \lambda^\alpha$. Using this estimator we have $\lambda^d \text{var}[\overline{Q}_{a,\lambda}(g; r)] = O(1)$. We note that these conditions are very similar to those given in Bandyopadhyay et al. (2015) (see condition C.4(i)).
3.4 Uniformly sampled locations

In many situations it is reasonable to suppose that the locations are uniformly distributed over a region. In this case, many of the results stated above can be simplified. These simplifications allow us to develop a simple method for estimating nuisance parameters in Section 5.

We first recall that if the locations are uniformly sampled then \( h_\lambda(s) = \lambda^{-d} I_{[-\lambda/2,\lambda/2]^d}(s) \), therefore the Fourier coefficients of \( h_\lambda(\cdot) \) are simply \( \gamma_0 = 1 \) and \( \gamma_j = 0 \) if \( j \neq 0 \). This implies that \( \Gamma_r \) (defined in Corollary 3.1) is such that \( \Gamma_r = 0 \) if \( r \neq 0 \) and \( \Gamma_0 = 1 \). Therefore if \( \{Z(s); s \in \mathbb{R}^d\} \) is a stationary spatial random process that satisfies the assumptions in Theorem 3.6 (with uniform sampling of the locations) and \( \|r\|_1 \ll \lambda \)

\[
\lambda^d \text{cov} \left[ \Re \widetilde{Q}_{a,\lambda}(g; r_1), \Re \widetilde{Q}_{a,\lambda}(g; r_2) \right] = \begin{cases} 
\frac{1}{2} [C_1 + D_1] & \text{if } r_1 = r_2 (= r) \\
O(\ell_{\lambda,a,n} + (\lambda^2 I_{S=NG})/n^2) & \text{if } r_1 \neq r_2 
\end{cases}
\]

\[
\lambda^d \text{cov} \left[ \Im \widetilde{Q}_{a,\lambda}(g; r_1), \Im \widetilde{Q}_{a,\lambda}(g; r_2) \right] = \begin{cases} 
\frac{1}{2} [C_1 + D_1] & \text{if } r_1 = r_2 (= r) \\
O(\ell_{\lambda,a,n} + (\lambda^2 I_{S=NG})/n^2) & \text{if } r_1 \neq r_2 
\end{cases}
\]

and \( \lambda^d \text{cov} \left[ \Re \widetilde{Q}_{a,\lambda}(g; r_1), \Im \widetilde{Q}_{a,\lambda}(g; r_2) \right] = O(\ell_{\lambda,a,n} + (\lambda^2 I_{S=NG})/n^2) \), where \( I_{S=NG} \) denotes the indicator variable and is one if the random field is Gaussian, else it is zero (see Appendix A.2). From the above we observe that \( \{\Re \widetilde{Q}_{a,\lambda}(g; r), \Im \widetilde{Q}_{a,\lambda}(g; r); r \in \mathbb{Z}^d\} \) is a ‘near uncorrelated’ sequence where in the case \( r \neq 0 \), \( \Re \widetilde{Q}_{a,\lambda}(g; r) \) and \( \Im \widetilde{Q}_{a,\lambda}(g; r) \) are consistent estimators of zero. It is this property we exploit in Section 5 for estimating the variance of \( \widetilde{Q}_{a,\lambda}(g; 0) \). Below we show asymptotic normality in the case of uniformly distributed locations.

**Theorem 3.7** \([\text{CLT on real and imaginary parts}]\) Suppose Assumptions 2.1, 2.2, 2.5(b,c) and 2.4(i) or 2.4(ii) hold. Let \( C_1 \) and \( C_2 \), be defined as in Corollary 3.1. We define the m-dimension complex random vectors \( \widetilde{Q}_m = (\widetilde{Q}_{a,\lambda}(g, r_1), \ldots, \widetilde{Q}_{a,\lambda}(g, r_m)) \), where \( r_1, \ldots, r_m \) are such that \( r_i \neq -r_j \) and \( r_i \neq 0 \). Under these conditions we have

\[
\frac{2\lambda^{d/2}}{C_1} \left( \frac{C_1}{C_1 + \Re C_2} \Re \widetilde{Q}_{a,\lambda}(g, 0), \Re \widetilde{Q}_m, \Im \widetilde{Q}_m \right) \xrightarrow{p} \mathcal{N}(0, I_{2m+1})
\]

with \( \frac{\log^2(a)}{\lambda^{d/2}} \to 0 \) as \( \lambda \to \infty \), \( n \to \infty \) and \( a \to \infty \).

**PROOF** See Appendix D, Subba Rao (2015b).

\[\square\]

4 Statistical inference

In this section we apply the results derived in the previous section to some of the examples discussed in Section 2.2.

We will assume the locations are uniformly distributed and the random field is Gaussian and Assumption 2.5(b,c) is satisfied. All the statistics considered below are bias corrected and based on \( \widetilde{Q}_{a,\lambda}(g; 0) \). Since we make a bias correction, let \( \hat{\sigma}_n^2 = \frac{1}{n} \sum_{j=1}^{n} Z(s_j)^2 \).

In Appendix F, Subba Rao (2015b), we state the results for the non-biased corrected statistics.
4.1 The Whittle likelihood

We first consider the Whittle likelihood described in Example 2.1(i), but with the ‘bias’ term removed. Let \( \tilde{\theta}_n = \arg \min_{\theta \in \Theta} \tilde{L}_n(\theta) \), where

\[
\tilde{L}_n(\theta) = \frac{1}{\lambda^d} \sum_{k=-C\lambda}^{C\lambda} \left( \log f_\theta(\omega_k) + \frac{|J_n(\omega_k)|^2}{f_\theta(\omega_k)} \right) - \frac{\sigma_n^2}{n} \sum_{k=-C\lambda}^{C\lambda} \frac{1}{f_\theta(\omega_k)}
\]

\[
= \tilde{Q}_{a,\lambda}(f_\theta(\omega_k)^{-1}; 0) + \sum_{k=-C\lambda}^{C\lambda} \log f_\theta(\omega_k)
\]

and \( \Theta \) is a compact parameter space. Let \( \theta_0 = \arg \min_{\theta \in \Theta} L(\theta) \), where

\[
L(\theta) = \frac{1}{(2\pi)^d} \int_{2\pi[-C,C]^d} \left[ \log f_\theta(\omega) + \frac{f(\omega)}{f_\theta(\omega)} \right] d\omega.
\]

We will assume that \( \theta_0 \) is such that for all \( \omega \in \mathbb{R}^d \), \( f(\omega) = f_{\theta_0}(\omega) \), where \( f(\omega) \) denotes the true spectral density of the stationary spatial random process and there does not exist another \( \theta \in \Theta \) such that for some \( \omega \in 2\pi[-C,C]^d \) \( f_{\theta_0}(\omega) = f_\theta(\omega) \). Under the assumption that \( \sup_{\theta \in \Theta} \sup_{\omega \in 2\pi[-C,C]^d} |\nabla_\theta f_\theta(\omega)^{-1}| < \infty \) and \( \sup_{\theta \in \Theta} \sup_{\omega \in 2\pi[-C,C]^d} |\nabla_\theta^3 f_\theta(\omega)^{-1}| < \infty \) and by using Theorem 3.1(i) and (22) we can show equicontinuity in probability of \( \tilde{L}_n(\theta) \) and \( \nabla_\theta^3 \tilde{L}_n(\theta) \). These two conditions imply consistency i.e. \( \tilde{\theta}_n \overset{P}{\to} \theta_0 \) and \( \nabla_\theta^2 \tilde{L}(\tilde{\theta}_n) \overset{P}{\to} V \) for any \( \theta_0 \) that lies between \( \tilde{\theta}_n \) and \( \theta_0 \) and

\[
V = \frac{2}{(2\pi)^d} \int_{2\pi[-C,C]^d} [\nabla_\theta \log f_\theta(\omega)] [\nabla_\theta \log f_\theta(\omega)'] |_{\theta=\theta_0} d\omega.
\]

Using these preliminary results we now derive the limiting distribution of \( \tilde{\theta}_n \).

By making a Taylor expansion of \( \tilde{L}_n \) about \( \theta_0 \) is clear that the asymptotic sampling results are determined by the the first derivative of \( \tilde{L}_n(\theta) \) at the true parameter

\[
\nabla_\theta \tilde{L}_n(\theta_0) = \tilde{Q}_{a,\lambda}(\nabla_\theta f_{\theta_0}(\cdot)^{-1}; 0) + \frac{1}{\lambda^d} \sum_{k=-a}^{a} \frac{1}{f_{\theta_0}(\omega_k)} \nabla_\theta f_{\theta_0}(\omega_k),
\]

where the second term on the right hand side of the above is deterministic and

\[
\tilde{Q}_{a,\lambda}(\nabla_\theta f_{\theta_0}(\cdot)^{-1}; 0) = \frac{1}{\lambda^d} \sum_{k=-a}^{a} \nabla_\theta f_{\theta_0}(\omega_k)^{-1} |J_n(\omega_k)|^2 - \frac{\sigma_n^2}{n} \sum_{k=-a}^{a} \nabla_\theta f_{\theta_0}(\omega_k)^{-1}.
\]

By applying Theorem 3.7 to \( \tilde{Q}_{a,\lambda}(\nabla_\theta f_{\theta_0}(\cdot)^{-1}; 0) \), we have \( \lambda^{d/2} \nabla_\theta \tilde{L}_n(\theta) \overset{P}{\to} \mathcal{N}(0, V) \). Altogether this gives

\[
\lambda^{d/2}(\tilde{\theta}_n - \theta_0) \overset{P}{\to} \mathcal{N}(0, V^{-1}),
\]

with \( \lambda^d/n \to 0 \) and \( \log^2(a)/\lambda^2 \to 0 \) as \( a \to \infty \) and \( \lambda \to \infty \). In the non-Gaussian case the limiting variance will change to \( V^{-1} W V^{-1} \), where an expression for \( W \) can be deduced from Theorem A.2. Note that the asymptotic sampling properties of the discretized Whittle likelihood estimator \( \tilde{\theta}_n \) are identical to the asymptotic sampling properties of the integrated Whittle likelihood estimator derived in Matsuda and Yajima (2009).
4.2 The nonparametric covariance estimator

We now consider the bias corrected version of nonparametric estimator defined in Example 2.2(ii). Let

\[
\tilde{c}_n(v) = T \left( \frac{2v}{\lambda} \right) \tilde{c}_n(v)
\]

where \( \tilde{c}_n(v) = \left( \frac{1}{\lambda^d} \sum_{k=-a}^{a} |J_n(\omega_k)|^2 \exp(\imath v^T \omega_k) - \frac{\gamma_n^2}{n} \sum_{k=-a}^{a} \exp(\imath v^T \omega_k) \right) \),

and \( T \) is the \( d \)-dimensional triangle kernel. It is clear the asymptotic sampling properties of \( \tilde{c}_n(v) \) are determined by \( \tilde{c}_n(v) \). Therefore, we first consider \( \tilde{c}_n(v) \). We observe that \( \tilde{c}_n(v) = \tilde{Q}_{a,\lambda}(e^{i\nu}; 0) \), thus we use the results in Section 3 to derive the asymptotic sampling properties of \( \tilde{c}_n(v) \). By using Theorem 3.1(ii) and Assumption 2.5 we have

\[
E[\tilde{c}_n(v)] = I \left( e^{i\nu}; \frac{a}{\lambda} \right) + O \left( \frac{\log \lambda}{\lambda} \right)
\]

\[
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\omega) \exp(\imath v^T \omega) d\omega + O \left( \frac{\lambda^\delta}{a} + \frac{\log \lambda}{\lambda} \right) = c(v) + O \left( \frac{\lambda^\delta}{a} + \frac{\log \lambda}{\lambda} \right),
\]

(24)

where we use the condition \( f(\omega) \leq \beta_\delta(\omega) \) to replace \( I \left( e^{i\nu}; \frac{a}{\lambda} \right) \) with \( I \left( e^{i\nu}; \infty \right) \). Therefore, if \( \lambda^{d+2}/a^\delta \to 0, \lambda^d/n \to 0 \) as \( a \to \infty \) and \( \lambda \to \infty \), then by using Theorem 3.7 we have

\[
\lambda^{d/2} [\tilde{c}_n(v) - c(v)] \overset{D}{\to} \mathcal{N}(0, \sigma(v)^2)
\]

(25)

where

\[
\sigma(v)^2 = \frac{1}{(2\pi)^d} \int_{2\pi[-\lambda a/\lambda, \lambda a/\lambda]^d} f(\omega)^2 \left[ 1 + \exp(\imath 2v^T \omega) \right] d\omega.
\]

We now derive the sampling properties of \( \tilde{c}_n(v) \). By using properties of the triangle kernel we have

\[
E[\tilde{c}_n(v)] = c(v) + O \left( \frac{\lambda^\delta}{a^\delta} + \frac{\log \lambda}{\lambda} \right) \left( 1 - \frac{\min_{1 \leq i \leq d} |v_i|}{\lambda} \right) + \frac{\max_{1 \leq i \leq d} |v_i|}{\lambda}
\]

and

\[
\lambda^{d/2} \left[ \frac{\tilde{c}_n(v) - T \left( \frac{2v}{\lambda} \right) c(v)}{T \left( \frac{2v}{\lambda} \right) \sigma(v)} \right] \overset{D}{\to} \mathcal{N}(0, 1),
\]

with \( \lambda^{d+2}/a^\delta \to 0, \lambda^d/n \to 0 \) and \( \frac{\log^2(a/\lambda)}{\lambda^{d/2}} \to 0 \) as \( a \to \infty \) and \( \lambda \to \infty \).

We mention that the Fourier transform of the bias corrected nonparametric covariance estimator is \( \int_{\mathbb{R}^d} \tilde{c}_n(v) e^{-i\nu^T \omega} dv = \sum_{k=-a}^{a} \left\{ |J_n(\omega_k)|^2 - \frac{\gamma_n^2}{n} \right\} \text{Sinc}^2[\lambda(\omega_k - \omega)] \). Therefore, when a bias correction is made the corresponding sample covariance may not be a non-negative definite function. The sampling properties of the non-negative definite nonparametric covariance function estimator defined in Example 2.2(ii), can be found in Appendix F, Subba Rao (2015b).
4.3 Parameter estimation using an $L_2$ criterion

In this section we consider the asymptotic sampling properties of the parameter estimator using the $L_2$ criterion. We will assume that there exists a $\theta_0 \in \Theta$, such that for all $\omega \in \mathbb{R}^d$, $f_{\theta_0}(\omega) = f(\omega)$ and there does not exist another $\theta \in \Theta$ such that for all $\omega \in \mathbb{R}^d f_{\theta_0}(\omega) = f_0(\omega)$. In addition, we assume $\sup_{\in \Theta} \int_{\mathbb{R}^d} \|\nabla f(\omega; \theta)\|^2 d\omega < \infty$, $\sup_{\in \Theta} \sup_{\omega \in \mathbb{R}^d} |\nabla f_0(\omega)| < \infty$ and $\sup_{\in \Theta} \sup_{\omega \in \mathbb{R}^d} |\nabla^2 f_0(\omega)| < \infty$.

We consider the bias corrected version of the $L_2$ criterion given in Example 2.2(iii). That is

$$L_n(\theta) = \frac{1}{\lambda^d} \sum_{k_1, \ldots, k_d = -a}^a (|J_n(\omega_k)|^2 - \hat{\sigma}_n^2 - f_0(\omega_k))^2,$$

and let $\hat{\theta}_n = \arg \min_{\in \Theta} L_n(\theta)$ ($\Theta$ is a compact set).

By adapting similar methods to those developed in this paper to quartic forms we can show that pointwise $L_n(\theta) \xrightarrow{P} L(\theta)$, where $L(\theta) = \int_{\mathbb{R}^d} [f(\omega) - f_0(\omega)] d\omega + \int_{\mathbb{R}^d} f(\omega)^2 d\omega$, which is minimised at $\theta_0$. Under the stated conditions we can show uniform convergence of $L_n(\theta)$, which implies $\hat{\theta}_n \xrightarrow{P} \theta_0$. Using this result the sampling properties of $\hat{\theta}_n$ are determined by $\nabla_{\theta} L_n(\theta)$ (given in equation (11)) and in particular its random component $\tilde{Q}_{a, \lambda}(-2\nabla_{\theta} f_0(\cdot); 0)$, where

$$\nabla_{\theta} L_n(\theta) = -\frac{2}{\lambda^d} \sum_{k_1, \ldots, k_d = -a}^a \Re \nabla f_0(\omega_k) \{ |J_n(\omega_k)|^2 - f_0(\omega_k) \} + \frac{2\hat{\sigma}_n^2}{\lambda^d} \sum_{k_1, \ldots, k_d = -a}^a \Re \nabla f_0(\omega_k)$$

$$= \tilde{Q}_{a, \lambda}(-2\nabla_{\theta} f_0(\cdot); 0) + \frac{2}{\lambda^d} \sum_{k_1, \ldots, k_d = -a}^a f_0(\omega_k) \nabla_{\theta} f_0(\omega_k).$$

Using the assumptions stated above and Theorem 3.3 we can show that for any $\theta_0$ between $\theta_0$ and $\hat{\theta}_n$ we have $\nabla_{\theta}^2 L_n(\theta_n) \xrightarrow{P} A$ where

$$A = \frac{2}{(2\pi)^d} \int_{\mathbb{R}^d} [\nabla_{\theta} f(\omega; \theta_0)] [\nabla_{\theta} f(\omega; \theta_0)]' d\omega,$$

as $\lambda^d/n \to 0$ with $\lambda \to \infty$ and $n \to \infty$. Thus $\lambda^{d/2}(\hat{\theta}_n - \theta_0) = A^{-1} \lambda^{d/2} \nabla L_n(\theta_0) + o_p(1)$, and it is clear the asymptotic sampling properties of $\hat{\theta}_n$ are determined $\nabla_{\theta} L_n(\theta_0) = \tilde{Q}_{a, \lambda}(-2\nabla_{\theta} f_0(\cdot); 0) + \frac{1}{\lambda^d} \sum_{k_1, \ldots, k_d = -a}^a \nabla_{\theta} f_0(\omega_k)^2$.

Thus by using Theorem 3.7 we have $\lambda^{d/2} \nabla_{\theta} L_n(\theta_0) \xrightarrow{D} N(0, B)$, where

$$B = \frac{4}{(2\pi)^d} \int_{\mathbb{R}^d} [f(\omega)]^2 [\nabla_{\theta} f_0(\omega)] [\nabla_{\theta} f_0(\omega)]' d\omega.$$

Therefore, by using the above we have

$$\lambda^{d/2}(\hat{\theta}_n - \theta_0) \xrightarrow{P} N(0, A^{-1} BA^{-1})$$

with $\lambda^d/n \to 0$ and $\log^2(a)/\lambda^{d/2} \to 0$ as $a \to \infty$ and $\lambda \to \infty$. 

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5 An estimator of the variance of $\tilde{Q}_{a,\lambda}(g; 0)$

The expression for the variance $\tilde{Q}_{a,\lambda}(g; 0)$ given in the examples above, is rather unwieldy and difficult to estimate directly. For example, in the case that the random field is Gaussian, one can estimate $C_1$ by replacing the integral with the sum $\sum_{k=-a}^{a} \text{ and the spectral density function with the squared bias corrected periodogram } (|J_a(\omega_k)|^2 - \hat{\sigma}_n^2)^2$ (see Bandyopadhyay et al. (2015), Lemma 7.5, though their estimator also holds for non-Gaussian spatial processes when $\tilde{Q}_{a,\lambda,a}$ is defined using the frequency grid described in Section 3.3.1).

In this section we describe an alternative, very simple method for estimating the variance of $\tilde{Q}_{a,\lambda}(g; 0)$ under the assumption the locations are uniformly distributed. For ease of presentation we will assume the spatial random field is Gaussian. However, exactly the same methodology holds when the spatial random fields is non-Gaussian. The methodology is motivated by the method of orthogonal samples for time series proposed in Subba Rao (2015a), where the idea is to define a sample which by construction shares some of the properties of the statistic of interest. In this section we show that $\{\tilde{Q}_{a,\lambda}(g; r); r \neq 0\}$ is an orthogonal sample associated with $\tilde{Q}_{a,\lambda}(g; r)$.

We first focus on estimating the variance of $\tilde{Q}_{a,\lambda}(g; 0)$. By using Theorem 3.1 we have $E[\tilde{Q}_{a,\lambda}(g; 0)] = I (g; \frac{d}{2}) + o(1)$ and

$$
\lambda^d \text{var}[\tilde{Q}_{a,\lambda}(g; 0)] = C_1 + O(\ell_{\lambda,a,n}).
$$

In contrast, we observe that if no elements of the vector $r$ are zero, then by Theorem 3.1 $E[\tilde{Q}_{a,\lambda}(g; r)] = O(\prod_{i=1}^{d} |\log \lambda + \log |r_i||/\lambda^d)$ (slightly slower rates are obtained when $r$ contains some zero). In other words, $\tilde{Q}_{a,\lambda}(g; r)$ is estimating zero. On the other hand, by (22) we observe that

$$
\lambda^d \text{cov} \left[ \Re \tilde{Q}_{a,\lambda}(g; r_1), \Re \tilde{Q}_{a,\lambda}(g; r_2) \right] = \begin{cases} 
\frac{1}{2} C_1 + O \left( \ell_{\lambda,a,n} \right) & r_1 = r_2 = (r_1) \\
O (\ell_{\lambda,a,n}) & r_1 \neq r_2, r_1 \neq -r_2 
\end{cases}.
$$

A similar result holds for $\{\Im \tilde{Q}_{a,\lambda}(g; r)\}$, furthermore we have $\lambda^d \text{cov} \left[ \Re \tilde{Q}_{a,\lambda}(g; r_1), \Im \tilde{Q}_{a,\lambda}(g; r_2) \right] = O(\ell_{\lambda,a,n})$.

In summary, if $||r||$ is not too large, then $\{\Re \tilde{Q}_{a,\lambda}(g; r), \Im \tilde{Q}_{a,\lambda}(g; r)\}$ are ‘near uncorrelated’ random variables whose variance is approximately the same as $Q_{a,\lambda}(g; 0)/\sqrt{2}$. This suggests that we can use $\{\Re \tilde{Q}_{a,\lambda}(g; r), \Im \tilde{Q}_{a,\lambda}(g; r); r \in S\}$ to estimate $\text{var}[\tilde{Q}_{a,\lambda}(g; 0)]$, where the set $S$ is defined as

$$
S = \{\text{r; } ||r|| \leq M, r_1 \neq r_2 \text{ and all elements of } r \text{ are non-zero}\}. \quad (27)
$$

This leads to the following estimator

$$
\tilde{V} = \frac{\lambda^d}{2|S|} \sum_{r \in S} \left( 2|\Re \tilde{Q}_{a,\lambda}(g; r)|^2 + 2|\Im \tilde{Q}_{a,\lambda}(g; r)|^2 \right) = \frac{\lambda^d}{|S|} \sum_{r \in S} |\tilde{Q}_{a,\lambda}(g; r)|^2, \quad (28)
$$

where $|S|$ denotes the cardinality of the set $S$. Note that we specifically select the set $S$ such that no element $r$ contains zero, this is to ensure that $E[\tilde{Q}_{a,\lambda}(g; r)]$ is small and does not bias the variance estimator.

In the following theorem we obtain a mean squared bound for $\tilde{V}$.
Theorem 5.1 Let $\bar{V}$ be defined as in (28), where $S$ is defined in (27). Suppose Assumptions 2.1, 2.2 and 2.5(a,b,c) hold and either Assumption 2.4(i) or (ii) holds. Then we have

$$E\left(\bar{V} - \lambda^d\text{var}[\bar{Q}_{a,\lambda}(g;0)]\right)^2 = O(|S|^{-1} + |M|\lambda^{-1} + \ell_{\lambda,a,n} + \lambda^{-d}\log^d(a))$$

as $\lambda \to \infty$, $a \to \infty$ and $n \to \infty$ (where $\ell_{a,\lambda,n}$ is defined in (35)).

PROOF. See Subba Rao (2015b), Section E. \(\square\)

Thus it follows from the above result that if the set $S$ grows at a rate such that $|M|\lambda^{-1} \to 0$ as $\lambda \to \infty$, then $\bar{V}$ is a mean square consistent estimator of $\lambda^d\text{var}[\bar{Q}_{a,\lambda}(g;0)]$.

We note that the same estimator can be used in the case that the random field is non-Gaussian, in this case to show consistency we need to place some conditions on the higher order cumulants of the spatial random process. However, it is a trickier to relax the assumption that the locations are uniformly distributed. This is because in the case of a non-uniform design $E\left[\bar{Q}_{a,\lambda}(g;r)\right]$ ($r \neq 0$) will not, necessarily, be estimating zero. We note that the same method can be used to estimate the variance of the non-bias corrected statistic $Q_{a,\lambda}(g;0)$.

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A Appendix

A.1 Proof of Theorem 3.1

It is clear from the motivation at the start of Section 3.1 that the sinc function plays an important role in the analysis of $Q_{a,\lambda}(g;r)$ and $\bar{Q}_{a,\lambda}(g;r)$. Therefore we now summarise some of its properties. It is well known that $\frac{1}{\lambda}\int_{\lambda/2}^{3\lambda/2} \exp(ix\omega)dx = \text{sinc}\left(\frac{\omega}{\pi}\right)$ and

$$\int_{-\infty}^{\infty} \text{sinc}(u)du = \pi \quad \text{and} \quad \int_{-\infty}^{\infty} \text{sinc}^2(u)du = \pi.$$ (29)

We next state a well know result that is an important component of the proofs in this paper.

Lemma A.1 [Orthogonality of the sinc function]

$$\int_{-\infty}^{\infty} \text{sinc}(u)\text{sinc}(u+x)du = \pi\text{sinc}(x)$$ (30)

and if $s \in \mathbb{Z}/\{0\}$ then

$$\int_{-\infty}^{\infty} \text{sinc}(u)\text{sinc}(u+s\pi)du = 0.$$ (31)

Therefore, by replacing the summand with the integral we obtain (12).

The above method cannot be used to prove (ii) since \( a/\lambda \to \infty \), this leads to bounds which may not converge. Therefore, as discussed in Section 3.1 we consider an alternative approach. To do this we expand \( \bar{Q}_{a,\lambda}(g; r) \) as a quadratic form to give

\[
\bar{Q}_{a,\lambda}(g; r) = \frac{1}{\eta^4} \sum_{j_1,j_2=1}^{n} \sum_{k=-a}^{a} g(\omega_k) Z(s_{j_1}) Z(s_{j_2}) \exp(i \omega_k (s_{j_1} - s_{j_2})) \exp(-i \omega_k' s_{j_2}).
\]

Taking expectations gives

\[
E[\bar{Q}_{a,\lambda}(g; r)] = c_2 \sum_{k=-a}^{a} g(\omega_k) E \left[ c(s_1 - s_2) \exp(i \omega_k (s_1 - s_2) - i s_2 \omega_r) \right]
\]

where \( c_2 = n(n-1)/2 \). In the case that \( d = 1 \) the above reduces to

\[
E[\bar{Q}_{a,\lambda}(g; r)] = c_2 \sum_{k=-a}^{a} g(\omega_k) E \left[ c(s_1 - s_2) \exp(i \omega_k (s_1 - s_2) - i s_2 \omega_r) \right]
\]

Replacing \( c(s_1 - s_2) \) with the Fourier representation of the covariance function gives

\[
E[\bar{Q}_{a,\lambda}(g; r)] = \frac{c_2}{2\pi} \sum_{k=-a}^{a} g(\omega_k) \int_{-\infty}^{\infty} f(\omega) \sin\left(\frac{\lambda \omega}{2} + k\pi \right) \sin\left(\frac{\lambda \omega}{2} + (k+r)\pi \right) d\omega.
\]

By a change of variables \( y = \lambda \omega/2 + k\pi \) and replacing the sum with an integral we have

\[
E[\bar{Q}_{a,\lambda}(g; r)] = \frac{c_2}{\pi \lambda} \sum_{k=-a}^{a} g(\omega_k) \int_{-\infty}^{\infty} f\left(\frac{2y}{\lambda} - \omega_k\right) \sin(y) \sin(y + r\pi) dy
\]

\[
= \frac{c_2}{\pi} \int_{-\infty}^{\infty} \sin(y) \sin(y + r\pi) \left( \frac{1}{\lambda} \sum_{k=-a}^{a} g(\omega_k) f\left(\frac{2y}{\lambda} - \omega_k\right) \right) dy
\]

\[
= \frac{c_2}{2\pi^2} \int_{-\infty}^{\infty} \sin(y) \sin(y + r\pi) \int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(u) f\left(\frac{2u}{\lambda} - u\right) du dy + O\left(\frac{1}{\lambda}\right),
\]
where the \( O(\lambda^{-1}) \) comes from Lemma E.1(ii) in Subba Rao (2015b). Replacing \( f(\frac{2y}{\lambda} - u) \) with \( f(-u) \) gives

\[
E[\tilde{Q}_{a,\lambda}(g; r)] = \frac{c_2}{2\pi^2} \int_{-\infty}^{\infty} \text{sinc}(y)\text{sinc}(y + r\pi) \int_{-2\pi/a/\lambda}^{2\pi/a/\lambda} g(u)f(-u)du + R_n + O(\lambda^{-1}),
\]

where

\[
R_n = \frac{c_2}{2\pi^2} \int_{-\infty}^{\infty} \text{sinc}(y)\text{sinc}(y + r\pi) \left( \int_{-2\pi/a/\lambda}^{2\pi/a/\lambda} g(u) \left( f(\frac{2y}{\lambda} - u) - f(-u) \right) du \right).
\]

By using Lemma C.2 in Subba Rao (2015b), we have \( |R_n| = O(\frac{\log \lambda + I(r \neq 0) \log |r|}{\lambda}) \). Therefore, by using Lemma A.1, replacing \( c_2 \) with one (which leads to the error \( O(n^{-1}) \)) and (29) we have

\[
E[\tilde{Q}_{a,\lambda}(g; r)] = \frac{c_2}{2\pi^2} \int_{-\infty}^{\infty} \text{sinc}(y)\text{sinc}(y + r\pi) \int_{-2\pi/a/\lambda}^{2\pi/a/\lambda} g(u)f(-u)du + O \left( \frac{1}{\lambda} + \frac{\log \lambda + I(r \neq 0) \log |r|}{\lambda} \right),
\]

which gives (13) in the case \( d = 1 \).

To prove the result for \( d > 1 \), we will only consider the case \( d = 2 \), as it highlights the difference from the \( d = 1 \) case. By substituting the spectral representation into (12) we have

\[
E[\tilde{Q}_{a,\lambda}(g; (r_1, r_2))] = \frac{c_2}{\pi^2 \lambda^2} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}, \omega_{k_2}) \int_{\mathbb{R}^2} f \left( \frac{u_1}{\lambda} - \omega_{k_1}, \frac{u_2}{\lambda} - \omega_{k_2} \right) \text{sinc}(u_1)\text{sinc}(u_1 + r_1\pi)\text{sinc}(u_2)\text{sinc}(u_2 + r_2\pi)du_1du_2.
\]

In the case that either \( r_1 = 0 \) or \( r_2 = 0 \) we can use the same proof given in the case \( d = 1 \) to give

\[
E[\tilde{Q}_{a,\lambda}(g; r)] = \frac{1}{(2\pi)^2 \pi^2 \lambda^2} \int_{[-a/\lambda, a/\lambda]^2} g(\omega_1, \omega_2)f(\omega_1, \omega_2) \int_{\mathbb{R}^d} \text{sinc}(u_1)\text{sinc}(u_1 + r_1\pi)\text{sinc}(u_2)\text{sinc}(u_2 + r_2\pi)du_1du_2 + R_n,
\]

where \( |R_n| = O(\frac{\log \lambda + I(\|r\|_1 \neq 0) \log \|r\|_1}{\lambda} + n^{-1}) \), which gives the desired result. However, in the case that both \( r_1 \neq 0 \) and \( r_2 \neq 0 \), we can use Lemma C.2, equation (49), Subba Rao (2015b) to obtain

\[
E[\tilde{Q}_{a,\lambda}(g; (r_1, r_2))] = \frac{c_2}{\pi^2 \lambda^2} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}, \omega_{k_2}) \int_{\mathbb{R}^d} f \left( \frac{u_1}{\lambda} - \omega_{k_1}, \frac{u_2}{\lambda} - \omega_{k_2} \right) \times
\]

\[
\text{sinc}(u_1)\text{sinc}(u_1 + r_1\pi)\text{sinc}(u_2)\text{sinc}(u_2 + r_2\pi)du_1du_2 = O \left( \prod_{i=1}^{l} \frac{\log \lambda + \log |r_i|}{\lambda^2} \right)
\]

thus we obtain the faster rate of convergence.

It is straightforward to generalize these arguments to \( d > 2 \).
A.2 Variance calculations in the case of uniformly sampled locations

We start by proving the results in Section 3 for the case that the locations are uniformly distributed. The proof here forms the building blocks for the case of non-uniform sampling of the locations, which can found in Section B, Subba Rao (2015b).

In Lemma E.2, Appendix E, Subba Rao (2015b) we show that if the spatial process is Gaussian and the locations are uniformly sampled then for a bounded frequency grid we have

\[
\lambda^d \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] = \begin{cases} 
C_1(\omega_r) + O(\frac{1}{\lambda} + \frac{\lambda^d}{n}) & r_1 = r_2 (= r) \\
O(\frac{1}{\lambda} + \frac{\lambda^d}{n}) & r_1 \neq r_2
\end{cases}
\]

and

\[
\lambda^d \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] = \begin{cases} 
C_2(\omega_r) + O(\frac{1}{\lambda} + \frac{\lambda^d}{n}) & r_1 = -r_2 (= r) \\
O(\frac{1}{\lambda} + \frac{\lambda^d}{n}) & r_1 \neq -r_2
\end{cases}
\]

where

\[
C_1(\omega_r) = C_{1,1}(\omega_r) + C_{1,2}(\omega_r) \quad \text{and} \quad C_2(\omega_r) = C_{2,1}(\omega_r) + C_{2,2}(\omega_r),
\]

with

\[
C_{1,1}(\omega_r) = \frac{1}{(2\pi)^d} \int_{2\pi[-a/\lambda,a/\lambda]^d} f(\omega) f(\omega + \omega_r) |g(\omega)|^2 d\omega,
\]

\[
C_{1,2}(\omega_r) = \frac{1}{(2\pi)^d} \int_{D_r} f(\omega) f(\omega + \omega_r) g(\omega) (-\omega - \omega_r) d\omega,
\]

\[
C_{2,1}(\omega_r) = \frac{1}{(2\pi)^d} \int_{2\pi[-a/\lambda,a/\lambda]^d} f(\omega) f(\omega + \omega_r) g(\omega) (-\omega) d\omega,
\]

\[
C_{2,2}(\omega_r) = \frac{1}{(2\pi)^d} \int_{D_r} f(\omega) f(\omega + \omega_r) g(\omega) g(\omega + \omega_r) d\omega.
\]

On the other other hand, if the frequency grid is unbounded, then under Assumptions 2.4(ii) and 2.5(b) we have

\[
\lambda^d \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] = A_1(\mathbf{r}_1, \mathbf{r}_2) + A_2(\mathbf{r}_1, \mathbf{r}_2) + O(\frac{\lambda^d}{n}),
\]

\[
\lambda^d \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] = A_3(\mathbf{r}_1, \mathbf{r}_2) + A_4(\mathbf{r}_1, \mathbf{r}_2) + O(\frac{\lambda^d}{n}),
\]

where

\[
A_1(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\pi^{2d} \lambda^d} \sum_{m=-2a}^{2a} \sum_{k=\max(-a,-a+m)}^{\min(a,a+m)} \int_{\mathbb{R}^d} f \left( \frac{2u}{\lambda} - \omega_k \right) g \left( \frac{2v}{\lambda} + \omega_k + \omega_{r_1} \right) \Sinc(u - m\pi) \Sinc(v + (m + \mathbf{r}_1 - \mathbf{r}_2)\pi) \Sinc(u) \Sinc(v) dudv
\]
and expressions for $A_2(r_1, r_2), \ldots, A_4(r_1, r_4)$ can be found in Lemma E.2, Subba Rao (2015b). Interestingly $\sup_x |A_1(r_1, r_2)| < \infty$, $\sup_x |A_2(r_1, r_2)| < \infty$, $\sup_x |A_3(r_1, r_2)| < \infty$ and $\sup_x |A_4(r_1, r_2)| < \infty$. Hence if $\lambda^d/n \to c$ ($c < \infty$) we have
\[
\lambda^d \sup_a \left| \text{cov} \left[ Q_{a,\lambda}(g; r_1), \overline{Q}_{a,\lambda}(g; r_2) \right] \right| < \infty \quad \text{and} \quad \lambda^d \sup_a \left| \text{cov} \left[ \overline{Q}_{a,\lambda}(g; r_1), \overline{Q}_{a,\lambda}(g; r_2) \right] \right| < \infty.
\]

We now obtain simplified expressions for the terms $A_1(r_1, r_2), \ldots, A_4(r_1, r_2)$ under the slightly stronger condition that Assumption 2.5(c) also holds.

**Lemma A.2** Suppose Assumptions 2.4(ii) and 2.5(b,c) hold. Then for $0 \leq |r_1|, |r_2| \leq C|a|$ (where $C$ is some finite constant) we have

\[
A_1(r_1, r_2) = \begin{cases} 
C_{1,1}(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = r_2 (= r) \\
O(\ell_{\lambda,a,n}) & r_1 \neq r_2
\end{cases}
\]

\[
A_2(r_1, r_2) = \begin{cases} 
C_{1,2}(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = r_2 (= r) \\
O(\ell_{\lambda,a,n}) & r_1 \neq r_2
\end{cases}
\]

\[
A_3(r_1, r_2) = \begin{cases} 
C_{2,1}(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = -r_2 (= r) \\
O(\ell_{\lambda,a,n}) & r_1 \neq -r_2
\end{cases}
\]

\[
A_4(r_1, r_2) = \begin{cases} 
C_{2,2}(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = -r_2 (= r) \\
O(\ell_{\lambda,a,n}) & r_1 \neq -r_2
\end{cases}
\]

and

\[
A_n(r_1, r_2, \ldots, a, n) = \begin{cases} 
C_{n,n}(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = \ldots = r_n (= r) \\
O(\ell_{\lambda,a,n}) & r_1 \neq \ldots \neq r_n
\end{cases}
\]

where $C_{1,1}(\omega_r), \ldots, C_{2,2}(\omega_r)$ (using $d = 1$) and $\ell_{\lambda,a,n}$ are defined in Lemma E.2.

**PROOF.** We first consider $A_1(r_1, r_2)$ and write it as

\[
A_1(r_1, r_2) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=-2a}^{2a} \text{sinc}(u) \text{sinc}(v) \text{sinc}(u - m\pi) \text{sinc}(v + (m + r_1 - r_2)\pi) H_{m,\lambda} \left( \frac{2u}{\lambda}, \frac{2v}{\lambda}; r_1 \right) dv du,
\]

where

\[
H_{m,\lambda} \left( \frac{2u}{\lambda}, \frac{2v}{\lambda}; r_1 \right) = \frac{1}{\lambda} \sum_{k=\max(-a,-a+m)}^{\min(a,a+m)} f \left( -\frac{2u}{\lambda} + \omega_k \right) f \left( \frac{2v}{\lambda} + \omega_k + \omega_r \right) g(\omega_k) g(\omega_k - \omega_m)
\]

noting that $f \left( \frac{2a}{\lambda} - \omega \right) = f \left( \omega - \frac{2a}{\lambda} \right)$.

If $f \left( \frac{-2a}{\lambda} + \omega_k \right)$ and $f \left( \frac{2a}{\lambda} + \omega_k + \omega_r \right)$ are replaced with $f(\omega_k)$ and $f(\omega_k + \omega_r)$ respectively, then we can exploit the orthogonality property of the sinc functions. This requires the following series of approximations.
(i) We start by defining a similar version of $A_1(r_1, r_2)$ but with the sum replaced with an integral. Let

$$B_1(r_1 - r_2; r_1) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=-2a}^{2a} \text{sinc}(u)\text{sinc}(u - m\pi)\text{sinc}(v)\text{sinc}(v + (m + r_1 - r_2)\pi)H_m\left(\frac{2u}{\lambda}, \frac{2v}{\lambda}; r_1\right) dudv,$$

where

$$H_m\left(\frac{2u}{\lambda}, \frac{2v}{\lambda}; r_1\right) = \frac{1}{2\pi} \int_{2\pi \max(-a,-a+m)/\lambda}^{2\pi \min(a,a+m)/\lambda} f(\omega - \frac{2u}{\lambda})f(\frac{2v}{\lambda} + \omega + \omega_r)g(\omega)\overline{g(\omega - \omega_m)}d\omega.$$

By using Lemma C.4, Subba Rao (2015b) we have

$$|A_1(r_1, r_2) - B_1(r_1 - r_2; r_1)| = O\left(\frac{\log^2 a}{\lambda}\right).$$

(ii) Define the quantity

$$C_1(r_1 - r_2; r_1)$$

$$= \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=-2a}^{2a} \text{sinc}(u)\text{sinc}(u - m\pi)\text{sinc}(v)\text{sinc}(v + (m + r_1 - r_2)\pi)H_m(0,0; r_1)dudv.$$

By using Lemma C.5, Subba Rao (2015b), we can replace $B_1(r_1 - r_2; r_1)$ with $C_1(r_1 - r_2; r_1)$ to give the replacement error

$$|B_1(r_1 - r_2; r_1) - C_1(r_1 - r_2; r_1)| = O\left(\log^2(a)\left[\frac{(\log a + \log \lambda)}{\lambda}\right]\right).$$

(iii) Finally, we analyze $C_1(r_1 - r_2; r_1)$. Since $H_m(0,0; r_1)$ does not depend on $u$ or $v$ we take it out of the integral to give

$$C_1(r_1 - r_2; r_1)$$

$$= \frac{1}{\pi^2} \sum_{m=-2a}^{2a} H_m(0,0; r_1) \left(\int_{\mathbb{R}} \text{sinc}(u)\text{sinc}(u - m\pi)du\right) \left(\int_{\mathbb{R}} \text{sinc}(v)\text{sinc}(v + (m + r_1 - r_2)\pi)dv\right)$$

$$= \frac{1}{\pi^2} H_0(0,0; r_1) \left(\int_{\mathbb{R}} \text{sinc}^2(u)du\right) \left(\int_{\mathbb{R}} \text{sinc}(v)\text{sinc}(v + (r_1 - r_2)\pi)dv\right),$$

where the last line of the above is due to orthogonality of the sinc function (see Lemma A.1). If $r_1 \neq r_2$, then by orthogonality of the sinc function we have $C_1(r_1 - r_2; r_1) = 0$. On the other hand if $r_1 = r_2$ we have

$$C_1(0; r) = \frac{1}{2\pi} \int_{-2\pi a/\lambda}^{2\pi a/\lambda} f(\omega)f(\omega + \omega_r)|g(\omega)|^2d\omega = C_{1,1}(\omega_r).$$

The proof for the remaining terms $A_2(r_1, r_2), A_3(r_1, r_2)$ and $A_4(r_1, r_2)$ is identical, thus we omit the details. \qed
Theorem A.1 Suppose Assumptions 2.1, 2.2, 2.4(ii) and 2.5(b,c) hold. Then for $0 \leq |r_1|, |r_2| \leq C|a|$ (where $C$ is some finite constant) we have

$$
\lambda^d \text{cov} \left[ Q_{a,\lambda}(g; r_1), Q_{a,\lambda}(g; r_2) \right] = \begin{cases} 
C_1(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = r_2 (= r) \\
O(\ell_{\lambda,a,n}) & r_1 \neq r_2 
\end{cases}
$$

$$
\lambda^d \text{cov} \left[ \overline{Q}_{a,\lambda}(g; r_1), \overline{Q}_{a,\lambda}(g; r_2) \right] = \begin{cases} 
C_2(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = -r_2 (= r) \\
O(\ell_{\lambda,a,n}) & r_1 \neq -r_2 
\end{cases},
$$

where $C_1(\omega_r)$ and $C_2(\omega_r)$ are defined in (79) and

$$
\ell_{\lambda,a,n} = \log^2(a) \left[ \log a + \log \lambda \right] + \frac{\lambda^d}{n},
$$

(35)

PROOF. By using Lemmas E.2 and A.2 we immediately obtain (in the case $d = 1$)

$$
\text{cov} \left[ \overline{Q}_{a,\lambda}(g; r_1), \overline{Q}_{a,\lambda}(g; r_2) \right] = \begin{cases} 
C_1(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = r_2 (= r) \\
O(\ell_{\lambda,a,n}) & r_1 \neq r_2
\end{cases}
$$

and

$$
\text{cov} \left[ \overline{Q}_{a,\lambda}(g; r_1), \overline{Q}_{a,\lambda}(g; r_2) \right] = \begin{cases} 
C_2(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = -r_2 (= r) \\
O(\ell_{\lambda,a,n}) & r_1 \neq -r_2
\end{cases}.
$$

This gives the result for $d = 1$. To prove the result for $d > 1$ we use the same procedure outlined in the proof of Lemma A.2 and the above. \(\square\)

The above theorem means that the variance for both the bounded and unbounded frequency grid are equivalent (up to the limits of an integral).

Using the above results and the Lipschitz continuity of $g(\cdot)$ and $f(\cdot)$ we can show that

$$
C_j(\omega_r) = C_j + O \left( \frac{\|r\|_1}{\lambda} \right),
$$

where $C_1$ and $C_2$ are defined in Corollary 3.1.

We now derive an expression for the variance for the non-Gaussian case.

**Theorem A.2** Let us suppose that $\{Z(s); s \in \mathbb{R}^d\}$ is a fourth order stationary spatial random field that satisfies Assumption 2.1(i). Suppose all the assumptions in Theorem 3.6 hold with the exception of Assumption 2.3 which is replaced with Assumption 2.2 (i.e. we assume the locations are uniformly sampled). Then for $\|r\|_1, \|r\|_2 << \lambda$ we have

$$
\lambda^d \text{cov} \left[ \overline{Q}_{a,\lambda}(g; r_1), \overline{Q}_{a,\lambda}(g; r_2) \right] = \begin{cases} 
C_1 + D_1 + O(\ell_{\lambda,a,n} + \frac{(a\lambda)^d}{n^2}) + \frac{\|r\|_1}{\lambda} & r_1 = r_2 (= r) \\
O(\ell_{\lambda,a,n} + \frac{(a\lambda)^d}{n^2}) & r_1 \neq r_2
\end{cases}
$$

(36)

$$
\lambda^d \text{cov} \left[ \overline{Q}_{a,\lambda}(g; r_1), \overline{Q}_{a,\lambda}(g; r_2) \right] = \begin{cases} 
C_2 + D_2 + O(\ell_{\lambda,a,n} + \frac{(a\lambda)^d}{n^2}) + \frac{\|r\|_1}{\lambda} & r_1 = -r_2 (= r) \\
O(\ell_{\lambda,a,n} + \frac{(a\lambda)^d}{n^2}) & r_1 \neq -r_2
\end{cases},
$$

(37)
where $C_1$ and $C_2$ are defined in Corollary 3.1 and

$$D_1 = \frac{1}{(2\pi)^2} \int_{2\pi[-a/\lambda, a/\lambda]^2} g(\omega_1)g(\omega_2) f_4(-\omega_1, -\omega_2, \omega_2) d\omega_1 d\omega_2$$

$$D_2 = \frac{1}{(2\pi)^2} \int_{2\pi[-a/\lambda, a/\lambda]^2} g(\omega_1)g(\omega_2) f_4(-\omega_1, -\omega_2, \omega_2) d\omega_1 d\omega_2.$$

**Proof** We prove the result for the notationally simple case $d = 1$. By using indecomposable partitions, conditional cumulants and (81) in Subba Rao (2015b) we have

$$\lambda \text{cov} \left[ Q_{a, \lambda}(g; r_1), Q_{a, \lambda}(g; r_2) \right] = A_1(r_1, r_2) + A_2(r_1, r_2) + B_1(r_1, r_2) + B_2(r_1, r_2) + O \left( \frac{\lambda}{n} \right), \quad (38)$$

where $A_1(r_1, r_2)$ and $A_2(r_1, r_2)$ are defined below equation (81) and

$$B_1(r_1, r_2) = \lambda c_4 \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1})g(\omega_{k_2}) E \left[ \kappa_4(s_2 - s_1, s_3 - s_1, s_4 - s_1) e^{i\omega_{k_1}} e^{-i\omega_{k_1} + r_1} e^{-i\omega_{k_2} + r_2} \right]$$

$$B_2(r_1, r_2) = \frac{\lambda}{n^4} \sum_{j_1, \ldots, j_4 \in D_3} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1})g(\omega_{k_2}) \times$$

$$E \left[ \kappa_4(s_{j_2} - s_{j_1}, s_{j_3} - s_{j_1}, s_{j_4} - s_{j_1}) e^{i\omega_{k_1}} e^{-i\omega_{k_1} + r_1} e^{-i\omega_{k_2} + r_2} \right]$$

with $c_4 = n(n-1)(n-2)(n-3)/n^2$

$D_3 = \{ j_1, \ldots, j_4; j_1 \neq j_2$ and $j_3 \neq j_4$ but some $j$’s are in common}. The limits of $A_1(r_1, r_2)$ and $A_2(r_1, r_2)$ are given in Lemma A.2, therefore, all that remains is to derive bounds for $B_1(r_1, r_2)$ and $B_2(r_1, r_2)$. We will show that $B_1(r_1, r_2)$ is the dominating term, whereas by placing sufficient conditions on the rate of growth of $a$, we will show that $B_2(r_1, r_2) \rightarrow 0$.

In order to analyze $B_1(r_1, r_2)$ we will use the result

$$\int_{\mathbb{R}^3} \text{sinc}(u_1 + u_2 + u_3 + (r_2 - r_1)\pi) \text{sinc}(u_2) \text{sinc}(u_3) du_1 du_2 du_3 = \begin{cases} \pi^3 & r_1 = r_2 \\ 0 & r_1 \neq r_2. \end{cases}, \quad (39)$$

which follows from Lemma A.1. In the following steps we will make a series of approximations which will allow us to apply (39).

We start by substituting the Fourier representation of the cumulant function

$$\kappa_4(s_1 - s_2, s_1 - s_3, s_1 - s_4) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} f_4(\omega_1, \omega_2, \omega_3) e^{i(s_1 - s_2)\omega_1} e^{i(s_1 - s_3)\omega_2} e^{i(s_1 - s_4)\omega_4} d\omega_1 d\omega_2 d\omega_3,$$

into $B_1(r_1, r_2)$ to give

$$B_1(r_1, r_2) = \frac{c_4}{(2\pi)^3} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1})g(\omega_{k_2}) \int_{\mathbb{R}^3} f_4(\omega_1, \omega_2, \omega_3) \int_{[-\lambda/2, \lambda/2]^3} e^{i\omega_{k_1} + r_1} e^{-i\omega_{k_2} + r_2} d\omega_{k_1}$$

$$e^{-i\omega_{k_1} + r_1} e^{-i\omega_{k_2} + r_2} ds_1 ds_2 ds_3 ds_4 d\omega_1 d\omega_2 d\omega_3$$

$$= \frac{c_4 \lambda}{(2\pi)^3} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1})g(\omega_{k_2}) \int_{\mathbb{R}^3} f_4(\omega_1, \omega_2, \omega_3) \text{sinc} \left( \frac{\lambda_{\omega_{k_1} + \omega_2 + \omega_3 + \omega_{k_1}}}{2} + k_1 \pi \right) \text{sinc} \left( \frac{\lambda_{\omega_{k_2}}}{{2}} + (k_1 + r_1) \pi \right)$$

$$\times \text{sinc} \left( \frac{\lambda_{\omega_{k_2}}}{{2}} + k_2 \pi \right) \text{sinc} \left( \frac{\lambda_{\omega_{k_2}}}{{2}} - (k_2 + r_2) \pi \right) d\omega_1 d\omega_2 d\omega_3.$$
Now we make a change of variables and let \( u_1 = \frac{\omega_1}{2} + (k_1 + r_1), u_2 = \frac{\omega_2}{2} + k_2 \pi \) and \( u_3 = \frac{\omega_3}{2} - (k_2 + r_2) \pi \), this gives

\[
B_1(r_1, r_2) = \frac{c_4}{\pi^3 \lambda^2} \sum_{k_1, k_2 = -a}^{a} \int_{\mathbb{R}^3} g(\omega_1) g(\omega_2) f_4 \left( \frac{2u_1}{\lambda} - \omega_{k_1 + r_1}, \frac{2u_2}{\lambda} - \omega_{k_2}, \frac{2u_3}{\lambda} + \omega_{k_2 + r_2} \right) \times
\]
\[
sinc (u_1 + u_2 + u_3 + (r_2 - r_1) \pi) \text{sinc}(u_1) \text{sinc}(u_2) \text{sinc}(u_3) du_1 du_2 du_3.
\]

Next we exchange the summand with a double integral and use Lemma E.1(iii) together with Lemma C.1, equation (44) (in Subba Rao (2015a)) to obtain

\[
B_1(r_1, r_2) = \frac{c_4}{(2\pi)^2 \lambda^3} \int_{2\pi[-a/a, a/a]^2} \int_{\mathbb{R}^3} g(\omega_1) g(\omega_2) f_4 \left( \frac{2u_1}{\lambda} - \omega_1 - \omega_{r_1}, \frac{2u_2}{\lambda} - \omega_2, \frac{2u_3}{\lambda} + \omega_1 + \omega_2 + \omega_{r_2} \right) \times
\]
\[
sinc (u_1 + u_2 + u_3 + (r_2 - r_1) \pi) \text{sinc}(u_1) \text{sinc}(u_2) \text{sinc}(u_3) du_1 du_2 du_3 \text{d} \omega_1 \text{d} \omega_2 + O \left( \frac{1}{\lambda} \right),
\]

By using Lemma C.3, we replace \( f_4 \left( \frac{2u_1}{\lambda} - \omega_1 - \omega_{r_1}, \frac{2u_2}{\lambda} - \omega_2, \frac{2u_3}{\lambda} + \omega_1 + \omega_2 + \omega_{r_2} \right) \) in the integral with \( f_1(-\omega_1 - \omega_{r_1}, -\omega_2, \omega_2 + \omega_{r_2}) \), this gives

\[
B_1(r_1, r_2) = \frac{c_4}{(2\pi)^2 \lambda^3} \int_{2\pi[-a/a, a/a]^2} \int_{\mathbb{R}^3} g(\omega_1) g(\omega_2) f_4 \left( -\omega_1 - \omega_{r_1}, -\omega_2, \omega_2 + \omega_{r_2} \right) \times
\]
\[
sinc (u_1 + u_2 + u_3 + (r_2 - r_1) \pi) \text{sinc}(u_1) \text{sinc}(u_2) \text{sinc}(u_3) du_1 du_2 du_3 \text{d} \omega_1 \text{d} \omega_2 + O \left( \frac{\log^3 \lambda}{\lambda} \right),
\]

where the last line follows from the orthogonality relation of the sinc function in equation (39).

Finally we make one further approximation. In the case \( r_1 = r_2 \) we replace \( f_4(-\omega_1 - \omega_{r_1}, -\omega_2, \omega_2 + \omega_{r_2}) \) with \( f_4(-\omega_1, -\omega_2, \omega_2) \) which by using the Lipschitz continuity of \( f_4 \) and Lemma C.1, equation (44) gives

\[
B_1(r_1, r_1) = \frac{c_4}{(2\pi)^2} \int_{2\pi[-a/a, a/a]^2} g(\omega_1) g(\omega_2) f_4(-\omega_1, -\omega_2, \omega_2) \text{d} \omega_1 \text{d} \omega_2 + O \left( \frac{\log^3 \lambda}{\lambda} + \frac{|r_1|}{\lambda} \right).
\]

Altogether this gives

\[
B_1(r_1, r_2) = \begin{cases} \frac{1}{(2\pi)^2} \int_{2\pi[-a/a, a/a]^2} g(\omega_1) g(\omega_2) f_4(-\omega_1, -\omega_2, \omega_2) \text{d} \omega_1 \text{d} \omega_2 + O \left( \frac{\log^3 \lambda}{\lambda} + \frac{|r_1| + |r_2|}{\lambda} \right) & r_1 = r_2, \\ O \left( \frac{\log^3 \lambda}{\lambda} \right) & r_1 \neq r_2. \end{cases}
\]

Next we show that \( B_2(r_1, r_2) \) is asymptotically negligible. To bound \( B_2(r_1, r_2) \) we decompose \( \mathcal{D}_3 \) into six sets, the set \( \mathcal{D}_{3,1} = \{ j_1, \ldots, j_4; j_1 = j_3, j_2 \text{ and } j_4 \text{ are different} \} \), \( \mathcal{D}_{3,2} = \{ j_1, \ldots, j_4; j_1 = j_4, j_2 \text{ and } j_3 \text{ are different} \} \), \( \mathcal{D}_{3,3} = \{ j_1, \ldots, j_4; j_2 = j_3, j_1 \text{ and } j_4 \text{ are different} \} \), \( \mathcal{D}_{3,4} = \{ j_1, \ldots, j_4; j_2 = j_4, j_1 \text{ and } j_3 \text{ are different} \} \), \( \mathcal{D}_{2,1} = \{ j_1, \ldots, j_4; j_1 = j_3 \text{ and } j_2 = j_4 \} \), \( \mathcal{D}_{2,2} = \{ j_1, \ldots, j_4; j_1 = j_4 \text{ and } j_2 = j_3 \} \).
\[ j_2 = j_3 \]. Using this decomposition we have \( B_2(r_1, r_2) = \sum_{j=1}^4 B_{2,(3,j)}(r_1, r_2) + \sum_{j=1}^2 B_{2,(2,j)}(r_1, r_2), \]

where

\[
B_{2,(3,1)}(r_1, r_2) = \frac{|D_{3,1}|\lambda}{n^4} \sum_{k_1, k_2=-a}^a g(\omega_{k_1})g(\omega_{k_2}) \times 
\mathbb{E}\left[ \kappa_4(s_{j_2} - s_{j_1}, 0, s_{j_4} - s_{j_3})e^{i\omega_{k_1} + r_1}e^{-i\omega_{k_2} + r_2} \right]
\]

for \( j = 2, 3, 4 \), \( B_{2,(3,j)}(r_1, r_2) \) are defined similarly,

\[
B_{2,(2,1)}(r_1, r_2) = \frac{|D_{2,1}|\lambda}{n^4} \sum_{k_1, k_2=-a}^a g(\omega_{k_1})g(\omega_{k_2}) \times 
\mathbb{E}\left[ \kappa_4(s_{j_2} - s_{j_1}, 0, s_{j_2} - s_{j_3})e^{i\omega_{k_1} + r_1}e^{-i\omega_{k_2} + r_2} \right]
\]

\( B_{2,(2,2)}(r_1, r_2) \) is defined similarly and \( |\cdot| \) denotes the cardinality of a set. By using identical methods to those used to bound \( B_1(r_1, r_2) \) we have

\[
|B_{2,(3,1)}(r_1, r_2)| \leq \frac{C\lambda}{n^4} \sum_{k_1, k_2=-a}^a |g(\omega_{k_1})g(\omega_{k_2})| \int_{\mathbb{R}^3} |f_4(\omega_1, \omega_2, \omega_3)| \left| \text{sinc}\left( \frac{\lambda(\omega_1 + \omega_3)}{2} + (k_2 - k_1)\pi \right) \right| \times \\
|\text{sinc}\left( \frac{\lambda\omega_1}{2} - (k_1 + \omega_1)\pi \right)| \times |\text{sinc}\left( \frac{\lambda\omega_2}{2} + (k_2 + \omega_2)\pi \right)| d\omega_1 d\omega_2 d\omega_3 = O\left( \frac{\lambda}{n} \right).
\]

Similarly we can show \( |B_{2,(3,j)}(r_1, r_2)| = O\left( \frac{\lambda}{n^2} \right) \) (for \( 2 \leq j \leq 4 \)) and

\[
|B_{2,(2,1)}(r_1, r_2)| \leq \frac{\lambda}{(2\pi)^3n^2} \sum_{k_1, k_2=-a}^a |g(\omega_{k_1})g(\omega_{k_2})| \int_{\mathbb{R}^3} |f_4(\omega_1, \omega_2, \omega_3)| \left| \text{sinc}\left( \frac{\lambda(\omega_1 + \omega_2)}{2} + (k_2 - k_1)\pi \right) \right| \times \\
|\text{sinc}\left( \frac{\lambda(\omega_1 + \omega_2)}{2} + (k_2 - k_1 + \omega_1 - \omega_2)\pi \right)| d\omega_1 d\omega_2 d\omega_3 = O\left( \frac{a\lambda}{n^2} \right).
\]

This immediately gives us (36). To prove (37) we use identical methods. Thus we obtain the result. \( \square \)

Note that the term \( B_{2,(2,1)}(r_1, r_2) \) in the proof above is important as it does not seem possible to improve on the bound \( O(a\lambda/n^2) \).

References


Supplementary material

B Proofs in the case of non-uniform sampling

In this section we prove the results in Section 3. Most of the results derived here are based on the methodology developed in the uniform sampling case.

**PROOF of Theorem 2.2** We prove the result for \( d = 1 \). It is straightforward to show

\[
\text{cov} \left[ J_n(\omega_1), J_n(\omega_2) \right] = \frac{c_2}{\lambda} \sum_{j_1,j_2=-\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \int_{-\lambda/2}^{\lambda/2} \int_{-\lambda/2}^{\lambda/2} c(s_1 - s_2) e^{i s_1 \omega_1} e^{i s_2 \omega_2} e^{i s_1 \omega_1} e^{-i s_2 \omega_2} ds_1 ds_2 + \frac{c(0) \gamma_{k_1-k_2} \lambda}{n}.
\]

Thus, by using the same arguments as those used in the proof of Theorem 2.1, Bandyopadhyay and Subba Rao (2015), we can show that

\[
\text{cov} \left[ J_n(\omega_1), J_n(\omega_2) \right] = \frac{c_2}{\lambda} \sum_{j_1,j_2=-\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \left( \int_{-\lambda/2}^{\lambda/2} e^{-i s(\omega_2 - \omega_{j_2} - \omega_{j_1} - \omega_{k_1})} ds \right) \left( \int_{-\lambda/2}^{\lambda/2} c(t) e^{it(\omega_1 + \omega_{k_1})} dt \right) + \frac{c(0) \gamma_{k_1-k_2} \lambda}{n} + O \left( \frac{1}{\lambda} \right),
\]

where to obtain the remainder \( O \left( \frac{1}{\lambda} \right) \) we use that \( \sum_{j=-\infty}^{\infty} |\gamma_j| < \infty \). Next, by using the identity

\[
\int_{-\lambda/2}^{\lambda/2} e^{-i s(\omega_2 - \omega_{j_2} - \omega_{j_1} - \omega_{k_1})} ds = \left\{ \begin{array}{ll} 0 & k_2 - j_2 \neq k_1 + j_1 \\ \lambda & k_2 - j_2 = k_1 + j_1 \end{array} \right.
\]

we have

\[
\text{cov} \left[ J_n(\omega_1), J_n(\omega_2) \right] = c_2 \sum_{j=-\infty}^{\infty} \gamma_j \gamma_{k_2-k_1-j} \int_{-\lambda/2}^{\lambda/2} c(t) e^{it(\omega_{j_1} + \omega_{k_1})} dt ds + \frac{c(0) \gamma_{k_1-k_2} \lambda}{n} + O \left( \frac{1}{\lambda} \right) + O(\lambda^{-1})
\]

\[
= \sum_{j=-\infty}^{\infty} \gamma_j \gamma_{k_2-k_1-j} f(\omega_{k_1+j}) + \frac{c(0) \gamma_{k_1-k_2} \lambda}{n} + O \left( \frac{1}{\lambda} + \frac{1}{n} \right).
\]

Finally, we replace \( f(\omega_{k_1+j}) \) with \( f(\omega_{k_1}) \) and use the Lipschitz continuity of \( f(\cdot) \) to give

\[
\left| \sum_{j=-\infty}^{\infty} \gamma_j \gamma_{k_2-k_1-j} [f(\omega_{k_1}) - f(\omega_{k_1+j})] \right| \leq \frac{C}{\lambda} \sum_{j=-\infty}^{\infty} |j| \cdot |\gamma_j \gamma_{k_2-k_1-j}| = O \left( \frac{1}{\lambda} \right),
\]

where the last line follows from \( |\gamma_j| \leq C |j|^{-(1+\delta)} I(j \neq 0) \). Altogether, this gives

\[
\text{cov} \left[ J_n(\omega_1), J_n(\omega_2) \right] = f(\omega_{k_1}) \sum_{j=-\infty}^{\infty} \gamma_j \gamma_{k_2-k_1-j} + \frac{c(0) \gamma_{k_1-k_2} \lambda}{n} + O \left( \frac{1}{\lambda} \right).
\]

This completes the proof for \( d = 1 \), the proof for \( d > 1 \) is the same. \( \square \)
**PROOF of Theorem 3.2** We prove the result for the case $d = 1$. Using the same method used to prove Theorem 3.1 (see the arguments in Section 3.1) we obtain

$$E\left[ \tilde{Q}_{a,\lambda}(g; r) \right] = \frac{c_2}{2\pi} \sum_{j_1, j_2 = -\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \sum_{k = -a}^{a} g(\omega_k) \int_{-\infty}^{\infty} f(\omega) \text{sinc} \left( \frac{\lambda \omega}{2} + (k + j_1)\pi \right) \text{sinc} \left( \frac{\lambda \omega}{2} + (k + r - j_2)\pi \right) d\omega.$$  By the orthogonality of the sinc function at integer shifts (and noting that $\text{sinc}(\frac{\lambda \omega}{2} + (k + j_1)\pi) \cdot \text{sinc}(\frac{\lambda \omega}{2} + (k + r - j_2)\pi)$)

$$E\left[ \tilde{Q}_{a,\lambda}(g; r) \right] = \frac{c_2}{\lambda \pi} \sum_{j_1, j_2 = -\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \sum_{k = -a}^{a} g(\omega_k) \int_{-\infty}^{\infty} f \left( \frac{2y}{\lambda} - \omega_k \right) \text{sinc}(y) \text{sinc}(y + (r - j_1 - j_2)\pi) dy.$$  Replacing sum with an integral and using Lemma E.1(ii) gives

$$E\left[ \tilde{Q}_{a,\lambda}(g; r) \right] = \frac{c_2}{2\pi^2} \sum_{j_1, j_2 = -\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(\omega) f \left( \frac{2y}{\lambda} - \omega \right) d\omega \int_{-\infty}^{\infty} \text{sinc}(y) \text{sinc}(y + (r - j_1 - j_2)\pi) dy + O \left( \frac{1}{\lambda} \right).$$  Next, replacing $f \left( \frac{2y}{\lambda} - \omega \right)$ with $f(-\omega)$ and we have

$$E\left[ \tilde{Q}_{a,\lambda}(g; r) \right] = \frac{c_2}{2\pi^2} \sum_{j_1, j_2 = -\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(\omega) f \left( \frac{2y}{a} - \omega \right) d\omega \int_{-\infty}^{\infty} \text{sinc}(y) \text{sinc}(y + (r - j_1 - j_2)\pi) dy + R_n + O \left( \frac{1}{\lambda} \right).$$  where

$$R_n = \frac{c_2}{2\pi^2} \sum_{j_1, j_2 = -\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(\omega) \left[ f \left( \frac{2y}{\lambda} - \omega \right) - f(-\omega) \right] \int_{-\infty}^{\infty} \text{sinc}(y) \text{sinc}(y + (r - j_1 - j_2)\pi) dy d\omega.$$  By using Lemma C.2, Subba Rao (2015b) we have

$$|R_n| \leq C \sum_{j_1, j_2 = -\infty}^{\infty} |\gamma_{j_1}| \cdot |\gamma_{j_2}| \frac{\log \lambda + \log |r - j_1 - j_2|}{\lambda} \leq C \sum_{j_1, j_2 = -\infty}^{\infty} |\gamma_{j_1}| \cdot |\gamma_{j_2}| \frac{\log \lambda + \log |r| + \log |j_1| + \log |j_2|}{\lambda},$$  noting that $C$ is a generic constant that changes between inequalities. This gives

$$E\left[ \tilde{Q}_{a,\lambda}(g; r) \right] = \frac{c_2}{2\pi^2} \sum_{j_1, j_2 = -\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(\omega) f(\omega) d\omega \int_{-\infty}^{\infty} \text{sinc}(y) \text{sinc}(y + (r - j_1 - j_2)\pi) dy + O \left( \frac{\log \lambda + \log |r|}{\lambda} \right).$$  Finally, by the orthogonality of the sinc function at integer shifts (and $f(-\omega) = f(\omega)$) we have

$$E\left[ \tilde{Q}_{a,\lambda}(g; r) \right] = \frac{1}{2\pi} \sum_{j = -\infty}^{\infty} \gamma_{j} \gamma_{r-j} \int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(\omega) f(\omega) d\omega + O \left( \frac{\log \lambda + \log |r|}{\lambda} + \frac{1}{n} \right),$$  40
thus we obtain the desired result.

PROOF of Theorem 3.3(i) To prove (i) we use Theorem 2.2 and Lemma D.1 which immediately gives the result.

PROOF of Theorem 3.3(ii) We first note that by using Lemma D.1 (generalized to non-uniform sampling), we can show that

\[
\lambda \text{cov} \left[ \widetilde{Q}_{a,\lambda}(g;r_1), \widetilde{Q}_{a,\lambda}(g;r_2) \right] = A_1(r_1, r_2) + A_2(r_1, r_2) + O \left( \frac{\lambda}{n} \right)
\]

(40)

where

\[
A_1(r_1, r_2) = \lambda \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) \overline{g(\omega_{k_2})} \text{cov} \left[ Z(s_1) \exp(is_1 \omega_{k_1}), Z(s_3) \exp(is_3 \omega_{k_2}) \right] \times
\]

\[
\text{cov} \left[ Z(s_2) \exp(-is_2 \omega_{k_1} + r_1), Z(s_4) \exp(-is_4 \omega_{k_2} + r_2) \right]
\]

\[
A_2(r_1, r_2) = \lambda \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) \overline{g(\omega_{k_2})} \text{cov} \left[ Z(s_1) \exp(is_1 \omega_{k_1}), Z(s_4) \exp(-is_4 \omega_{k_2} + r_2) \right] \times
\]

\[
\text{cov} \left[ Z(s_2) \exp(-is_2 \omega_{k_1} + r_1), Z(s_3) \exp(is_3 \omega_{k_2}) \right].
\]

We first analyze \( A_1(r_1, r_2) \). Conditioning on the locations \( s_1, \ldots, s_4 \) gives

\[
A_1(r_1, r_2) = \sum_{j_1, \ldots, j_4 = -\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) \overline{g(\omega_{k_2})} \frac{1}{\lambda^3} \int_{[-\lambda/2, \lambda/2]^4} c(s_1 - s_3)c(s_2 - s_4) dsdsdsds.
\]

By using the spectral representation theorem and integrating out \( s_1, \ldots, s_4 \) we can write the above as

\[
A_1(r_1, r_2) = \frac{1}{\lambda^2} \sum_{j_1, \ldots, j_4 = -\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) \overline{g(\omega_{k_2})} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)f(y) \text{sinc} \left( \frac{\lambda x}{2} + (k_1 + j_1)\pi \right) \times \text{sinc} \left( \frac{\lambda y}{2} - (r_1 + k_1 - j_2)\pi \right) \text{sinc} \left( \frac{\lambda x}{2} + (k_2 - j_3)\pi \right) \times \text{sinc} \left( \frac{\lambda y}{2} - (r_2 + k_2 + j_4)\pi \right) dx dy
\]

\[
\times \text{sinc} \left( \frac{2u}{\lambda} - \omega_{k_1 + j_1} \right) f \left( \frac{2v}{\lambda} + \omega_{k_1 + r_1 - j_2} \right)
\]

\[
\times \text{sinc} \left( u + (k_2 - k_1 - j_1 - j_3)\pi \right) \text{sinc} \left( v + (k_2 - k_1 + r_1 - j_2)\pi \right) dudv.
\]

(41)

By making a change of variables \( m = k_1 - k_2 \) we have

\[
|A_1(r_1, r_2)| \leq \frac{1}{\lambda^2} \sum_{j_1, \ldots, j_4 = -\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \sum_{m = -\infty}^{\infty} \sum_{k_1 = -a}^{a} g(\omega_{k_1}) \overline{g(\omega_{m-k_1})} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| f \left( \frac{2u}{\lambda} - \omega_{k_1 + j_1} \right) f \left( \frac{2v}{\lambda} + \omega_{k_1 + r_1 - j_2} \right) \right| \times \text{sinc} \left( u + (m + j_1 + j_3)\pi \right) \text{sinc} \left( v + (m - r_2 - j_4 + r_1 - j_2)\pi \right) dudv.
\]

41
Now by following the same series of bounds used to prove Lemma E.2(iii) we have
\[
|A_1(r_1, r_2)| \leq \frac{1}{\pi^2 \lambda} \sum_{\gamma_{j_1}, j_2, j_3} \sup_{\omega} |g(\omega)|^2 |f|_2^2 \\
\times \sum_{m=-\infty}^{\infty} \int_{\mathbb{R}^d} \left| \sin(u - (m + j_1 + j_3)\pi) \sin(v + (m - r_2 + r_1 - j_4)\pi) \sin(u) \sin(v) \right| du dv < \infty.
\]
Similarly we can bound \( A_2(r_1, r_2) \) and \( \lambda \text{cov} \left[ \overline{Q}_{a, \lambda}(g; r_1), \overline{Q}_{a, \lambda}(g; r_2) \right] \), thus giving the required result.
\[\Box\]

**PROOF of Theorem 3.3(iii)** The proof uses the expansion (40). Using this as a basis, we will show that
\[
A_1(r_1, r_1) = U_1(r_1, r_2; \omega_1, \omega_r) + O(\ell_{\lambda, a, n}) \\
A_2(r_1, r_1) = U_2(r_1, r_2; \omega_1, \omega_2) + O(\ell_{\lambda, a, n}).
\]
We find an approximation for \( A_1(r_1, r_2) \) starting with the expansion given in (41). We use the same proof as that used to prove Lemma A.2 to approximate the terms inside the sum \( \sum_{j_1, \ldots, j_4} \). More precisely we let \( m = k_1 - k_2 \), replace \( \omega_{k_1} \) with \( \omega \) and by using the same methodology given in the proof of Lemma A.2 (and that \( \sum_j |\gamma_j| < \infty \)), we have
\[
A_1(r_1, r_2) = \frac{1}{2\pi^3} \sum_{j_1, \ldots, j_4} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \sum_{m=-a}^{2a} \int_{\mathbb{R}^d} \sin(u - (m + j_1 + j_3)\pi) \sin(v - (m - r_2 - j_4)\pi) du dv + O(\ell_{\lambda, a, n}).
\]
By orthogonality of the sinc function we see that the above is zero unless \( m = -j_1 - j_3 \) and \( m = r_2 - r_1 + j_2 + j_4 \) (and using that \( f(-\omega - \omega_{j_1}) = f(\omega + \omega_{j_1}) \)), therefore
\[
A_1(r_1, r_2) = \frac{1}{2\pi} \sum_{j_1, \ldots, j_4} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \int_{\mathbb{R}^d} \sin(u - (m + j_1 + j_3)\pi) \sin(v - (m - r_2 - j_4)\pi) du dv + O(\ell_{\lambda, a, n}).
\]
This gives us \( U_{1,1}(r_1, r_2; \omega_1, \omega_2) \). Next we consider \( A_2(r_1, r_2) \)
\[
A_2(r_1, r_2) = \frac{1}{\lambda^3} \sum_{j_1, j_2, j_3, j_4} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \sum_{k_1, k_2} \exp(-is_1 \omega_{k_1} + j_1) \exp(-is_2 \omega_{k_2} + j_2) \exp(-is_3 \omega_{k_2} + j_3) \int_{\mathbb{R}^2} \sin(x) \sin(y) \sin \left( \frac{\lambda x}{2} + (k_1 + j_1)\pi \right) \sin \left( \frac{\lambda y}{2} + (k_2 + j_2)\pi \right) dx dy.
\]
Making a change of variables \( u = \frac{\lambda v}{2} + (k_1 + j_1)\pi \) and \( v = \frac{\lambda u}{2} - (k_1 + r_1 - j_2)\pi \) we have

\[
A_2(r_1, r_2) = \frac{1}{\pi^2} \sum_{j_1, j_2, j_3, j_4 \in \mathbb{Z}} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \int_{\mathbb{R}^2} f\left(\frac{2u}{\lambda} - \omega_{k_1 + j_1}\right) f\left(\frac{2v}{\lambda} + \omega_{k_1 + r_1 - j_2}\right) \times \\
sinc(u) \text{sinc}(u - (k_2 + r_2 + j_4 + k_1 + j_1)\pi) \text{sinc}(v) \text{sinc}(v + (k_2 - j_3 + k_1 + r_1 - j_2)\pi) \text{dudv}.
\]

Again by using the same proof as that given in Lemma A.2 to approximate the terms inside the sum \( \sum_{j_1,\ldots,j_4} \) (setting \( m = k_1 + k_2 \) and replacing \( \omega_{k_1} \) with \( \omega \)), we can approximate \( A_2(r_1, r_2) \) with

\[
A_2(r_1, r_2) = \frac{1}{2\pi} \sum_{j_1, j_2, j_3, j_4 \in \mathbb{Z}} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \sum_{m = -2a}^{2a} \int_{\mathbb{R}^2} g(\omega) g(-\omega + \omega_m) \int_{\mathbb{R}^2} f(-\omega - j_1) f(\omega + \omega_{r_1 - j_2}) \times \\
sinc(u) \text{sinc}(u - (m + r_2 + j_4 + j_1)\pi) \text{sinc}(v) \text{sinc}(v + (m - j_3 + r_1 - j_2)\pi) \text{dudv} + O(\ell_{a,n}).
\]

Using the orthogonality of the sinc function, the inner integral is non-zero when \( m - j_3 + r_1 - j_2 = 0 \) and \( m + r_2 + j_4 + j_1 = 0 \). Setting \( m = -r_1 + j_2 + j_3 \), this implies

\[
A_2(r_1, r_2) = \frac{1}{2\pi} \sum_{j_1, j_2, j_3, j_4 = r_1 - r_2} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \int_{2\pi} g(\omega) g(-\omega + \omega_m) \int_{2\pi} f(\omega + \omega_{j_1}) f(\omega + \omega_{r_1 - j_2}) \times \\
\text{dudv} + O(\ell_{a,n}),
\]

thus giving us \( U_{1,2}(r_1, r_2; \omega_{r_1}, \omega_{r_2}) \).

By using Lemma D.1, we can show that

\[
\lambda \text{cov}\left[\widetilde{Q}_{a,\lambda}(g; r_1), \widetilde{Q}_{a,\lambda}(g; r_2)\right] = A_3(r_1, r_2) + A_4(r_1, r_2) + O\left(\frac{\lambda}{n}\right)
\]

where

\[
A_3(r_1, r_2) = \lambda \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \text{cov}\left[Z(\omega_1) \text{exp}(is_1 \omega_{k_1}), Z(\omega_3) \text{exp}(-is_3 \omega_{k_2})\right] \times \\
\text{cov}\left[Z(\omega_2) \text{exp}(-is_2 \omega_{k_1 + r_1}), Z(\omega_4) \text{exp}(is_4 \omega_{k_2 + r_2})\right]
\]

\[
A_4(r_1, r_2) = \lambda \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \text{cov}\left[Z(\omega_1) \text{exp}(is_1 \omega_{k_1}), Z(\omega_4) \text{exp}(is_4 \omega_{k_2 + r_2})\right] \times \\
\text{cov}\left[Z(\omega_2) \text{exp}(-is_2 \omega_{k_1 + r_1}), Z(\omega_3) \text{exp}(-is_3 \omega_{k_2})\right].
\]
By following the same proof as that used to prove $A_1(r_1, r_2)$ we have

$$A_3(r_1, r_2) = \frac{\lambda}{(2\pi)^2} \sum_{j_1, j_2, j_3, j_4 = -\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) f(y) \text{sinc} \left( \frac{\lambda x}{2} + (k_1 + j_1) \pi \right) \times \text{sinc} \left( \frac{\lambda y}{2} - (k_2 + j_3) \pi \right) \text{sinc} \left( \frac{\lambda y}{2} - (r_1 + k_1 - j_2) \pi \right) \text{sinc} \left( \frac{\lambda y}{2} + (r_2 + j_4 - j_1 - j_2) \pi \right) dx dy$$

$$= \frac{1}{\pi^2 \lambda} \sum_{j_1, j_2, j_3, j_4 = -\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) f(y) \text{sinc} \left( \frac{2u}{\lambda} - \omega_{k_1 + j_1} \right) f \left( \frac{2v}{\lambda} + \omega_{k_1 + r_1 - j_2} \right) \times \text{sinc}(u) \text{sinc}(u - (k_2 + k_1 + j_1 + j_3) \pi) \text{sinc}(v) \text{sinc}(v - (r_2 + k_2 - j_4 + r_1 + k_1 - j_2) \pi) dudv$$

Again using the method used to bound $A_1(r_1, r_2)$ gives

$$A_3(r_1, r_2) = \frac{1}{2\pi^3} \sum_{j_1 + j_2 + j_3 + j_4 = r_1 + r_2} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \sum_{m = -2a}^{2a} g(\omega_{m}) \text{sinc} \left( \frac{2u}{\lambda} - \omega_{k_1 + j_1} \right) f(\omega + \omega_{j_1}) f(\omega + \omega_{r_1 - j_2}) d\omega + O(\ell_{\lambda, a,n}) = U_{1,2}(r_1, r_2; \omega_{r_1}, \omega_{r_2}) + O(\ell_{\lambda, a,n}).$$

Finally we consider $A_4(r_1, r_2)$. Using the same expansion as the above we have

$$A_4(r_1, r_2)$$

$$= \frac{1}{\lambda^2} \sum_{j_1, j_2, j_3, j_4 \in \mathbb{Z}} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(s_1 - s_4) c(s_2 - s_3) \exp(is_1 \omega_{k_1 + j_1}) \exp(-is_2 \omega_{k_2 + r_2 - j_3}) \exp(-is_3 \omega_{k_1 + r_1 - j_2}) ds_1 ds_2 ds_3 ds_4$$

$$= \frac{1}{\pi^2 \lambda} \sum_{j_1, j_2, j_3, j_4 \in \mathbb{Z}} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \int_{\mathbb{R}^2} f(x) f(y) \text{sinc} \left( \frac{\lambda x}{2} + (k_1 + j_1) \pi \right) \times \text{sinc} \left( \frac{\lambda y}{2} + (k_2 + r_2 - j_3) \pi \right) \text{sinc} \left( \frac{\lambda y}{2} - (k_1 + r_1 - j_2) \pi \right) \text{sinc} \left( \frac{\lambda y}{2} - (k_2 + j_3 + k_1 - r_1 - j_2) \pi \right) dx dy$$

$$= \frac{1}{2\pi^3} \sum_{j_1, j_2, j_3, j_4 \in \mathbb{Z}} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \sum_{m = -2a}^{2a} g(\omega) g(\omega - \omega_m) \int_{\mathbb{R}^2} f(-\omega - \omega_{j_1}) f(\omega + \omega_{r_1 - j_2}) d\omega \times \text{sinc}(u) \text{sinc}(u - (r_2 + j_4 - m - j_1) \pi) \text{sinc}(v) \text{sinc}(v - (j_3 - m - r_1 + j_2) \pi) dudv + O(\ell_{\lambda, a,n})$$

$$= \frac{1}{2\pi} \sum_{j_1 + j_2 + j_3 + j_4 = r_1 + r_2} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \int_{\mathbb{R}^2} f(\omega + \omega_{j_1}) f(\omega + \omega_{r_1 - j_2}) d\omega + O(\ell_{\lambda, a,n}).$$
This gives the desired result. \[\square\]

We now obtain and approximation of $U_1$ and $U_2$.

**Proof of Corollary 3.1** By using Lipschitz continuity of $g(\cdot)$ and $f(\cdot)$ and $|\gamma_j| \leq C \prod_{i=1}^d |j_i|^{-(1+\delta)} I(j_i \neq 0)$ we obtain the result.

**Proof of Theorem 3.6** We prove the result for the case $d = 1$ and using $A_1(r_1, r_2), \ldots, A_4(r_1, r_2)$ defined in proof of Theorem 3.3. The proof is identical to the proof of Theorem A.2. Following the same notation in proof of Theorem A.2 we have

$$
\lambda \text{cov} \left[ \tilde{Q}_{a, \lambda}(g; r_1), \tilde{Q}_{a, \lambda}(g; r_2) \right] = A_1(r_1, r_2) + A_2(r_1, r_2) + B_1(r_1, r_2) + B_2(r_1, r_2) + O \left( \frac{\lambda}{n} \right),
$$

with $|B_2(r_1, r_2)| = O((a \lambda)^2 / n^2)$ and the main term involving the trispectral density is

$$
B_1(r_1, r_2) = \lambda c_4 \sum_{k_1, k_2 = -a}^a g(\omega_{k_1}) g(\omega_{k_2}) E \left[ \kappa_4(s_1 - s_2, s_1 - s_3, s_1 - s_4) e^{i \omega_1} e^{-i \omega_2} e^{i \omega_3} e^{i \omega_4} \right]
$$

$$
= \frac{c_4}{(2\pi)^3} \sum_{j_1, \ldots, j_4 = -\infty}^\infty g(\omega_{k_1}) g(\omega_{k_2}) \int_{\mathbb{R}^4} f_4(\omega_1, \omega_2, \omega_3) e^{i \omega_1} e^{i \omega_2} e^{i \omega_3} e^{i \omega_4} ds_1 ds_2 ds_3 ds_4 d\omega_1 d\omega_2 d\omega_3.
$$

Now we make a change of variables and let $u_1 = \frac{\lambda \omega_1}{2} + (k_1 + r_1 - j_2)$, $u_2 = \frac{\lambda \omega_2}{2} + (k_2 + r_2 + j_4)\pi$ and $u_3 = \frac{\lambda \omega_3}{2} + (k_2 + r_2 + j_4)\pi$, this gives

$$
B_1(r_1, r_2) = \frac{c_4}{\pi^2} \sum_{j_1, \ldots, j_4 = -\infty}^\infty g(\omega_{k_1}) g(\omega_{k_2}) \int_{\mathbb{R}^4} f_4 \left( \frac{2u_1}{\lambda} - \omega_{k_1 + r_1 - j_2}, \frac{2u_2}{\lambda} - \omega_{k_2 + r_2 + j_4}, \frac{2u_3}{\lambda} + \omega_{k_2 + r_2 + j_4} \right) \times \text{sinc}(u_1 + u_2 + u_3 + (r_2 - r_1 + j_1 + j_2 + j_3 + j_4)\pi) \text{sinc}(u_1) \text{sinc}(u_2) \text{sinc}(u_3) du_1 du_2 du_3.
$$

Next we exchange the summand with a double integral and use Lemma E.1(iii) together with Lemma C.1, equation (44) to obtain

$$
B_1(r_1, r_2) = \frac{c_4}{\pi^2} \sum_{j_1, \ldots, j_4 = -\infty}^\infty g(\omega_{k_1}) g(\omega_{k_2}) \int_{\mathbb{R}^2} f_4 \left( \frac{2u_1}{\lambda} - \omega_{k_1 + r_1 - j_2}, \frac{2u_2}{\lambda} - \omega_{k_2 + r_2 + j_4}, \frac{2u_3}{\lambda} + \omega_{k_2 + r_2 + j_4} \right) \times \text{sinc}(u_1 + u_2 + u_3 + (r_2 - r_1 + j_1 + j_2 + j_3 + j_4)\pi) \text{sinc}(u_1) \text{sinc}(u_2) \text{sinc}(u_3) du_1 du_2 du_3 + O \left( \frac{1}{\lambda} \right).
$$

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By using Lemma C.3, we replace $f_4\left(\frac{2\pi}{\lambda} - \omega_1 - \omega_{r_1-j_2}, \frac{2\pi}{\lambda} - \omega_2 - \omega_{j_3}, \frac{2\pi}{\lambda} + \omega_{r_2+j_4}\right)$ with $f_4(-\omega_1 - \omega_{r_1-j_2}, -\omega_2 - \omega_{j_3}, \omega_2 + \omega_{r_2+j_4})$, to give

$$B_1(r_1,r_2)$$

$$= \frac{c_4}{\pi^3(2\pi)^2} \sum_{j_1,..,j_4=-\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \int_{2\pi[-a/\lambda,a/\lambda]^2} g(\omega_1)g(\omega_2) \times$$

$$f_4(-\omega_1 - \omega_{r_1-j_2} - \omega_2 - \omega_{j_3}, \omega_2 + \omega_{r_2+j_4}) \sin(cu_1 + u_2 + u_3 + (r_2 - r_1 + j_1 + j_2 + j_3 + j_4)\pi) \times \sin(c(u_1)\sin(c(u_2))\sin(c(u_3))du_1du_2du_3d\omega_1d\omega_2 + O\left(\frac{\log^3(\lambda)}{\lambda}\right)$$

$$= \frac{c_4}{(2\pi)^2} \sum_{j_1+j_2+j_3+j_4=r_2-r_1} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \int_{2\pi[-a/\lambda,a/\lambda]^2} g(\omega_1)g(\omega_2) \times$$

$$f_4(-\omega_1 - \omega_{r_1-j_2} - \omega_2 - \omega_{j_3}, \omega_2 + \omega_{r_2+j_4}) d\omega_1d\omega_2 + O\left(\frac{\log^3(\lambda)}{\lambda}\right),$$

where the last line follows from (39).

To obtain an expression for $\lambda\text{cov}\left[\bar{Q}_{a,\lambda}(g;r_1), \bar{Q}_{a,\lambda}(g;r_2)\right]$ we note that

$$\lambda\text{cov}\left[\bar{Q}_{a,\lambda}(g;r_1), \bar{Q}_{a,\lambda}(g;r_2)\right] = A_3(r_1,r_2) + A_4(r_1,r_2) + B_3(r_1,r_2) + B_4(r_1,r_2) + O\left(\frac{\lambda}{n}\right),$$

just as in the proof of Theorem A.2 we can show that $|B_4(r_1,r_2)| = O((\lambda a)^d/n^2)$ and the leading term involving the trispectral density is

$$B_3(r_1,r_2)$$

$$= \frac{\lambda c_4}{(2\pi)^3} \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})g(\omega_{k_2})E\left[\kappa_4(s_1 - s_2, s_1 - s_3, s_1 - s_4)e^{is_1\omega_{k_1}}e^{-is_2\omega_{k_1} + r_1}e^{is_3\omega_{k_2}}e^{-is_4\omega_{k_2} + r_2}\right]$$

$$= \frac{c_4}{(2\pi)^3} \sum_{j_1,..,j_4=-\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})g(\omega_{k_2}) \int_{\mathbb{R}^3} f_4(\omega_1,\omega_2,\omega_3) \int_{\mathbb{R}^3} e^{is_1(\omega_{1} + \omega_{2} + \omega_{3} + \omega_{k_1} + j_1)}$$

$$e^{-is_2(\omega_{k_1} + r_1 - j_2)}e^{-is_3(\omega_{k_2} - r_2 + j_3)}e^{is_4(-\omega_{k_2} + r_2 - j_4)}ds_1ds_2ds_3ds_4d\omega_1d\omega_2d\omega_3$$

$$= \frac{c_4\lambda}{(2\pi)^3} \sum_{j_1,..,j_4=-\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})g(\omega_{k_2}) \int_{\mathbb{R}^3} f_4(\omega_1,\omega_2,\omega_3)\sin\left(\frac{\lambda(\omega_{1} + \omega_{2} + \omega_{3})}{2} + (k_1 + j_1)\pi\right)$$

$$\times \sin\left(\frac{\lambda_1}{2} + (k_1 + r_1 - j_2)\pi\right) \sin\left(\frac{\lambda_2}{2} - (k_2 + j_3)\pi\right) \sin\left(\frac{\lambda_3}{2} + (k_2 + r_2 - j_4)\pi\right) d\omega_1d\omega_2d\omega_3.$$
Finally, by replacing \( f_4(-\omega_1 - \omega_{r_1-j_2}, -\omega_2 - \omega_{j_3}, \omega_2 + \omega_{r_2+j_4}) \) with \( f_4(-\omega_1, -\omega_2, \omega_2) \) in \( B_1(r_1, r_2) \) and \( f_4(-\omega_1 - \omega_{r_1-j_2}, \omega_2 + \omega_{j_3}, -\omega_2 - \omega_{r_2+j_4}) \) with \( f_4(-\omega_1, \omega_2, -\omega_2) \) and using the pointwise Lipschitz continuity of \( f_4 \) and that \( |\gamma_j| \leq CI(j \neq 0)|j|^{-1+\delta} \) we obtain \( B_1(r_1, r_2) = D_1 + O \left( \frac{\log^3(\lambda)}{\lambda} + \frac{1}{\lambda} \right) \) and \( B_3(r_1, r_2) = D_3 + O \left( \frac{\log^3(\lambda)}{\lambda} + \frac{1}{\lambda} \right) \). Thus giving the required result.

\[ \Box \]

C Technical Lemmas

We first prove Lemma A.1, then state four lemmas, which form an important component in the proofs of this paper. Through out this section we use \( C \) to denote a finite generic constant. It is worth mentioning that many of these results build on the work of T. Kawata (see Kawata (1959)).

**PROOF of Lemma A.1** We first prove (30). By using partial fractions and the definition of the sinc function we have

\[
\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(u)\sin(u+x)}{u} du
\]

For the second integral we make a change of variables \( u' = u + x \), this gives

\[
\int_{-\infty}^{\infty} \frac{\sin(u)\sin(u+x)}{u} du = \frac{1}{x} \int_{-\infty}^{\infty} \frac{\sin(u)\sin(u+x)}{u} du = \frac{2\sin(x)}{x} \int_{-\infty}^{\infty} \frac{\cos(u)\sin(u)}{u} du = \frac{\pi \sin(x)}{x}.
\]

To prove (31), it is clear that for \( x = s\pi \) (with \( s \in \mathbb{Z}\{0\} \)) \( \frac{\pi \sin(s\pi)}{s\pi} = 0 \), which gives the result.

The following result is used to obtain bounds for the variance and higher order cumulants.

**Lemma C.1** Define the function \( \ell_p(x) = C/e \) for \( |x| \leq e \) and \( \ell_p(x) = C \log^p |x|/|x| \) for \( |x| \geq e \).

(i) We have

\[
\int_{-\infty}^{\infty} \frac{\sin x}{x} \frac{\sin x + y}{|x+y|} dx 
\]

\[
\leq \begin{cases} 
C \frac{\log |y|}{|y|} & |y| \geq e \\
C & |y| < e
\end{cases} = \ell_1(y),
\]

and

\[
\int_{-\infty}^{\infty} \frac{\sin x}{x} \ell_p(x+y) dx \leq \ell_{p+1}(y)
\]

and

\[
\int_{\mathbb{R}^p} \left| \text{sinc} \left( \sum_{j=1}^{p} x_j \right) \prod_{j=1}^{p} \text{sinc}(x_j) \right| dx_1 \ldots dx_p \leq C,
\]

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(ii) \[
\sum_{m=-a}^{a} \int_{-\infty}^{\infty} \left| \frac{\sin^2(x)}{x(x+m\pi)} \right| \, dx \leq C \log^2 a
\]

(iii) \[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=-a}^{a} \frac{\sin^2(x)}{x(x+m\pi)} \left| \frac{\sin^2(y)}{y(y+m\pi)} \right| \, dx \, dy \leq C, \tag{45}
\]

(iv) \[
\sum_{m_1,\ldots,m_q=0}^{a} \int_{\mathbb{R}^q} \prod_{j=1}^{q-1} \text{sinc}(x_j) \text{sinc}(x_j + m_j \pi) \times \left| \text{sinc}(x_q) \text{sinc}(x_q + \pi \sum_{j=1}^{q-1} m_j) \right| \prod_{j=1}^{q} dx_j \leq C \log^2(q-2)(a),
\]

where \(C\) is a finite generic constant which is independent of \(a\).

PROOF. We first prove (i), equation (42). It is clear that for \(|y| \leq \epsilon\) that \(\int_{-\infty}^{\infty} \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|} \, dx \leq C\). Therefore we now consider the case \(|y| > \epsilon\), without loss of generality we prove the result for \(y > \epsilon\).

Partitioning the integral we have

\[
\int_{-\infty}^{\infty} \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|} \, dx = I_1 + I_2 + I_3 + I_4 + I_5,
\]

where

\[
I_1 = \int_{-\infty}^{0} \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|} \, dx \quad I_2 = \int_{-\infty}^{0} \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|} \, dx
\]

\[
I_3 = \int_{-y}^{0} \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|} \, dx \quad I_4 = \int_{-\infty}^{-2y} \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|} \, dx
\]

\[
I_5 = \int_{y}^{\infty} \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|} \, dx.
\]

To bound \(I_1\) we note that for \(y > 1\) and \(x > 0\) that \(|\sin(x+y)/(x+y)| \leq 1/y\), thus

\[
I_1 = \frac{1}{y} \int_{0}^{y} \frac{|\sin(x)|}{|x|} \, dx \leq C \log \frac{y}{y}.
\]

To bound \(I_2\), we further partition the integral

\[
I_2 = \int_{-\infty}^{-y/2} \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|} \, dx + \int_{-y/2}^{0} \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|} \, dx \leq \frac{2}{y} \int_{-y}^{-y/2} \frac{|\sin(x+y)|}{|(x+y)|} \, dx + \frac{2}{y} \int_{-y/2}^{0} \frac{|\sin(x)|}{|x|} \, dx \leq C \log \frac{y}{y}.
\]
To bound $I_3$, we use the bound

$$I_3 \leq \frac{1}{y} \int_{-2y}^{-y} \frac{\sin(x + y)}{|x + y|} dx \leq C \frac{\log y}{y}.$$

To bound $I_4$ we use that for $y > 0$, $\int_{y}^{\infty} x^{-2} dx \leq C|y|^{-1}$, thus

$$I_4 \leq \int_{-\infty}^{-y} \frac{1}{x^2} dx \leq C|y|^{-1}$$

and using a similar argument we have $I_5 \leq C|y|^{-1}$. Altogether, this gives (42).

We now prove (43). It is clear that for $|y| \leq e$ that $\int_{-\infty}^{\infty} |\sin(x)| \ell_p(x + y) dx \leq C$. Therefore we now consider the case $|y| > e$, without loss of generality we prove the result for $y > e$. As in (42) we partition the integral

$$\int_{-\infty}^{\infty} |\sin(x)| \ell_p(x + y) dx = I_1 + \ldots + I_5,$$

where $I_1, \ldots, I_5$ are defined in the same way as $I_1, \ldots, I_5$ just with $|\sin(x)| \ell_p(x + y)$ replacing $|\sin(x)| |x+y|^{-1}$. To bound $I_1$ we note that

$$I_1 = \int_{0}^{y} |\sin(x)| \ell_p(x + y) dx \leq \frac{\log^p(y)}{y} \int_{0}^{y} |\sin(x)| dx \leq \frac{\log^{p+1}(y)}{y},$$

we use similar method to show $I_2 \leq C \frac{\log^{p+1}(y)}{y}$ and $I_3 \leq C \frac{\log^{p+1}(y)}{y}$. Finally to bound $I_4$ and $I_5$ we note that by using a change of variables $x = yz$, we have

$$I_5 = \int_{y}^{\infty} |\sin(x)| \log^p(x + y) dx \leq \int_{y}^{\infty} \frac{\log^p(x + y)}{x(x + y)} dx = \frac{1}{y} \int_{1}^{\infty} \frac{\log(y) + \log(z + 1)}{z(z + 1)} dz \leq C \frac{\log^p(y)}{y}.$$

Similarly we can show that $I_4 \leq C \frac{\log^p(y)}{y}$. Altogether, this gives the result.

To prove (44) we recursively apply (43) to give

$$\int_{\mathbb{R}^p} |\sin(x_1 + \ldots + x_p)| \prod_{j=1}^{p} |\sin(x_j)| dx_1 \ldots dx_p \leq \int_{\mathbb{R}^{p-1}} |\ell_1(x_1 + \ldots + x_{p-1})| \prod_{j=1}^{p-1} |\sin(x_j)| dx_1 \ldots dx_{p-1} \leq \int_{\mathbb{R}} |\ell_{p-1}(x) |\sin(x_1)| dx_1 = O(1),$$

thus we have the required the result.

To bound (ii), without loss of generality we derive a bound over $\sum_{m=1}^{a}$, the bounds for $\sum_{-a}^{m}$ is identical. Using (42) we have

$$\sum_{m=1}^{a} \int_{-\infty}^{\infty} \frac{\sin^2(x)}{|x(x + m\pi)|} dx = \sum_{m=1}^{a} \int_{-\infty}^{\infty} \frac{|\sin(x)| |\sin(x + m\pi)|}{|x(x + m\pi)|} dx \leq \sum_{m=1}^{a} \ell_1(m\pi) = C \sum_{m=1}^{a} \frac{\log(m\pi)}{m\pi} \leq C \log(a\pi) \sum_{m=1}^{a} \frac{1}{m\pi} = C \log(a\pi) \log(a) \leq C \log^2 a.$$
Thus we have shown (ii).

To prove (iii) we use (42) to give
\[
\sum_{m=-a}^{a} \left( \int_{-\infty}^{\infty} \frac{\sin^2(x)}{|x + m\pi|} \, dx \right) \left( \int_{-\infty}^{\infty} \frac{\sin^2(y)}{|y + m\pi|} \, dy \right) \leq C \sum_{m=-a}^{a} \left( \frac{\log m}{m} \right)^2 \leq C.
\]

To prove (iv) we apply (42) to each of the integrals this gives
\[
\sum_{m_1, \ldots, m_{q-1} = -a}^{a} \int_{\mathbb{R}^q} \left| \prod_{j=1}^{q-1} \operatorname{sinc}(x_j) \operatorname{sinc}(x_j + m_j \pi) \operatorname{sinc}(x_q + \pi \sum_{j=1}^{q-1} m_j) \right| \prod_{j=1}^{q} dx_j \leq \sum_{m_1, \ldots, m_{q-1} = -a}^{a} \ell_1(m_1 \pi) \ldots \ell_1(m_{q-1} \pi) \ell_1(\pi \sum_{j=1}^{q-1} m_j) \leq C \log^2(q-2) \, a,
\]
thus we obtain the desired result. \(\square\).

The proofs of Theorem 3.1, Theorems A.1 (ii), Theorem 2.2 involve integrals of \(\operatorname{sinc}(u)\operatorname{sinc}(u + m\pi)\), where \(|m| \leq a + |r_1| + |r_2|\) and that \(a \to \infty\) as \(\lambda \to \infty\). In the following lemma we obtain bounds for these integrals. We note that all these results involve an inner integral difference of the form
\[
\int_{-b}^{b} g(\omega) \left[ h \left( \omega + \frac{2u}{\lambda} \right) - h(\omega) \right] \, d\omega \, du.
\]
By using the mean value theorem heuristically it is clear that \(O(\lambda^{-1})\) should come out of the integral however the 2u in the numerator of \(h \left( \omega + \frac{2u}{\lambda} \right)\) makes the analysis quite involved.

**Lemma C.2** Suppose \(h\) is a function which is absolutely integrable and \(|h'(\omega)| \leq \beta(\omega)\) (where \(\beta\) is a monotonically decreasing function that is absolutely integrable) and \(m \in \mathbb{Z}\) and \(g(\omega)\) is a bounded function. \(b\) can take any value, and the bounds given below are independent of \(b\). Then we have
\[
\left| \int_{-\infty}^{\infty} \operatorname{sinc}(u) \operatorname{sinc}(u + m\pi) \int_{-b}^{b} g(\omega) \left[ h \left( \omega + \frac{2u}{\lambda} \right) - h(\omega) \right] \, d\omega \, du \right| \leq C \frac{\log(\lambda) + I(m \neq 0) \log(|m|)}{\lambda},
\]
where \(C\) is a finite constant independent of \(m\) and \(b\). If \(g(\omega)\) is a bounded function with a bounded first derivative, then we have
\[
\left| \int_{-\infty}^{\infty} \operatorname{sinc}(u) \operatorname{sinc}(u + m\pi) \int_{-b}^{b} h(\omega) \left( g(\omega + \frac{2u}{\lambda}) - g(\omega) \right) \, d\omega \right| \leq C \frac{\log(\lambda) + I(m \neq 0) \log(|m|)}{\lambda}.
\]
In the case of double integrals, we assume \(h(\cdot, \cdot)\) is such that \(\int_{\mathbb{R}^2} |h(\omega_1, \omega_2)| \, d\omega_1 \, d\omega_2 < \infty\), and
\[
\left| \frac{\partial h(\omega_1, \omega_2)}{\partial \omega_1} \right| \leq \beta(\omega_1) \beta(\omega_2), \quad \left| \frac{\partial h(\omega_1, \omega_2)}{\partial \omega_2} \right| \leq \beta(\omega_1) \beta(\omega_2) \quad \text{and} \quad \left| \frac{\partial^2 h(\omega_1, \omega_2)}{\partial \omega_1 \partial \omega_2} \right| \leq \beta(\omega_1) \beta(\omega_2) \quad \text{(where } \beta \text{ is a...} \right.
\]
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monotonically decreasing function that is absolutely integrable) and \(g(\cdot, \cdot)\) is a bounded function. Then if \(m_1 \neq 0\) and \(m_2 \neq 0\) we have
\[
\left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sin(u_1) \sin(u_1 + m_1 \pi) \sin(u_2) \sin(u_2 + m_2 \pi)}{\lambda} \right| \, du_1 du_2 \leq \frac{C \prod_{i=1}^{2} \log(\lambda) + \log(|m_i|)}{\lambda^2}.
\]
and
\[
\left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sin(u_1) \sin(u_1 + m_1 \pi) \sin(u_2) \sin(u_2 + m_2 \pi)}{\lambda} \right| \, du_1 du_2 \leq \frac{C \prod_{i=1}^{2} \log(\lambda) + \log(|m_i|)}{\lambda^2}.
\]
where \(a \to \infty\) as \(\lambda \to \infty\).

PROOF. To simplify the notation in the proof, we’ll prove (46) for \(m > 0\) (the proof for \(m \leq 0\) is identical).

The proof is based on considering the cases that \(|u| \leq \lambda\) and \(|u| > \lambda\) separately. For \(|u| \leq \lambda\) we apply the mean value theorem to the difference \(h(\omega + \frac{2u}{\lambda}) - h(\omega)\) and for \(|u| > \lambda\) we exploit that the integral \(\int_{|u|>|\lambda|} \sin(u) \sin(u + m \pi) \, du\) decays as \(\lambda \to \infty\). We now make these argument precise. We start by partitioning the integral
\[
\int_{-\infty}^{\infty} \sin(u) \sin(u + m \pi) \int_{-\infty}^{b} g(\omega) \left(h(\omega + \frac{2u}{\lambda}) - h(\omega)\right) \, d\omega \, du = I_1 + I_2,
\]
where
\[
I_1 = \int_{|u|>\lambda} \sin(u) \sin(u + m \pi) \int_{-\infty}^{b} g(\omega) \left(h(\omega + \frac{2u}{\lambda}) - h(\omega)\right) \, d\omega \, du
\]
and
\[
I_2 = \int_{|u|<\lambda} \sin(u) \sin(u + m \pi) \int_{-b}^{b} g(\omega) \left(h(\omega + \frac{2u}{\lambda}) - h(\omega)\right) \, d\omega \, du.
\]
We further partition the integral \(I_1 = I_{11} + I_{12} + I_{13}\), where
\[
I_{11} = \int_{|u|<\lambda} \sin(u) \sin(u + m \pi) \int_{-b}^{b} g(\omega) \left(h(\omega + \frac{2u}{\lambda}) - h(\omega)\right) \, d\omega \, du
\]
and \(I_{1} = I_{21} + I_{22} + I_{23}\), where
\[
I_{21} = \int_{|u|>\lambda} \sin(u) \sin(u + m \pi) \int_{-b}^{\min(\{|u|/\lambda, b\})} g(\omega) \left(h(\omega + \frac{2u}{\lambda}) - h(\omega)\right) \, d\omega \, du
\]
and
\[
I_{22} = \int_{|u|<\lambda} \sin(u) \sin(u + m \pi) \int_{-b}^{\min(\{|u|/\lambda, b\})} g(\omega) \left(h(\omega + \frac{2u}{\lambda}) - h(\omega)\right) \, d\omega \, du.
\]
We start by bounding $I_1$. Taking absolutes of $I_{11}$, and using that $h(\omega)$ is absolutely integrable we have

$$|I_{11}| \leq 2\Gamma \int_{\lambda}^{\infty} \frac{\sin^2(u)}{u(u + m\pi)} du,$$

where $\Gamma = \sup_u |g(u)| \int_0^{\infty} |h(u)| du$. Since $m > 0$, it is straightforward to show that $\int_{\lambda}^{\infty} \frac{\sin^2(u)}{u(u + m\pi)} du \leq C\lambda^{-1}$, where $C$ is some finite constant. This implies $|I_{11}| \leq 2CT\lambda^{-1}$. Similarly it can be shown that $|I_{13}| \leq 2CT\lambda^{-1}$. To bound $I_{12}$ we note that

$$|I_{12}| \leq \frac{2\Gamma}{\lambda} \int_{-\lambda - m\pi}^{-\lambda} \frac{\sin^2(u)}{|u + m\pi|} du \leq \frac{2\Gamma}{\lambda} \times \left\{ \begin{array}{ll}
\log \left( \frac{\lambda - m\pi}{\lambda - m\pi} \right) & m\pi < \lambda \\
\log \lambda + \log(m\pi - \lambda) & m\pi > \lambda.
\end{array} \right.$$

Thus, we have $|I_{12}| \leq 2\Gamma\lambda^{-1}[\log \lambda + \log m]$. Altogether, the bounds for $I_{11}, I_{12}, I_{13}$ give

$$|I_1| \leq \frac{C(\log \lambda + \log m)}{\lambda}.$$

To bound $I_2$ we apply the mean value theorem to $h(\omega + \frac{2u}{\lambda}) - h(\omega) = \frac{2u}{\lambda} h'(\omega + \zeta(\omega, u)\frac{2u}{\lambda})$, where $0 \leq \zeta(\omega, u) \leq 1$. Substituting this into $I_{23}$ gives

$$|I_{23}| \leq \frac{2}{\lambda} \int_{-\lambda}^{\lambda} \frac{\sin^2(u)}{|u + m\pi|} \int_{\min(\frac{4u}{\lambda}, 1/\lambda, b)}^{b} \left|h'(\omega + \zeta(\omega, u)\frac{2u}{\lambda})\right| d\omega du.$$

Since the limits of the inner integral are greater than $4u/\lambda$, and the derivative is bounded by $\beta(\omega)$, this means $|h'(\omega + \zeta(\omega, u)\frac{2u}{\lambda})| \leq \max[\beta(\omega), \beta(\omega + \frac{2u}{\lambda})] = \beta(\omega)$. Altogether, this gives

$$|I_{23}| \leq \frac{2}{\lambda} \left( \int_{\min(\frac{4u}{\lambda}, 1/\lambda, b)}^{b} \beta(\omega)d\omega \right) \int_{-\lambda}^{\lambda} \frac{\sin^2(u)}{|u + m\pi|} du \leq \frac{2\Gamma\log(\lambda + m\pi)}{\lambda}.$$

Using the same method we obtain $I_{22} \leq \frac{2\Gamma\log(\lambda + m\pi)}{\lambda}$. Finally, to bound $I_{21}$, we cannot bound $h'(\omega + \zeta(\omega, u)\frac{2u}{\lambda})$ by a monotonic function since $\omega$ and $\omega + \frac{2u}{\lambda}$ can have different signs. Therefore we simply bound $h'(\omega + \zeta(\omega, u)\frac{2u}{\lambda})$ with a constant, this gives

$$|I_{21}| \leq \frac{8C}{\lambda^2} \int_{-\lambda}^{\lambda} \frac{|u|\sin^2(u)}{|u + m\pi|} du \leq \frac{16C}{\lambda}.$$

Altogether, the bounds for $I_{21}, I_{22}, I_{23}$ give

$$|I_2| \leq CT\frac{\log \lambda + \log(m) + \log(\lambda + m\pi)}{\lambda}.$$

Finally, we recall that if $\lambda > 2$ and $m\pi > 2$, then $(\lambda + m\pi) < \lambda m\pi$, thus $\log(\lambda + m\pi) \leq \log \lambda + \log(m\pi)$, Therefore, we obtain (46). The proof of (47) is similar, but avoids some of the awkward details that are required to prove (46).

To prove (48) we note that both $m_1 \neq 0$ and $m_2 \neq 0$ (if one of these values were zero the slower bound given in (46) only holds). Without loss of generality we will prove the result for $m_1 > 0$ and
$m_2 > 0$. We first note since $m_1 \neq 0$ and $m_2 \neq 0$, by orthogonality of the sinc function at integer shifts (see Lemma A.1) we have

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{sinc}(u_1) \text{sinc}(u_1 + m_1 \pi) \text{sinc}(u_2) \text{sinc}(u_2 + m_2 \pi) \int_{-b}^{b} \int_{-b}^{b} g(\omega_1, \omega_2) \times
\left[ h \left( \omega_1 + \frac{2u_1}{\lambda}, \omega_2 + \frac{2u_2}{\lambda} \right) - h(\omega_1, \omega_2) \right] d\omega_1 d\omega_2 du_1 du_2
$$

$$
= \int_{-\infty}^{\infty} \text{sinc}(u_2) \text{sinc}(u_2 + m_2 \pi) \int_{-\infty}^{\infty} \text{sinc}(u_1) \text{sinc}(u_1 + m_1 \pi) \int_{-b}^{b} \int_{-b}^{b} g(\omega_1, \omega_2) \times
h \left( \omega_1 + \frac{2u_1}{\lambda}, \omega_2 + \frac{2u_2}{\lambda} \right) d\omega_1 d\omega_2 du_1 du_2,
$$

(51)

since in the last line $h(\omega_1, \omega_2)$ comes outside the integral over $u_1$ and $u_2$. To further the proof, we note that for $m_2 \neq 0$ we have

$$
\int_{-\infty}^{\infty} \text{sinc}(u_2) \text{sinc}(u_2 + m_2 \pi) \int_{|u_1| > \lambda} \text{sinc}(u_1) \text{sinc}(u_1 + m_1 \pi) \int_{-b}^{b} g(\omega_1, \omega_2) \times
h \left( \omega_1 + \frac{2u_1}{\lambda}, \omega_2 \right) d\omega_1 d\omega_2 du_1 du_2 = 0.
$$

(52)

We use this zero-equality at relevant parts in the proof.

We subtract (52) from (51), and use the same decomposition and notation used in (50) we decompose the integral in (51) over $u_1$ into $\int_{|u_1| > \lambda} + \int_{|u_1| \leq \lambda}$ to give

$$
= \int_{-\infty}^{\infty} \text{sinc}(u_2) \text{sinc}(u_2 + m_2 \pi) \left( \int_{-\infty}^{\infty} \text{sinc}(u_1) \text{sinc}(u_1 + m_1 \pi) \int_{-b}^{b} g(\omega_1, \omega_2) \times
\left[ h \left( \omega_1 + \frac{2u_1}{\lambda}, \omega_2 + \frac{2u_2}{\lambda} \right) - h(\omega_1, \omega_2) \right] d\omega_1 d\omega_2 du_1 du_2
\right)
$$

$$
= \int_{-\infty}^{\infty} \text{sinc}(u_2) \text{sinc}(u_2 + m_2 \pi) \left( \int_{-b}^{b} \left( \int_{-\infty}^{\infty} \text{sinc}(u_1) \text{sinc}(u_1 + m_1 \pi) \int_{-b}^{b} g(\omega_1, \omega_2) \times
\left[ h \left( \omega_1 + \frac{2u_1}{\lambda}, \omega_2 + \frac{2u_2}{\lambda} \right) - h(\omega_1, \omega_2) \right] d\omega_1 d\omega_2 du_1 du_2
\right)
$$

$$
= \int_{-\infty}^{\infty} \text{sinc}(u_2) \text{sinc}(u_2 + m_2 \pi) \int_{-b}^{b} [I_1(u_2, \omega_2) + I_2(u_2, \omega_2)] d\omega_2 du_2 = J_1 + J_2,
$$

where

$$
I_1(u_2, \omega_2) = \int_{|u_1| > \lambda} \text{sinc}(u_1) \text{sinc}(u_1 + m_1 \pi) \times
\int_{-b}^{b} g(\omega_1, \omega_2) \left[ h \left( \omega_1 + \frac{2u_1}{\lambda}, \omega_2 + \frac{2u_2}{\lambda} \right) - h(\omega_1, \omega_2) \right] d\omega_1 du_1
$$

$$
I_2(u_2, \omega_2) = \int_{|u_1| \leq \lambda} \text{sinc}(u_1) \text{sinc}(u_1 + m_1 \pi) \int_{-b}^{b} g(\omega_1, \omega_2) \frac{\partial h}{\partial \omega_1} \left( \omega_1 + \zeta_1(\omega_1, u_1) \frac{2u_1}{\lambda}, \omega_2 + \frac{2u_2}{\lambda} \right) d\omega_1 du_1
$$

and $|\zeta_1(\omega_1, u_1)| \leq 1$. Note that the expression for $I_2(u_2, \omega_2)$ applies the mean value theorem to the difference $h \left( \omega_1 + \frac{2u_1}{\lambda}, \omega_2 + \frac{2u_2}{\lambda} \right) - h(\omega_1 + \frac{2u_1}{\lambda}, \omega_2)$. Next to bound $J_1$ we decompose the outer
integral over $u_2$ into $\int_{|u_2|>\lambda}$ and $\int_{|u_2|\leq \lambda}$ to give $J_1 = J_{11} + J_{12}$, where

\[
J_{11} = \int_{|u_2|>\lambda} \int_{|u_1|>\lambda} \frac{\text{sinc}(u_1)\text{sinc}(u_1 + m_1\pi)\text{sinc}(u_2)\text{sinc}(u_2 + m_2\pi)}{\lambda} \times \\
\int_{-b}^{b} \int_{-b}^{b} g(\omega_1, \omega_2) \left[ h \left( \frac{2u_1}{\lambda}, \omega_1 + \frac{2u_2}{\lambda} \right) - h \left( \frac{2u_1}{\lambda}, \omega_1 \right) \right] d\omega_1 d\omega_2 du_1 du_2 \omega_2
\]

\[
J_{12} = \int_{|u_2|>\lambda} \int_{|u_1|\leq \lambda} \frac{\text{sinc}(u_1)\text{sinc}(u_1 + m_1\pi)}{\lambda} \times \\
\int_{|u_2|\leq \lambda} \text{sinc}(u_2)\text{sinc}(u_2 + m_2\pi) \int_{-b}^{b} g(\omega_1, \omega_2) \frac{\partial h}{\partial \omega_2} \left( \frac{2u_1}{\lambda}, \omega_2 + \frac{2u_2}{\lambda} \right) \right] d\omega_1 d\omega_2 du_1 du_2,
\]

and $|\zeta_2(\omega_2, u_2)| < 1$. Note that the expression for $J_{12}$ is obtained by applying the mean value theorem to $h \left( \frac{2u_1}{\lambda}, \omega_1 + \frac{2u_2}{\lambda} \right) - h \left( \frac{2u_1}{\lambda}, \omega_1 \right)$. Now by using the same methods used to bound $I_1$ and $I_2$ in (50) we can show that $|J_{11}| = O(\lambda^{-2})$, $|J_{12}| = O(\left[ \log(\lambda) + \log|m_2|/\lambda^2 \right])$ and $|J_{21}| = O(\lambda^{-2})$. To bound $J_2$ we again use that $m_2 \neq 0$ and orthogonality of the sinc function to add the additional term $\frac{\partial h}{\partial \omega_1} \left( \frac{2u_1}{\lambda}, \omega_2 + \frac{2u_2}{\lambda} \right) \right] d\omega_1 d\omega_2 du_1 du_2$. Altogether this proves (48).

By decomposing the outer integral of $J_2$ over $u_2$ into $\int_{|u_2|>\lambda}$ and $\int_{|u_2|\leq \lambda}$ we have $J_2 = J_{21} + J_{22}$, where

\[
J_{21} = \int_{|u_2|>\lambda} \int_{|u_1|\leq \lambda} \frac{\text{sinc}(u_2)\text{sinc}(u_2 + m_2\pi)}{\lambda} \int_{|u_1|\leq \lambda} \frac{\text{sinc}(u_1)\text{sinc}(u_1 + m_1\pi)}{\lambda} \times \\
\int_{-b}^{b} g(\omega_1, \omega_2) \left[ \frac{\partial h}{\partial \omega_1} \left( \frac{2u_1}{\lambda}, \omega_2 \right) - \frac{\partial h}{\partial \omega_1} \left( \frac{2u_1}{\lambda}, \omega_2 \right) \right] d\omega_1 d\omega_2 du_1 du_2
\]

and

\[
J_{22} = \int_{|u_2|\leq \lambda} \int_{|u_1|\leq \lambda} \frac{\text{sinc}(u_2)\text{sinc}(u_2 + m_2\pi)}{\lambda} \int_{|u_1|\leq \lambda} \frac{\text{sinc}(u_1)\text{sinc}(u_1 + m_1\pi)}{\lambda} \times \\
\int_{-b}^{b} g(\omega_1, \omega_2) \frac{\partial^2 h}{\partial \omega_1 \partial \omega_2} \left( \frac{2u_1}{\lambda}, \omega_2 + \frac{2u_2}{\lambda} \right) \right] d\omega_1 d\omega_2 du_1 du_2
\]

with $|\zeta_3(\omega_2, u_2)| < 1$. Note that the expression for $J_{22}$ is obtained by applying the mean value theorem to $\frac{\partial^2 h}{\partial \omega_1 \partial \omega_2} \left( \frac{2u_1}{\lambda}, \omega_2 \right) - \frac{\partial h}{\partial \omega_1} \left( \omega_1 + \zeta_3(\omega_1, u_1) \right)$.

Again by using the methods used to bound $I_1$ and $I_2$ in (50) we can show that $|J_{21}| = O(\left[ \log(\lambda) + \log|m_1|/\lambda^2 \right])$ and $|J_{22}| = O(\left[ \log(\lambda) + \log|m_i|/\lambda^2 \right])$. Altogether this proves (48).

The proof of (49) follows exactly the same method used to prove (48) the only difference is that the summand rather than the integral makes the notation more cumbersome. For this reason we omit the details. □
The following result is used in to obtain expression for the fourth order cumulant term (in the case that the spatial random field is not Gaussian). It is used in the proofs of Theorems 3.6 and A.2.

**Lemma C.3** Suppose $h$ is a function which is absolutely integrable and $|h'(\omega)| \leq \beta(\omega)$ (where $\beta$ is a monotonically decreasing function that is absolutely integrable), $m \in \mathbb{Z}$ and $g(\omega)$ is a bounded function. Then we have

\[
\begin{align*}
\int_{\mathbb{R}^3} \text{sinc}(u_1 + u_2 + u_3 + m\pi)\text{sinc}(u_1)\text{sinc}(u_2)\text{sinc}(u_3) & \int_{-a/\lambda}^{a/\lambda} g(\omega) \left[ h \left( \frac{2u_1}{\lambda} - \omega \right) - h(\omega) \right] d\omega du_1 du_2 du_3 \\
& = O \left( \frac{[\log(\lambda)] + \log |m|)^3}{\lambda} \right). \quad (53)
\end{align*}
\]

\[
\begin{align*}
\int_{\mathbb{R}^2} \text{sinc}(u_1 + u_2 + m\pi)\text{sinc}(u_1)\text{sinc}(u_2) & \int_{-a/\lambda}^{a/\lambda} g(\omega) \left[ h \left( \frac{2u_1}{\lambda} - \omega \right) - h(\omega) \right] d\omega du_1 du_2 \\
& = O \left( \frac{[\log(\lambda)] + \log |m|)^3}{\lambda} \right). \quad (54)
\end{align*}
\]

**PROOF.** The proof of (53) is very similar to the proof of Lemma C.2. We start by partitioning the integral over $u_1$ into $\int_{|u_1| \leq \lambda}$ and $\int_{|u_1| > \lambda}$

\[
\begin{align*}
\int_{\mathbb{R}^3} \text{sinc}(u_1 + u_2 + u_3 + m\pi)\text{sinc}(u_1)\text{sinc}(u_2)\text{sinc}(u_3) & \times \\
& \int_{-a/\lambda}^{a/\lambda} g(\omega) \left( h \left( \frac{2u_1}{\lambda} - \omega \right) - h(\omega) \right) d\omega du_1 du_2 du_3 = I_1 + I_2,
\end{align*}
\]

where

\[
\begin{align*}
I_1 &= \\
& \int_{|u_1| > \lambda} \int_{\mathbb{R}^3} \text{sinc}(u_1 + u_2 + u_3 + m\pi)\text{sinc}(u_1)\text{sinc}(u_2)\text{sinc}(u_3) \\
& \times \int_{-a/\lambda}^{a/\lambda} g(\omega) \left( h \left( \frac{2u_1}{\lambda} - \omega \right) - h(\omega) \right) d\omega du_1 du_2 du_3
\end{align*}
\]

\[
\begin{align*}
I_2 &= \\
& \int_{|u_1| \leq \lambda} \int_{\mathbb{R}^3} \text{sinc}(u_1 + u_2 + u_3 + m\pi)\text{sinc}(u_1)\text{sinc}(u_2)\text{sinc}(u_3) \\
& \times \int_{-a/\lambda}^{a/\lambda} g(\omega) \left( h \left( \frac{2u_1}{\lambda} - \omega \right) - h(\omega) \right) d\omega du_1 du_2 du_3.
\end{align*}
\]

Taking absolutes of $I_1$ and using Lemma C.1, equations (42) and (43) we have

\[
|I_1| \leq 4\Gamma \int_{|u_1| > \lambda} |\text{sinc}(u_1 + m\pi)|d\xi_2(u_1) du_1 \leq C \int_{|u| > \lambda} \frac{\log^2(u)}{|u|} \times \frac{|\text{sinc}(u + m\pi)|}{|u + m\pi|} du,
\]

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where \( \Gamma = \sup_\omega |g(\omega)| \int_0^\infty |h(\omega)| d\omega \) and \( C \) is a finite constant (which has absorbed \( \Gamma \)). Decomposing the above integral we have

\[
|I_1| \leq C \int_{|u| > \lambda} \frac{\log^2(u)}{|u|} \left| \frac{\text{sinc}(u + m\pi)}{|u + m\pi|} \right| du = I_{11} + I_{12},
\]

where

\[
I_{11} = \int_{\lambda < |u| \leq \lambda(1 + |m|)} \frac{\log^2(u)}{|u|} \left| \frac{\text{sinc}(u + m\pi)}{|u + m\pi|} \right| du
\]

\[
I_{12} = \int_{|u| > \lambda(1 + |m|)} \frac{\log^2(u)}{|u|} \left| \frac{\text{sinc}(u + m\pi)}{|u + m\pi|} \right| du.
\]

We first bound \( I_{11} \)

\[
I_{11} \leq \frac{\log^2[\lambda(1 + |m|\pi)]}{\lambda} \int_{\lambda < |u| \leq \lambda(1 + |m|)} \left| \frac{\text{sinc}(u + m\pi)}{|u + m\pi|} \right| du
\]

\[
\leq 2C\frac{\log^2[\lambda(1 + |m|\pi)]}{\lambda} \times \log(\lambda + m\pi) = C \left( \frac{\log |m| + \log \lambda^3}{\lambda} \right)
\]

To bound \( I_{12} \) we make a change of variables \( u = \lambda z \), the above becomes

\[
I_{12} \leq C \frac{\int |\log \lambda + \log z|^2}{\lambda} \left| \frac{1}{z(\lambda + \frac{a}{\lambda})} \right| dz = O \left( \frac{\log^2(\lambda)}{\lambda} \right).
\]

Altogether the bounds for \( I_{11} \) and \( I_{12} \) give \( |I_1| \leq C(\log |m| + \log \lambda)^3/\lambda \).

To bound \( I_2 \), just as in Lemma C.2, equation (46), we decompose it into three parts \( I_2 = I_{21} + I_{22} + I_{23} \), where using Lemma C.1, equations (42) and (43) we have the bounds

\[
|I_{21}| \leq \int_{|u| \leq \lambda} |\text{sinc}(u + m\pi)| |\ell_2(u)| \int_{\min(a,4u)/\lambda}^{\min(a,4u)}/\lambda |g(\omega)| \left| h \left( \frac{2u}{\lambda} - \omega \right) - h(-\omega) \right| d\omega du
\]

\[
|I_{22}| \leq \int_{|u| \leq \lambda} |\text{sinc}(u + m\pi)| |\ell_2(u)| \int_{-\min(a,4u)}/\lambda^{-\min(a,4u)}/\lambda |g(\omega)| \left| h \left( \frac{2u}{\lambda} - \omega \right) - h(-\omega) \right| d\omega du
\]

\[
|I_{23}| \leq \int_{|u| \leq \lambda} |\text{sinc}(u + m\pi)| |\ell_2(u)| \int_{\min(a,4u)/\lambda}^{a}/\lambda |g(\omega)| \left| h \left( \frac{2u}{\lambda} - \omega \right) - h(-\omega) \right| d\omega du.
\]

Using the same method used to bound \( |I_{21}|, |I_{22}|, |I_{23}| \) in Lemma C.2, we have \( |I_{21}|, |I_{22}|, |I_{23}| \leq C(\log(\lambda) + \log(|m|))^3/\lambda \). Having bounded all partitions of the integral, we have the result.

The proof of (54) is identical and we omit the details.

\[\square\]

C.1 Lemmas required to prove Lemma A.2 and Theorem A.1

In this section we give the proofs of the three results used in Lemma A.2 (which in turn proves Theorem A.1).

**Lemma C.4** Suppose Assumptions 2.4(ii) and 2.5(b,c) holds. Then for \( r_1, r_2 \in \mathbb{Z} \) we have

\[
|A_1(r_1, r_2) - B_1(r_1 - r_2; r_1)| = O \left( \frac{\log^2(a)}{\lambda} \right)
\]

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PROOF. To obtain a bound for the difference we use Lemma E.1(ii) to give

\[ |A_1(r_1, r_2) - B_1(r_1 - r_2; r_1)| \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=-2a}^{2a} |\text{sinc}(u)\text{sinc}(u - m\pi)\text{sinc}(v)\text{sinc}(v + (m + r_1 - r_2)\pi)| \]

\[ H_m(\frac{2u}{\lambda}, \frac{2v}{\lambda}; r_1) - H_m(\frac{2u}{\lambda}, \frac{2v}{\lambda}; r_1) \]

\[ \leq C/\lambda \]

\[ \leq \frac{C}{\lambda} \sum_{m=-2a}^{2a} \int_{-\infty}^{\infty} |\text{sinc}(u)\text{sinc}(u - m\pi)| du \int_{-\infty}^{\infty} |\text{sinc}(v)\text{sinc}(v + (m + r_1 - r_2)\pi)| dv < \infty \]

\[ \leq \frac{C}{\lambda} \sum_{m=-2a}^{2a} \int_{-\infty}^{\infty} |\text{sinc}(u)\text{sinc}(u - m\pi)| du \text{ (from Lemma C.1(ii))} \]

\[ = O\left(\frac{\log^2 a}{\lambda}\right), \]

thus giving the desired result. \[ \square \]

**Lemma C.5** Suppose Assumptions 2.4(ii) and 2.5(b,c) holds (with 0 \( \leq |r_1|, |r_2| \leq C|a| \)). Then we have

\[ |B_1(s; r) - C_1(s; r)| = O\left(\log^2(a) \left(\frac{\log a}{\lambda} \right)\right). \]

Note, if we relax the assumption on \( r_1, r_2 \) to \( r_1, r_2 \in \mathbb{Z} \) then the above bound requires the additional term \( \log^2(a)[I(r_1 \neq 0) \log |r_1| + I(r_2 \leq 0) \log |r_2|]/\lambda. \)

PROOF. Taking differences, it is easily seen that

\[ B_1(s, r) - C_1(s, r) \]

\[ = \int_{\mathbb{R}^2} \sum_{m=-2a}^{2a} \text{sinc}(u)\text{sinc}(u - m\pi)\text{sinc}(v)\text{sinc}(v + (m + s)\pi) \]

\[ \times \frac{1}{(2\pi)} \int_{2\pi \max(-a,a+m)/\lambda}^{2\pi \min(a,a+m)/\lambda} g(\omega)g(\omega + \omega_m) \left[ f(\omega - \frac{2u}{\lambda})f(\omega + \frac{2v}{\lambda}) - f(\omega) \right] d\omega d\nu \]

\[ = I_1 + I_2 \]

where

\[ I_1 = \int_{\mathbb{R}^2} \sum_{m=-2a}^{2a} \text{sinc}(u)\text{sinc}(u - m\pi)\text{sinc}(v)\text{sinc}(v + (m + s)\pi) \]

\[ \times \frac{1}{(2\pi)} \int_{2\pi \max(-a,a+m)/\lambda}^{2\pi \min(a,a+m)/\lambda} g(\omega)g(\omega + \omega_m) f(\omega + \frac{2v}{\lambda}) \left[ f(\omega - \frac{2u}{\lambda}) - f(\omega) \right] d\omega d\nu \]

\[ = \int_{\mathbb{R}} \sum_{m=-2a}^{2a} \text{sinc}(v)\text{sinc}(v + (m + s)\pi) D_m(v) dv \]
and

\[ I_2 = \int_{R^2} \sum_{m=-2a}^{2a} \text{sinc}(u)\text{sinc}(u - m\pi)\text{sinc}(v)\text{sinc}(v + (m + s)\pi) \]

\[ \times \frac{1}{(2\pi)} \int_{2\pi \text{ max}(-a,-a+m)/\lambda}^{2\pi \text{ min}(a,a+m)/\lambda} g(\omega)g(\omega + \omega_m)f(\omega) \left[ f(\omega + \frac{2v}{\lambda} + \omega_r) - f(\omega + \omega_r) \right] d\omega dv du \]

\[ = \sum_{m=-2a}^{2a} d_m \int_{R} \text{sinc}(u)\text{sinc}(u - m\pi) du \]

with

\[ D_m(v) = \int_{R} \text{sinc}(u)\text{sinc}(u - m\pi) \frac{1}{(2\pi)} \int_{2\pi \text{ max}(-a,-a+m)/\lambda}^{2\pi \text{ min}(a,a+m)/\lambda} g(\omega)g(\omega + \omega_m)f(\omega) \]

\[ \times \left[ f(\omega - \frac{2u}{\lambda}) - f(\omega) \right] d\omega du \]

and

\[ d_m = \int_{R} \text{sinc}(v)\text{sinc}(v + (m + s)\pi) \frac{1}{(2\pi)} \int_{2\pi \text{ max}(-a,-a+m)/\lambda}^{2\pi \text{ min}(a,a+m)/\lambda} g(\omega)g(\omega + \omega_m)f(\omega) \]

\[ \times \left[ f(\omega + \frac{2v}{\lambda} + \omega_r) - f(\omega + \omega_r) \right] d\omega dv. \]

Since the functions \( f(\cdot) \) and \( g(\cdot) \) satisfy the conditions stated in Lemma C.2, the lemma can be used to show that

\[ \max_{|m| \leq a} \sup_v |D_m(v)| \leq C \left( \frac{\log \lambda + \log a}{\lambda} \right) \]

and

\[ \max_{|m| \leq a} |d_m| \leq C \left( \frac{\log \lambda + \log a}{\lambda} \right). \]

Substituting these bounds into \( I_1 \) and \( I_2 \) give

\[ |I_1| \leq C \left( \frac{\log \lambda + \log a}{\lambda} \right) \sum_{m=-2a}^{2a} \int_{-\infty}^{\infty} |\text{sinc}(v)\text{sinc}(v + (m + s)\pi)|dv \]

\[ |I_2| \leq C \left( \frac{\log \lambda + \log a}{\lambda} \right) \sum_{m=-2a}^{2a} \int_{-\infty}^{\infty} |\text{sinc}(u)\text{sinc}(u - m\pi)|du. \]

Therefore, by using Lemma C.1(ii) we have

\[ |I_1| \text{ and } |I_2| = O\left( \frac{\log^2(a) \log a + \log \lambda}{\lambda} \right). \]

Since \( |B_1(s;r) - C_1(s;r)| \leq |I_1| + |I_2| \) this gives the desired result. \( \square \)
D Approximations to the covariance and cumulants of $\widetilde{Q}_{a,\lambda}(g; r)$

In this section, our objective is to obtain bounds for $\text{cum}_4(\widetilde{Q}_{a,\lambda}(g; r_1), \ldots, \widetilde{Q}_{a,\lambda}(g; r_q))$, these results will be used to prove the asymptotic expression for the variance of $\widetilde{Q}_{a,\lambda}(g; r)$ (given in Section A.2) and asymptotic normality of $\widetilde{Q}_{a,\lambda}(g; r)$. Fox and Taqqu (1987), Dahlhaus (1989), Giraitis and Surgailis (1990) (see also Taqqu and Peccati (2011)) have developed techniques for dealing with the cumulants of sums of periodograms of Gaussian (discrete time) time series, and one would have expected that these results could be used here. However, in our setting there are a few differences that we now describe (i) despite the spatial random being Gaussian the locations are randomly sampled, thus the composite process $Z(s)$ is not Gaussian (we can only exploit the Gaussianity when we condition on the location) (ii) the random field is defined over $\mathbb{R}^d$ (not $\mathbb{Z}^d$) (iii) the number of terms in the sums $\widetilde{Q}_{a,\lambda}(\cdot)$ is not necessarily the sample size. Unfortunately, these differences make it difficult to apply the above mentioned results to our setting. Therefore, in this section we consider cumbersome notation we focus on the case that the locations are from a uniform distribution.

As a simple motivation we first consider $\text{var}[\widetilde{Q}_{a,\lambda}(1, 0)]$. By using indecomposable partitions we have

\begin{align*}
\text{var}[\widetilde{Q}_{a,\lambda}(1, 0)] &= \frac{1}{n^4} \sum_{j_1, j_2, j_3, j_4=1}^{n} \sum_{i, j_1 \neq j_2, j_1 \neq j_4}^{a} \text{cov} [Z(s_{j_1})Z(s_{j_2}) \exp(i\omega_{k_1}(s_{j_1} - s_{j_2})), Z(s_{j_3})Z(s_{j_4}) \exp(i\omega_{k_2}(s_{j_3} - s_{j_4}))] \\
&= \frac{1}{n^4} \sum_{j_1, j_2, j_3, j_4=1}^{n} \sum_{i, j_1 \neq j_2, j_1 \neq j_4}^{a} \left( \text{cum}[Z(s_{j_1})e^{i\omega_{k_1} s_{j_1}}, Z(s_{j_2})e^{-i\omega_{k_2} s_{j_2}}] \text{cum}[Z(s_{j_2})e^{-i\omega_{k_1} s_{j_2}}, Z(s_{j_3})e^{i\omega_{k_2} s_{j_3}}] \\
&\quad + \text{cum}[Z(s_{j_1})e^{i\omega_{k_1} s_{j_1}}, Z(s_{j_3})e^{-i\omega_{k_2} s_{j_3}}] \text{cum}[Z(s_{j_3})e^{-i\omega_{k_1} s_{j_3}}, Z(s_{j_4})e^{i\omega_{k_2} s_{j_4}}] \\
&\quad + \text{cum}[Z(s_{j_1})e^{i\omega_{k_1} s_{j_1}}, Z(s_{j_4})e^{-i\omega_{k_2} s_{j_4}}] \text{cum}[Z(s_{j_4})e^{-i\omega_{k_1} s_{j_4}}, Z(s_{j_2})e^{i\omega_{k_2} s_{j_2}}] \right).
\end{align*}

In order to evaluate the covariances in the above we condition on the locations $\{s_j\}$. To evaluate the fourth order cumulant of the above we appeal to a generalisation of the conditional variance method. This expansion was first derived in Brillinger (1969), and in the general setting it is stated as

\begin{align*}
\text{cum}(Y_1, Y_2, \ldots, Y_q) = \sum_\pi \text{cum}[\text{cum}(Y_{\pi_1}|s_1, \ldots, s_q), \ldots, \text{cum}(Y_{\pi_q}|s_1, \ldots, s_q)],
\end{align*}

where the sum is over all partitions $\pi$ of $\{1, \ldots, q\}$ and $\{\pi_1, \ldots, \pi_b\}$ are all the blocks in the partition $\pi$. We use (56) to evaluate $\text{cum}[Z(s_{j_1})e^{i\omega_{k_1} s_{j_1}}, \ldots, Z(s_{j_q})e^{i\omega_{k_q} s_{j_q}}]$, where $Y_i = Z(s_{j_i})e^{i\omega_{k_i}}$ and we condition on the locations $\{s_j\}$. Using this decomposition we observe that because the spatial process is Gaussian, $\text{cum}[Z(s_{j_1})e^{i\omega_{k_1} s_{j_1}}, \ldots, Z(s_{j_q})e^{i\omega_{k_q} s_{j_q}}]$ can only be composed of cumulants of covariances conditioned on the locations. Moreover, if $s_1, \ldots, s_q$ are independent then by using the same reasoning we see that $\text{cum}[Z(s_1)e^{i\omega_{k_1}}, \ldots, Z(s_q)e^{i\omega_{k_q}}] = 0$. Therefore,
\[ \text{cum}[Z(s_{j_1})e^{i\theta_{j_1}}, Z(s_{j_2})e^{-i\theta_{j_2}}, Z(s_{j_3})e^{i\theta_{j_3}}, Z(s_{j_4})e^{-i\theta_{j_4}}] \] will only be non-zero if some elements of \(s_{j_1}, s_{j_2}, s_{j_3}, s_{j_4}\) are dependent. Using these rules we have

\[
\begin{align*}
\text{var}[\tilde{Q}_{\alpha,\lambda}(1,0)] &= \frac{1}{n^4} \sum_{j_1,j_2,j_3,j_4 \in \mathcal{D}_4} \sum_{k_1,k_2=-a}^{a} \left( \text{cum}[Z(s_{j_1})e^{i\theta_{j_1}}, Z(s_{j_3})e^{-i\theta_{j_3}}, Z(s_{j_4})e^{i\theta_{j_4}}] \text{cum}[Z(s_{j_2})e^{-i\theta_{j_2}}, Z(s_{j_3})e^{i\theta_{j_3}}] \right) \\
&+ \frac{1}{n^4} \sum_{j_1,j_2,j_3,j_4 \in \mathcal{D}_3} \sum_{k_1,k_2=-a}^{a} \left( \text{cum}[Z(s_{j_1})e^{i\theta_{j_1}}, Z(s_{j_3})e^{-i\theta_{j_3}}, Z(s_{j_4})e^{i\theta_{j_4}}] \text{cum}[Z(s_{j_2})e^{-i\theta_{j_2}}, Z(s_{j_3})e^{i\theta_{j_3}}] \right) \\
&+ \frac{1}{n^4} \sum_{j_1,j_2,j_3,j_4 \in \mathcal{D}_3} \sum_{k_1,k_2=-a}^{a} \left( \text{cum}[Z(s_{j_1})e^{i\theta_{j_1}}, Z(s_{j_2})e^{-i\theta_{j_2}}, Z(s_{j_3})e^{i\theta_{j_3}}] \text{cum}[Z(s_{j_4})e^{-i\theta_{j_4}}, Z(s_{j_3})e^{i\theta_{j_3}}] \right) \\
&+ \frac{1}{n^4} \sum_{j_1,j_2,j_3,j_4 \in \mathcal{D}_3} \sum_{k_1,k_2=-a}^{a} \left( \text{cum}[Z(s_{j_1})e^{i\theta_{j_1}}, Z(s_{j_2})e^{-i\theta_{j_2}}, Z(s_{j_3})e^{i\theta_{j_3}}] \text{cum}[Z(s_{j_4})e^{-i\theta_{j_4}}, Z(s_{j_3})e^{i\theta_{j_3}}] \right)
\end{align*}
\] (57)

where \(\mathcal{D}_4 = \{j_1, \ldots, j_4 = \text{all } j_s \text{ are different}\}, \mathcal{D}_3 = \{j_1, \ldots, j_4; \text{two } j_s \text{ are the same but } j_1 \neq j_2 \text{ and } j_3 \neq j_4\}\) (noting that by definition of \(\tilde{Q}_{\alpha,\lambda}(1,0)\) more than two elements in \(\{j_1, \ldots, j_4\}\) cannot be the same). We observe that \(|\mathcal{D}_4| = O(n^4)\) and \(|\mathcal{D}_3| = O(n^3)\), where \(|\cdot|\) denotes the cardinality of a set. We will show that the second and third terms are asymptotically negligible with respect to the first term. To show this we require the following lemma.

**Lemma D.1** Suppose Assumptions 2.1, 2.2, 2.4 and 2.5(b,c) hold (note we only use Assumption 2.5(c) to get ‘neater expressions in the proofs’ it is not needed to obtain the same order). Then we have

\[
\sup_{a} \sum_{k_1,k_2=-a}^{a} \text{cum}[Z(s_1)e^{i\theta_{1}}, Z(s_2)e^{-i\theta_{2}}, Z(s_3)e^{i\theta_{3}}, Z(s_4)e^{-i\theta_{4}}] = O(1) \] (58)

\[
\sup_{a} \sum_{k_1,k_2=-a}^{a} \text{cum}[Z(s_1)e^{i\theta_{1}}, Z(s_2)e^{i\theta_{2}}] \text{cum}[Z(s_2)e^{-i\theta_{2}}, Z(s_3)e^{-i\theta_{3}}] = O(1). \] (59)

**PROOF.** To show (58) we use conditional cumulants (see (56)). By using the conditional cumulant
expansion and Gaussianity of $Z(s)$ conditioned on the location we have

$$
\sum_{k_1, k_2 = -a}^{a} \text{cum}[Z(s_1)e^{is_1\omega_{k_1}}, Z(s_2)e^{-is_2\omega_{k_1}}, Z(s_3)e^{-is_3\omega_{k_2}}, Z(s_4)e^{is_4\omega_{k_2}}] = 0
$$

Writing $I_1$ in terms of expectations and using the spectral representation of the covariance we have

$$
I_1 = \sum_{k_1, k_2 = -a}^{a} \left( \mathbb{E}[c(s_1 - s_2)c(s_1 - s_3)e^{is_1(\omega_{k_1} + \omega_{k_2})}e^{-is_3\omega_{k_2}}e^{-is_2\omega_{k_1}}] - \mathbb{E}[c(s_1 - s_2)e^{is_1\omega_{k_1} - is_2\omega_{k_1}}] \times \mathbb{E}[c(s_1 - s_3)e^{is_1\omega_{k_2} - is_3\omega_{k_2}}] \right)
$$

$$
= \frac{1}{(2\pi)^2} \sum_{k_1, k_2 = -a}^{a} \int \int f(x)f(y)\text{sinc}\left(\frac{\lambda}{2}(x+y) + (k_1 + k_2)\right)\text{sinc}\left(\frac{\lambda}{2}x + k_1\pi\right)\text{sinc}\left(\frac{\lambda}{2}y + k_2\pi\right) dxdy - \frac{1}{(2\pi)^2} \sum_{k_1, k_2 = -a}^{a} \int \int f(x)f(y)\text{sinc}\left(\frac{\lambda}{2}x + k_2\pi\right)\text{sinc}\left(\frac{\lambda}{2}x + k_2\pi\right)\text{sinc}\left(\frac{\lambda}{2}y + k_1\pi\right)\text{sinc}\left(\frac{\lambda}{2}y + k_1\pi\right) dxdy
$$

$$
= E_1 - E_2.
$$

To bound $E_1$ we make a change of variables $u = \frac{\lambda x}{2} + k_1\pi, v = \frac{\lambda y}{2} + k_2\pi$, and replace sum with integral (and use Lemma E.1) to give

$$
E_1 = \frac{1}{(2\pi)^2\lambda^2} \int \int \sum_{k_1, k_2 = -a}^{a} f\left(\frac{2u}{\lambda} - \omega_{k_1}\right) f\left(\frac{2v}{\lambda} - \omega_{k_2}\right)\text{sinc}(u+v)\text{sinc}(u)\text{sinc}(v) dudv
$$

$$
= \frac{1}{(2\pi)^2\lambda^2} \int \int \text{sinc}(u+v)\text{sinc}(u)\text{sinc}(v) \left( \int_{-2\pi/\lambda}^{2\pi/\lambda} f\left(\frac{2u}{\lambda} - \omega_{1}\right) f\left(\frac{2v}{\lambda} - \omega_{2}\right) d\omega_{1} d\omega_{2} \right) dudv + O\left(\frac{1}{\lambda}\right).
$$

Let $G(\frac{2u}{\lambda}) = \frac{1}{2\pi} \int_{-2\pi/\lambda}^{2\pi/\lambda} f\left(\frac{2u}{\lambda} - \omega\right) d\omega$, then substituting this into the above and using equation (44) in Lemma C.1 we have

$$
E_1 = \int \int \text{sinc}(u+v)\text{sinc}(u)\text{sinc}(v) G\left(\frac{2u}{\lambda}\right) G\left(\frac{2v}{\lambda}\right) dudv + O\left(\frac{1}{\lambda}\right) = O(1).
$$

To bound $E_2$ we use a similar technique and Lemma C.1(iii) to give $E_2 = O\left(\frac{1}{\lambda}\right)$. Altogether, this gives $I_1 = O(1)$. The same proof can be used to show that $I_2 = O(1)$. Altogether this gives (58).
To bound (59), we observe that if \( k_1 \neq -k_2 \), then 
\[ \sum_{k_1, k_2 = -a}^a \text{cum} \left[ Z(s_1) e^{i \omega_{k_1}}, Z(s_1) e^{i \omega_{k_2}} \right] = 0 \]
otherwise 
\[ \sum_{k_1, k_2 = -a}^a \text{cum} \left[ Z(s_1) e^{i \omega_{k_1}}, Z(s_1) e^{i \omega_{k_2}} \right] = c(0) \].
Using this, (59) can be reduced to
\[
\sum_{k_1, k_2 = -a}^a \text{cum} \left[ Z(s_1) e^{i \omega_{k_1}}, Z(s_1) e^{i \omega_{k_2}} \right] = c(0) \int_{-\infty}^\infty \sum_{k = -a}^a \frac{f(x) \text{sinc} \left( \frac{\lambda x}{2} + k \pi \right) \text{sinc} \left( \frac{\lambda x}{2} + k \pi \right) dx}{2\pi} \quad \text{(change variables using } \omega = \frac{\lambda x}{2} + k \pi) 
\]
\[
= c(0) \int_{-\infty}^\infty \frac{1}{\pi \lambda} \sum_{k = -a}^a f \left( \frac{2\omega}{\lambda} - \omega_k \right) \text{sinc}^2(\omega) d\omega 
\]
\[
= \frac{c(0)}{2\pi^2} \int_{-\infty}^\infty \text{sinc}^2(\omega) \left( \int_{-2\pi a/\lambda}^{2\pi a/\lambda} f \left( \frac{2\omega}{\lambda} - x \right) dx \right) d\omega + O \left( \frac{1}{\lambda} \right) = O(1), 
\]
thus proving (59).

We now derive an expression for \( \text{var}[Q_{a,\lambda}(1,0)] \), by using Lemma D.1 we have
\[
\text{var}[Q_{a,\lambda}(1,0)] = \frac{1}{n^4} \sum_{j_1, j_2, j_3, j_4 \in D_4} \sum_{k_1, k_2 = -a}^a \left( \text{cum} \left[ Z(s_{j_1}) e^{i \omega_{j_1}}, Z(s_{j_3}) e^{-i \omega_{j_3}} \right] \text{cum} \left[ Z(s_{j_2}) e^{-i \omega_{j_2}}, Z(s_{j_4}) e^{i \omega_{j_4}} \right] \right. 
\]
\[+\text{cum} \left[ Z(s_{j_1}) e^{i \omega_{j_1}}, Z(s_{j_4}) e^{i \omega_{j_4}} \right] \text{cum} \left[ Z(s_{j_2}) e^{-i \omega_{j_2}}, Z(s_{j_3}) e^{-i \omega_{j_3}} \right] \left. \right) + O \left( \frac{1}{n} \right). \tag{60} 
\]
In Lemma E.2 we have shown that the covariance terms above are of order \( O(\lambda^{-1}) \), thus dominating the fourth order cumulant terms which is of order \( O(n^{-1}) \) (so long as \( \lambda << n \)).

**Lemma D.2** Suppose that \( \{Z(s); s \in \mathbb{R}^d\} \) is a Gaussian random process and \( \{s_j\} \) are iid random variables. Then the following hold:

(i) As we mentioned above, by using that the spatial process is Gaussian and (56),
\[ \text{cum} \left[ Z(s_{j_1}) e^{i \omega_{j_1}}, \ldots, Z(s_{j_q}) e^{i \omega_{j_q}} \right] \] can be written as the sum of products of cumulants of the spatial covariance conditioned on location. Therefore, it is easily seen that the odd order cumulant
\[ \text{cum}_{2q+1} \left[ Z(s_{j_1}) \exp(i \omega_{k_1}), \ldots, Z(s_{j_{2q+1}} \exp(i \omega_{k_{2q+1}})) \right] = 0 \] for all \( q \) and regardless of \( \{s_j\} \) being dependent or not.

(ii) By the same reasoning given above, we observe that if more than \( (q+1) \) locations \( \{s_j; j = 1, \ldots, 2q\} \) are independent, then
\[ \text{cum}_{2q} \left[ Z(s_{j_1}) \exp(i \omega_{k_1}), \ldots, Z(s_{j_{2q}} \exp(i \omega_{k_{2q}})) \right] = 0. \]
Lemma D.3 Suppose Assumptions 2.1, 2.2 2.4 and 2.5(b) are satisfied, and \( d = 1 \). Then we have

\[
\text{cum}_3[\tilde{Q}_{a,\lambda}(g, r)] = O\left(\frac{\log^2(a)}{\lambda^2}\right)
\]

(61)

with \( \lambda^d/(\log^2(a)n) \to 0 \) as \( \lambda \to \infty \), \( n \to \infty \) and \( a \to \infty \).

**PROOF.** We prove the result for \( \text{cum}_3[\tilde{Q}_{a,\lambda}(1, 0)] \), noting that the proof is identical for general \( g \) and \( r \). We first expand \( \text{cum}_3[\tilde{Q}_{a,\lambda}(1, 0)] \) using indecomposable partitions. Using Lemma D.2(i) we note that the third order cumulant is zero, therefore

\[
\text{cum}_3[\tilde{Q}_{a,\lambda}(1, 0)]
= \frac{1}{n^6} \sum_{j \in \mathcal{D}} \sum_{k_1,k_2,k_3=-a}^{a} \text{cum}\left[Z(s_{j_1})Z(s_{j_2})\exp(i\omega_{k_1}(s_{j_1} - s_{j_2}))\right] Z(s_{j_3})Z(s_{j_4})\exp(i\omega_{k_2}(s_{j_3} - s_{j_4})) Z(s_{j_5})Z(s_{j_6})\exp(i\omega_{k_3}(s_{j_5} - s_{j_6}))
\]

\[
= \frac{1}{n^6} \sum_{j \in \mathcal{D}} \sum_{\pi(2,2,2) \in \mathcal{P}_{2,2,2}} A^j_{2,2,2}(\pi(2,2,2)) + \frac{1}{n^6} \sum_{j \in \mathcal{D}} \sum_{\pi(4,2) \in \mathcal{P}_{4,2}} A^j_{4,2}(\pi(4,2)) + \frac{1}{n^6} \sum_{j \in \mathcal{D}} A^j_6
\]

\[
= B_{2,2,2} + B_{4,2} + B_6,
\]

where \( \mathcal{D} = \{j_1, \ldots, j_6 \in \{1, \ldots, n\} \text{ but } j_1 \neq j_2, j_3 \neq j_4, j_5 \neq j_6\} \), \( A^j_{2,2,2} \) consists of only the product of cumulants of order two and \( \mathcal{P}_{2,2,2} \) is the set of all cumulants of order two from the set of indecomposable partitions of \( \{(1,2), (3,4), (5,6)\} \), \( A^j_{4,2} \) consists of only the product of 4th and 2nd order cumulants and \( \mathcal{P}_{4,2} \) is the set of all 4th order and 2nd order cumulant indecomposable partitions of \( \{(1,2), (3,4), (5,6)\} \), finally \( A^j_6 \) is the 6th order cumulant. Examples of \( A \)'s are given below

\[
A^j_{2,2,2}(\pi(2,2,2),1) = \sum_{k_1,k_2,k_3=-a}^{a} \text{cum}\left[Z(s_{j_1})e^{i\omega_{k_1}}Z(s_{j_2})e^{i\omega_{k_2}}\right] \text{cum}\left[Z(s_{j_3})e^{-i\omega_{k_1}}Z(s_{j_4})e^{i\omega_{k_2}}\right] \times \text{cum}\left[Z(s_{j_5})e^{-i\omega_{k_2}}Z(s_{j_6})e^{-i\omega_{k_3}}\right]
\]

\[
= \sum_{k_1,k_2,k_3=-a}^{a} \exp[i(s_{j_1} - s_{j_3})e^{i\omega_{k_1}} + s_{j_2} + s_{j_4} + s_{j_5} + s_{j_6}]] \exp[i(-s_{j_2} - s_{j_4} + s_{j_5} + s_{j_6} - s_{j_3})]
\]

\[
A^j_{4,2}(\pi(4,2),1) = \sum_{k_1,k_2,k_3=-a}^{a} \text{cum}\left[Z(s_{j_1}) \exp(-i\omega_{k_2})Z(s_{j_3}) \exp(-i\omega_{k_2})Z(s_{j_4}) \exp(-i\omega_{k_2})Z(s_{j_5}) \exp(i\omega_{k_2})Z(s_{j_6}) \exp(i\omega_{k_2})\right]
\]

\[
= \sum_{k_1,k_2,k_3=-a}^{a} \text{cum}\left[Z(s_{j_1}) \exp(i\omega_{k_1})Z(s_{j_2}) \exp(-i\omega_{k_1})Z(s_{j_3}) \exp(-i\omega_{k_1})Z(s_{j_4}) \exp(i\omega_{k_1})Z(s_{j_5}) \exp(i\omega_{k_1})Z(s_{j_6}) \exp(-i\omega_{k_1})\right]
\]

\[
A^j_6 = \sum_{k_1,k_2,k_3=-a}^{a} \text{cum}\left[Z(s_{j_1}) \exp(-i\omega_{k_2})Z(s_{j_2}) \exp(-i\omega_{k_2})Z(s_{j_3}) \exp(-i\omega_{k_2})Z(s_{j_4}) \exp(-i\omega_{k_2})Z(s_{j_5}) \exp(-i\omega_{k_2})Z(s_{j_6}) \exp(-i\omega_{k_2})\right]
\]

where \( j = (j_1, \ldots, j_6) \).
Bound for $B_{222}$

We will show that $B_{222}$ is the leading term in $\text{cum}_3(\tilde{Q}_{a,\lambda}(g;0))$. The set $\mathcal{D}$ is split into four sets, $\mathcal{D}_6$ where all the elements of $\tilde{j}$ are different, and for $3 \leq i \leq 5$, $\mathcal{D}_i$ where $i$ elements in $\tilde{j}$ are different, such that

$$B_{2,2,2} = \frac{1}{n^6} \sum_{i=0}^{3} \sum_{\tilde{j} \in \mathcal{D}_{6-i}} \sum_{\pi(2,2,2) \in \mathcal{P}(2,2,2)} A_{2,2,2}^j(\pi(2,2,2)).$$

We start by bounding the partition given in (62), we later explain how the same bounds can be obtained for other indecomposable partitions in $\mathcal{P}_{2,2,2}$. By using the spectral representation of the covariance and that $|\mathcal{D}_6| = O(n^6)$, it is straightforward to show that

$$\frac{1}{n^6} \sum_{\tilde{j} \in \mathcal{D}_6} A_{2,2,2}^j(\pi(2,2,2),1)$$

$$= \sum_{k_1,k_2,k_3} \frac{c_6}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{[-\lambda/2,\lambda/2]^6} f(x)f(y)f(z) \exp(i\omega_{k_1}(s_1-s_2)) \times$$

$$\exp(i\omega_{k_2}(s_3-s_4)) \exp(i\omega_{k_3}(s_5-s_6))$$

$$\exp(iz(s_1-s_3)) \exp(iz(s_4-s_6)) \exp(i(s_2-s_5)) \prod_{j=1}^{3} ds_{2j-1} ds_2 dx dy dz$$

$$= \frac{c_6}{(2\pi)^3} \sum_{k_1,k_2,k_3} \int_{\mathbb{R}^3} f(x)f(y)f(z) \frac{\lambda x}{2} + k_1 \pi) \frac{\lambda z}{2} - k_1 \pi) \times$$

$$\frac{\lambda y}{2} + k_2 \pi) \frac{\lambda z}{2} - k_2 \pi) \frac{\lambda y}{2} + k_3 \pi) dxdydz,$$

(65)

where $c_6 = n(n-1) \ldots (n-5)/n^6$. By changing variables $x = \frac{\lambda x}{2} + k_1 \pi$, $y = \frac{\lambda y}{2} - k_2 \pi$ and $z = \frac{\lambda z}{2} - k_3 \pi$ we have

$$\frac{1}{n^6} \sum_{\tilde{j} \in \mathcal{D}_6} A_{2,2,2}^j(\pi(2,2,2),1)$$

$$= \frac{c_6}{(2\pi)^3} \sum_{k_1,k_2,k_3} \frac{1}{\lambda^3} \int_{\mathbb{R}^3} f\left(\frac{2x}{\lambda} - \omega_{k_1}\right) f\left(\frac{2y}{\lambda} + \omega_{k_2}\right) f\left(\frac{2z}{\lambda} + \omega_{k_3}\right) \frac{\lambda y}{2} + k_2 \pi) \frac{\lambda z}{2} - k_3 \pi) dxdydz.$$

(66)

In order to understand how this case can generalise to other partitions in $\mathcal{P}_1$, we represent the $k$s inside the sinc function using the the linear equations

$$\begin{pmatrix}
-1 & -1 & 0 \\
-1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
k_1 \\
k_2 \\
k_3
\end{pmatrix}$$

(67)
where we observe that the above is a rank two matrix. Based on this we make the following change of variables \( k_1 = k_1, m_1 = k_2 + k_1 \) and \( m_2 = k_1 - k_3 \), and rewrite the sum as

\[
\frac{1}{n^6} \sum_{j \in D_6} A_{2,2,2}^j(\pi_{(2,2,2)},1)
= \frac{c_6}{(2\pi)^3} \sum_{k_1,m_1,m_2} \frac{1}{\lambda^3} \int_{\mathbb{R}^3} f \left( \frac{2x}{\lambda} - \omega_{k_1} \right) f \left( \frac{2y}{\lambda} + \omega_{m_1-k_1} \right) f \left( \frac{2z}{\lambda} + \omega_{k_1-m_2} \right) \text{sinc}(x)\text{sinc}(y)\text{sinc}(z) \text{sinc}(x-m_1\pi)\text{sinc}(y-(m_2-m_1)\pi)\text{sinc}(z-m_2\pi) \, dx \, dy \, dz
= \frac{c_6}{\lambda^2 \pi^2} \sum_{m_1,m_2} \int_{\mathbb{R}^3} \text{sinc}(x)\text{sinc}(x-m_1\pi)\text{sinc}(y)\text{sinc}(y+(m_1-m_2)\pi)\text{sinc}(z)\text{sinc}(z-m_2\pi) \\
\times \frac{1}{\lambda} \sum_{j_1} f \left( \frac{2x}{\lambda} - \omega_{k_1} \right) f \left( \frac{2y}{\lambda} + \omega_{m_1-k_1} \right) f \left( \frac{2z}{\lambda} - \omega_{k_1-m_1} \right) \, dx \, dy \, dz. \tag{68}
\]

Finally, we apply Lemma C.1(iv) to give

\[
\frac{1}{n^6} \sum_{j \in D_6} A_{2,2,2}^j(\pi_{(2,2,2)},1) = O \left( \frac{\log^2 a}{\lambda^2} \right). \tag{69}
\]

The above only gives the bound for one partition of \( \mathcal{P}_1 \), but we now show that the same bound applies to all the other partitions. Looking back at (66) and comparing with (68), the reason that only one of the three \( \lambda \)'s in the denominator of (66) gets ‘swallowed’ is because the matrix in (67) has rank two. Therefore, there are two independent \( m \) terms in the sinc function of (66), thus by applying Lemma C.1(iv) the sum \( \sum_{m_1,m_2} \) only grows with rate \( O(\log^2 a) \). Moreover, it can be shown that all indecomposable partitions of \( \mathcal{P}_{2,2,2} \) correspond to rank two matrices (for a proof see equation (A.13) in Deo and Chen (2000)). Thus all indecomposable partitions in \( \mathcal{P}_{2,2,2} \) will have the same order, which altogether gives

\[
\frac{1}{n^6} \sum_{j \in D_6} \sum_{\pi_{(2,2,2)} \in \mathcal{P}_1} A_{2,2,2}^j(\pi_{(2,2,2)}) = O \left( \frac{\log^2 a}{\lambda^2} \right).
\]

Now we consider the case that \( j \in D_5 \). In this case, there are two ‘typical’ cases \( j = (j_1, j_2, j_3, j_4, j_1, j_6) \), which gives

\[
A_{2,2,2}^j(\pi_{(2,2,2)},1) = \text{cum} \left[ Z(s_{j_1})e^{i\omega_{j_1}} \omega_{k_1}, Z(s_{j_3})e^{i\omega_{j_3}} \omega_{k_2} \right] \text{cum} \left[ Z(s_{j_2})e^{-i\omega_{j_2}} \omega_{k_1}, Z(s_{j_1})e^{i\omega_{j_1}} \omega_{k_3} \right] \text{cum} \left[ Z(s_{j_4})e^{-i\omega_{j_4}} \omega_{k_2}, Z(s_{j_6})e^{i\omega_{j_6}} \omega_{k_3} \right]
\]

and \( j = (j_1, j_2, j_1, j_4, j_5, j_6) \), which gives

\[
A_{2,2,2}^j(\pi_{(2,2,2)},1) = \text{cum} \left[ Z(s_{j_1})e^{i\omega_{j_1}} \omega_{k_1}, Z(s_{j_1})e^{i\omega_{j_1}} \omega_{k_2} \right] \text{cum} \left[ Z(s_{j_2})e^{-i\omega_{j_2}} \omega_{k_1}, Z(s_{j_5})e^{i\omega_{j_5}} \omega_{k_3} \right] \text{cum} \left[ Z(s_{j_4})e^{-i\omega_{j_4}} \omega_{k_2}, Z(s_{j_6})e^{i\omega_{j_6}} \omega_{k_3} \right].
\]

Using the same method used to bound (69), when \( j = (j_1, j_2, j_3, j_4, j_1, j_6) \) \( A_{2,2,2}^j(\pi_{(2,2,2)},1) = O \left( \frac{\log^2 a}{\lambda^2} \right) \). However, when \( j = (j_1, j_2, j_1, j_4, j_5, j_6) \) we use the same proof used to prove (59) to
we have $A^j_{2,2,2}(\pi(2,2,2),1) = O(\frac{1}{n})$. As we get similar expansions for all $j \in \mathcal{D}_5$ and $|\mathcal{D}_5| = O(n^5)$ we have

$$
\frac{1}{n^6} \sum_{j \in \mathcal{D}_5} \sum_{\pi(2,2,2) \in \mathcal{P}_{2,2,2}} A^j_{2,2,2}(\pi(2,2,2)) = O\left(\frac{1}{\lambda n} + \frac{\log^2(a)}{n \lambda^2}\right).
$$

Similarly we can show that

$$
\frac{1}{n^6} \sum_{j \in \mathcal{D}_4} \sum_{\pi(2,2,2) \in \mathcal{P}_{2,2,2}} A^j_{2,2,2}(\pi(2,2,2)) = O\left(\frac{1}{\lambda n^2} + \frac{\log^2(a)}{n^2 \lambda^2}\right).
$$

and

$$
\frac{1}{n^6} \sum_{j \in \mathcal{D}_3} \sum_{\pi(2,2,2) \in \mathcal{P}_{2,2,2}} A^j_{2,2,2}(\pi(2,2,2)) = O\left(\frac{1}{n^3} + \frac{\log^2(a)}{n^3 \lambda^2}\right).
$$

Therefore, if $n >> \lambda/\log^2(a)$ we have $B_{2,2,2} = O(\frac{\log^2(a)}{\lambda^2})$.

**Bound for $B_{1,2}$**

To bound $B_{1,2}$ we consider the ‘typical’ partition given in (63). Since $A^j_{1,2}(\pi(4,2,1))$ involves fourth order cumulants by Lemma D.2(ii) it will be zero in the case that the $j$ are all different. Therefore, only a maximum of five terms in $j$ can be different, which gives

$$
\frac{1}{n^6} \sum_{j \in \mathcal{D}} A^j_{1,2}(\pi(4,2,1)) = \frac{1}{n^6} \sum_{i=1}^{3} \sum_{j \in \mathcal{D}_{6-i}} A^j_{1,2}(\pi(4,2,1)).
$$

We will show that for $j \in \mathcal{D}_5$, $A^j_{1,2}(\pi(4,2,1))$ will not be as small as $O(\log^2(a)/\lambda^2)$, however, this will be compensated by $|\mathcal{D}_5| = O(n^5)$ (noting that $|\mathcal{D}_6| = O(n^6)$). Let $j = (j_1, j_2, j_3, j_4, j_5, j_6)$, then expanding the fourth order cumulant in $A^j_{1,2}(\pi(4,2,1))$ and using conditional cumulants (see (56)) we have

$$
A^j_{1,2}(\pi(4,2,1)) = \sum_{k_1, k_2, k_3 = -a}^a \text{cum}[Z(s_{j_4}) \exp(-is_{j_4} \omega_{k_2}), Z(s_{j_5}) \exp(-is_{j_5} \omega_{k_3})] \\
+ \sum_{k_1, k_2, k_3 = -a}^a \text{cum}[Z(s_{j_1}) \exp(is_{j_1} \omega_{k_1}), Z(s_{j_2}) \exp(-is_{j_2} \omega_{k_2}), Z(s_{j_3}) \exp(is_{j_3} \omega_{k_2}), Z(s_{j_5}) \exp(is_{j_5} \omega_{k_3})] \\
+ \sum_{k_1, k_2, k_3 = -a}^a \left\{ \text{cum}\left[c(s_1 - s_2) e^{i \omega_{k_1}}(s_1 - s_2)s_1, c(s_3 - s_4) e^{i \omega_{k_2} + i \omega_{k_3}} \right] \\
+ \text{cum}\left[c(s_1 - s_3) e^{i s_1 \omega_{k_1} + i s_1 \omega_{k_2}}, c(s_1 - s_2) e^{i s_1 \omega_{k_2} - i s_2 \omega_{k_1}} \right] \\
+ \text{cum}\left[c(0) e^{i s_1 \omega_{k_1} + i s_1 \omega_{k_3}}, c(s_2 - s_3) e^{-i s_2 \omega_{k_3} + i s_3 \omega_{k_1}} \right] \right\} E[c(s_4 - s_6) e^{-i s_4 \omega_{k_3} - i s_6 \omega_{k_3}}] = A^j_{1,2}(\pi(4,2,1), \Omega_1) + A^j_{1,2}(\pi(4,2,1), \Omega_2) + A^j_{1,2}(\pi(4,2,1), \Omega_3),
$$

(70)
where we use the notation $\Omega$ to denote the partition of the fourth order cumulant into its conditional cumulants. To bound each term we expand the covariances as expectations, this gives

$$A_{i,2}^{(4,2),1}(\pi_{(4,2),1}, \Omega_1) = \sum_{k_1,k_2,k_3=-a} \left\{ \mathbf{E}[c(s_1 - s_2)c(s_3 - s_1)e^{i\omega_{k_1}(s_1 - s_2)}e^{i\omega_{k_2} + i\omega_{k_3}}]\mathbf{E}[c(s_4 - s_6)e^{-i\omega_{k_2} - i\omega_{k_3}}] \right\}

E[c(s_1 - s_2)e^{i\omega_{k_1}(s_1 - s_2)}]E[c(s_3 - s_1)e^{i\omega_{k_2} + i\omega_{k_3}}]E[c(s_4 - s_6)e^{-i\omega_{k_2} - i\omega_{k_3}}] \right\}

= A_{i,2}^{(4,2),1}(\pi_{(4,2),1}, \Omega_1, \Pi_1) + A_{i,2}^{(4,2),1}(\pi_{(4,2),1}, \Omega_1, \Pi_2),$$

where we use the notation $\Pi$ to denote the expansion of the cumulants of the spatial covariances expanded into expectations. To bound $A_{i,2}^{(4,2),1}(\pi_{(4,2),1}, \Omega_1, \Pi_1)$ we use the spectral representation theorem to give

$$A_{i,2}^{(4,2),1}(\pi_{(4,2),1}, \Omega_1, \Pi_1) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \sum_{k_1,k_2,k_3=-a} f(x)f(y)f(z)\text{sinc}\left(\frac{\lambda(x - y)}{2} + (k_1 + k_3)\pi\right) \text{sinc}\left(\frac{\lambda x}{2} + k_1\pi\right) \times

\text{sinc}\left(\frac{\lambda y}{2} + k_2\pi\right) \text{sinc}\left(\frac{\lambda z}{2} - k_2\pi\right) \text{sinc}\left(\frac{\lambda z}{2} + k_3\pi\right) dx dy dz.$$

By changing variables

$$A_{i,2}^{(4,2),1}(\pi_{(4,2),1}, \Omega_1, \Pi_1) = \frac{1}{\pi^3\lambda^3} \sum_{k_1,k_2,k_3=-a} \int_{\mathbb{R}^3} f\left(\frac{2u}{\lambda} - \omega_{k_1}\right) f\left(\frac{2v}{\lambda} - \omega_{k_2}\right) f\left(\frac{2w}{\lambda} - \omega_{k_3}\right) \times

\text{sinc}(u - v + (k_2 + k_3)\pi)\text{sinc}(u)\text{sinc}(v)\text{sinc}(w)\text{sinc}(w - (k_2 + k_3)\pi)dw dv dw.$$

Just as in the bound for $B_{2,2,2}$, we represent the $k$s inside the sinc function as a set of linear equations

$$\begin{pmatrix}
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
k_1 \\
k_2 \\
k_3
\end{pmatrix},
(71)$$

observing the matrix has rank one. We make a change of variables $m = k_2 + k_3$, $k_1 = k_1$ and $k_2 = k_2$ to give

$$A_{i,3}^{(4,2),1}(\pi_{(4,2),1}, \Omega_1, \Pi_1) = \frac{1}{\pi^3\lambda} \int_{\mathbb{R}^3} \sum_{k_1,k_2} f\left(\frac{2u}{\lambda} - \omega_{k_1}\right) f\left(\frac{2v}{\lambda} - \omega_{k_2}\right) f\left(\frac{2w}{\lambda} - \omega_{m-k_2}\right) \times

\sum_{m} \text{sinc}(u - v + m\pi)\text{sinc}(u)\text{sinc}(v)\text{sinc}(w)\text{sinc}(w - m\pi)dw dv dw,

= \frac{2^3}{\lambda} \int_{\mathbb{R}^3} \sum_{m} G_{\lambda,m} \left(\frac{2u}{\lambda}, \frac{2v}{\lambda}, \frac{2w}{\lambda}\right) \text{sinc}(u - v + m\pi)\text{sinc}(u)\text{sinc}(v)\text{sinc}(w)\text{sinc}(w - m\pi)dw dv dw,$$
where \( G_{\lambda,m}(\frac{2u}{\lambda}, \frac{2v}{\lambda}, \frac{2w}{\lambda}) = \sum_{k_1,k_2} \frac{1}{\lambda^2} f(\frac{2u}{\lambda} - \omega_{k_1}) f(\frac{2v}{\lambda} - \omega_{k_2}) f(\frac{2w}{\lambda} - \omega_{m-k_2}). \) Taking absolutes gives
\[
|A_{4,2}^j(\pi_{(4,2),1}, \Omega_1, \Pi_1)| \leq C \frac{1}{\lambda} \int_{\mathbb{R}^3} \sum_m |\text{sinc}(u - v + m\pi) \text{sinc}(w - m\pi) \text{sinc}(u) \text{sinc}(v) \text{sinc}(w)| \, dudvdw
\]
Since the above contains \( m \) in the sinc function we use Lemma C.1(i), equations (42) and (43), to show
\[
|A_{4,2}^j(\pi_{(4,2),1}, \Omega_1, \Pi_1)| \leq C \frac{1}{\lambda} \int_{\mathbb{R}^3} \sum_m |\text{sinc}(u - v + m\pi) \text{sinc}(w - m\pi) \text{sinc}(u) \text{sinc}(v) \text{sinc}(w)| \, dudvdw
\]
\[
\leq C \frac{1}{\lambda} \sum m \ell_1(m\pi) \ell_2(m\pi) = O(\lambda^{-1}),
\]
where the functions \( \ell_1(\cdot) \) and \( \ell_2(\cdot) \) are defined in Lemma C.1(i). Thus \( |A_{4,2}^j(\pi_{(4,2),1}, \Omega_1, \Pi_1)| = O(\frac{1}{\lambda}). \)

We use the same method used to bound (69) to show that \( |A_{4,2}^j(\pi_{(4,2),1}, \Omega_1, \Pi_2)| = O(\frac{\log^2(a)}{\lambda^2}) \) and \( |A_{4,2}^j(\pi_{(4,2),1}, \Omega_2)| = O(\frac{1}{\lambda}). \) Furthermore, it is straightforward to see that by the independence of \( s_1 \) and \( s_2 \) and \( s_3 \) that \( A_{4,2}^j(\pi_{(4,2),1}, \Omega_3) = 0 \) (recalling that \( A_{4,2}^j(\pi_{(4,2),1}, \Omega_3) \) is defined in equation (70)). Thus altogether we have for \( j = (j_1,j_2,j_3,j_4,j_5,j_6) \) and the partition \( \pi_{(4,2),1} \), that \( |A_{4,2}^j(\pi_{(4,2),1})| = O(\frac{1}{\lambda}). \) However, it is important to note for all other \( j \in \mathcal{D}_5 \) and partitions in \( \mathcal{P}_{4,2} \) the same method will lead to a similar decomposition given in (70) and the rank one matrix given in (71). The rank one matrix means one ‘free’ \( m \) in the sinc functions and this \( |A_{4,2}^j(\pi_{4,2})| = O(\frac{1}{\lambda}) \) for all \( j \in \mathcal{D}_5 \) and \( \pi_{4,2} \in \mathcal{P}_{4,2} \). Thus, since \( |\mathcal{D}_5| = O(n^5) \) we have
\[
\frac{1}{n^6} \sum_{j \in \mathcal{D}_5} \sum_{\pi_{4,2} \in \mathcal{P}_{4,2}} A_{4,2}^j(\pi_{2}) = O \left( \frac{1}{\lambda n} + \frac{\log^2(a)}{\lambda^2 n} \right) = O \left( \frac{\log^2(a)}{\lambda^2} \right)
\]
if \( n >> \lambda/\log^2(a) \) (ie. \( \frac{\lambda}{n \log^2(a)} \rightarrow 0 \)). For \( j \in \mathcal{D}_4 \) and \( j \in \mathcal{D}_3 \) we use the same argument, noting that the number of free \( m \)'s in the sinc functions goes down but to compensate, \( |\mathcal{D}_4| = O(n^4) \) and \( |\mathcal{D}_3| = O(n^3) \). Therefore, if \( n >> \log^2(a)/\lambda \), then
\[
B_{4,2} = \frac{1}{n^6} \sum_{i=1}^{3} \sum_{j \in \mathcal{D}_{6-i}} \sum_{\pi_{4,2} \in \mathcal{P}_{4,2}} A_{4,2}^j(\pi_{2}) = O \left( \frac{\log^2(a)}{\lambda^2} \right).
\]

**Bound for \( B_6 \)**

Finally, we bound \( B_6 \). By using Lemma D.2(ii) we observe that \( A_{6}^j(k_1,k_2,k_3) = 0 \) if more than four elements of \( j \) are different. Thus
\[
B_6 = \frac{1}{n^6} \sum_{i=1}^{3} \sum_{j \in \mathcal{D}_{6-i}} A_{6}^j.
\]
We start by considering the case that \( \mathbf{j} = (j_1, j_2, j_1, j_4, j_1, j_6) \) (three elements in \( \mathbf{j} \) are the same), then by using conditional cumulants we have

\[
A^j_6 = \sum_{k_1, k_2, k_3 = -a}^{a} \sum_{\Omega \in \mathcal{R}} A^j_6(\Omega),
\]

where \( \mathcal{R} \) is the set of all pairwise partitions of \( \{1, 2, 4, 1, 6\} \), for example

\[
A^j_6(\Omega_1) = \sum_{k_1, k_2, k_3 = -a}^{a} \sum_{\Omega \in \mathcal{R}} A^j_6(\Omega),
\]

We will first bound the above and then explain how this generalises to the other \( \Omega \in \mathcal{R} \) and \( \mathbf{j} \in \mathcal{D}_4 \). Expanding the above third order cumulant in terms of expectations gives

\[
A^j_6(\Omega_1)
= \sum_{k_1, k_2, k_3 = -a}^{a} \left\{ E[c(s_1 - s_2)e^{i\omega_k_1(s_1 - s_2)}c(s_1 - s_4)e^{i\omega_k_2(s_1 - s_4)}c(s_1 - s_6)e^{i\omega_k_3(s_1 - s_6)}] - 
E[c(s_1 - s_2)e^{i\omega_k_1(s_1 - s_2)}]E[c(s_1 - s_4)e^{i\omega_k_2(s_1 - s_4)}c(s_1 - s_6)e^{i\omega_k_3(s_1 - s_6)}] - 
E[c(s_1 - s_2)e^{i\omega_k_1(s_1 - s_2)}c(s_1 - s_4)e^{i\omega_k_2(s_1 - s_4)}]E[c(s_1 - s_6)e^{i\omega_k_3(s_1 - s_6)}] - 
E[c(s_1 - s_2)e^{i\omega_k_1(s_1 - s_2)}c(s_1 - s_4)e^{i\omega_k_2(s_1 - s_4)}]E[c(s_1 - s_6)e^{i\omega_k_3(s_1 - s_6)}] - 
2E[c(s_1 - s_2)e^{i\omega_k_1(s_1 - s_2)}]E[c(s_1 - s_4)e^{i\omega_k_2(s_1 - s_4)}]E[c(s_1 - s_6)e^{i\omega_k_3(s_1 - s_6)}] \right\}
= \sum_{\ell = 1}^{5} A^j_6(\Omega_1, \Pi_\ell).
\]

We observe that for \( 2 \leq \ell \leq 5 \), \( A^j_6(\Omega_1, \Pi_\ell) \) resembles \( A^j_4(\pi_{4, 2, 1}, \Omega_1, \Pi_1) \) defined in (71), thus the same proof used to bound the terms in (71) can be used to show that for \( 2 \leq \ell \leq 5 \), \( A^j_6(\Omega_1, \Pi_\ell) = O(\frac{1}{\chi}) \).

However, the first term \( A^j_6(\Omega_1, \Pi_1) \) involves just one expectation, and is not included in the previous cases. By using the spectral representation theorem we have

\[
A^j_6(\Omega_1, \Pi_1)
= \sum_{k_1, k_2, k_3 = -a}^{a} E[c(s_1 - s_2)e^{i\omega_k_1(s_1 - s_2)}c(s_1 - s_4)e^{i\omega_k_2(s_1 - s_4)}c(s_1 - s_6)e^{i\omega_k_3(s_1 - s_6)}]
= \frac{1}{(2\pi)^3} \sum_{k_1, k_2, k_3 = -a}^{a} \int \int \int f(x)f(y)f(z) \sin\left(\frac{\lambda(x + y + z)}{2} + (k_1 + k_2 + k_3)\pi\right) \sin\left(\frac{\lambda y}{2} + k_2\pi\right) \sin\left(\frac{\lambda z}{2} + k_3\pi\right) dxdydz
= \frac{2^3}{(2\pi)^3 \lambda^3} \sum_{k_1, k_2, k_3 = -a}^{a} \int \int \int f(\frac{2u}{\lambda} - \omega_k_1)f(\frac{2v}{\lambda} - \omega_k_2)f(\frac{2w}{\lambda} - \omega_k_3) \times \sin(u + v + w) \sin(u) \sin(v) \sin(w) duduwdw.
\]

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It is obvious that the ks within the sinc function correspond to a rank zero matrix, and thus $A_6(\Omega_1, \Pi_1) = O(1)$. Therefore, $A_6^j(\Omega_1) = O(1)$. A similar bound holds for all $j \in D_4$, this we have
\[
\frac{1}{n} \sum_{j \in D_4} A_6^j(\Omega_1) = O\left(\frac{1}{n^2}\right),
\]
since $|D_4| = O(n^4)$. Indeed, the same argument applies to the other partitions $\Omega$ and $j \in D_3$, thus altogether we have
\[
B_6 = \frac{1}{n} \sum_{\Omega \in R} \sum_{i=2}^3 \sum_{j \in D_{6-i}} A_6^j(\Omega) = O\left(\frac{1}{n^2}\right).
\]

Altogether, using the bounds derived for $B_{2,2}, B_{4,2}$ and $B_6$ we have
\[
\text{cum}_4(\tilde{Q}_{a,\lambda}(1,0)) = O\left(\frac{\log^2(a)}{\lambda^2} + \frac{1}{n\lambda} + \frac{\log^2(a)}{\lambda^2 n} + \frac{1}{n^2}\right) = O\left(\frac{\log^2(a)}{\lambda^2}\right),
\]
where the last bound is due to the conditions on $a, n$ and $\lambda$. This gives the result. \(\Box\)

We now generalize the above results to higher order cumulants.

**Lemma D.4** Suppose Assumptions 2.1, 2.2, 2.4 and 2.5(b) are satisfied, and $d = 1$. Then for $q \geq 3$ we have
\[
\text{cum}_q[\tilde{Q}_{a,\lambda}(g, r)] = O\left(\frac{\log^2(q-2)(a)}{\lambda^{q-1}}\right),
\]
(72)
\[
\text{cum}_q[\tilde{Q}_{a,\lambda}(g_1), \ldots, \tilde{Q}_{a,\lambda}(g_q)] = O\left(\frac{\log^2(q-2)(a)}{\lambda^{q-1}}\right)
\]
and in the case $d > 1$, we have
\[
\text{cum}_q[\tilde{Q}_{a,\lambda}(g, r_1), \ldots, \tilde{Q}_{a,\lambda}(g, r_q)] = O\left(\frac{\log^2(q-2)(a)}{\lambda^{d(q-1)}}\right)
\]
with $\lambda^d/(\log^2(a)n) \to 0$ as $\lambda \to \infty$, $n \to \infty$ and $a \to \infty$.

**PROOF.** The proof essentially follows the same method used to bound the second and third cumulants. We first prove (72). To simplify the notation we prove the result for $g = 1$ and $r = 0$, noting that the proof in the general case is identical. Expanding out $\text{cum}_q[\tilde{Q}_{a,\lambda}(1,0)]$ and using indecomposable partitions gives
\[
\text{cum}_q[\tilde{Q}_{a,\lambda}(1,0)]
= \frac{1}{n^{2q}} \sum_{j_1, \ldots, j_{2q} \in D_{k_1, \ldots, k_{2q}}} \sum_a \text{cum} \left[ Z(s_{j_1})Z(s_{j_2})e^{i\omega_{k_1}(s_{j_1}-s_{j_2})}, \ldots, Z(s_{j_{2q-1}})Z(s_{j_{2q}})e^{i\omega_{k_q}(s_{j_{2q-1}}-s_{j_{2q}})} \right]
= \frac{1}{n^{2q}} \sum_{\pi} \sum_{b \in B_{q}} \sum_{\pi_{2q} \in \mathcal{P}_{2b}} A_{2b}^j(\pi_{2b})
\]

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By a change of variables we get identical arguments to those used for \( \text{var}(2A_\phi) \) pairwise decomposition therefore cumulant \( \text{cum}((62)-(64)) \). Let \( t \) be a sequence of positive integers which sums to \( q \). For example, if \( q = 3 \), then one example of an element of \( B_3 \) is \( \bar{b} = (1, 1, 1) \) and \( P_{2b} = P_{(2,2,2)} \) corresponds to all pairwise indecomposable partitions of \( \{(1, 2), (3, 4), \ldots, (2q-1, 2q)\} \). Finally, \( A_{2q}^j(\pi_{2q}) \) corresponds to the product of one indecomposable partition of the cumulant \( \text{cum}[Z(s_{j_1})e^{i\omega_{s_1}}(s_{j_1})] \), where the cumulants are of order \( 2b_1, 2b_2, \ldots, 2b_m \) (examples, in the case \( q = 3 \) are given in equation (62)-(64)). Let

\[
B_{2q} = \frac{1}{n_{2q}} \sum_{j \in D} \sum_{\pi_{2q} \in P_{2q}} A_{2q}^j(\pi_{2q}),
\]

therefore \( \text{cum}_q[\bar{Q}_a, \lambda(1, 0)] = \sum_{\bar{b} \in B_3} B_{2\bar{b}} \).

Just as in the proof of Lemma D.3, we will show that under the condition \( n > > \lambda/\log^2(a) \), the pairwise decomposition \( B_{2,\ldots,2} \) is the denominating term. We start with a ‘typical’ decomposition \( \pi_{(2,\ldots,2),1} \in P_{2,\ldots,2} \),

\[
A_{2,\ldots,2}^j(\pi_{(2,\ldots,2),1}) = \sum_{k_1,\ldots,k_q = -a}^a \text{cum}[Z(s_1)e^{i\omega_{s_1}}, Z(s_{2q})e^{i\omega_{s_{2q}}}] \prod_{c=1}^{q-1} \text{cum}[Z(s_{2c})e^{-is_{2c}\omega_{k_c}}, Z(s_{2c+1})e^{is_{2c+1}\omega_{k_{c+1}}}].
\]

and

\[
\frac{1}{n_{2q}} \sum_{j \in D} A_{2,\ldots,2}^j(\pi_{(2,\ldots,2),1}) = \frac{1}{n_{2q}} \sum_{i=0}^{q-1} \sum_{j \in D_{2q-i}} A_{2,\ldots,2}^j(\pi_{(2,\ldots,2),1}),
\]

where \( D_{2q} \) denotes the set where all elements of \( j \) are different and \( D_{2q-i} \) denotes the set that \( (2q-i) \) elements in \( j \) are different. We first consider the case that \( j = (1, 2, \ldots, 2q) \in D_{2q} \). Using identical arguments to those used for \( \text{var}[\bar{Q}_a, \lambda(1, 0)] \) and \( \text{cum}_3[\bar{Q}_a, \lambda(1, 0)] \) we can show that

\[
A_{2,\ldots,2}^j(\pi_{(2,\ldots,2),1}) = \sum_{k_1,\ldots,k_q = -a}^a \int_{\mathbb{R}^q} f(x_q) \text{sinc} \left( \frac{2x_q}{\lambda} + k_1\pi \right) \text{sinc} \left( \frac{2x_q}{\lambda} + k_q\pi \right) \times
\]

\[
\prod_{c=1}^{q-1} f(x_c) \text{sinc} \left( \frac{2x_c}{\lambda} - k_c\pi \right) \text{sinc} \left( \frac{2x_c}{\lambda} - k_{c+1}\pi \right) \prod_{c=1}^q dx_c.
\]

By a change of variables we get

\[
A_{2,\ldots,2}^j(\pi_{(2,\ldots,2),1}) = \frac{1}{\lambda^q} \sum_{k_1,\ldots,k_q = -a}^a \int_{\mathbb{R}^q} \prod_{c=1}^{q-1} f \left( \frac{\lambda u_c}{2} + \omega_c \right) \text{sinc}(u_c)\text{sinc}(u_c - (k_{c+1} - k_c)\pi)
\]

\[
\times f \left( \frac{\lambda u_q}{2} - \omega_1 \right) \text{sinc}(u_q)\text{sinc}(u_q + (k_q - k_1)\pi) \prod_{c=1}^q dx_c.
\]
As in the proof of the third order cumulant we can rewrite the ks in the above as a matrix equation
\[
\begin{pmatrix}
-1 & 1 & 0 & \ldots & 0 & 0 \\
0 & -1 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & 0 & \ldots & \ldots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
k_1 \\
k_2 \\
\vdots \\
k_q
\end{pmatrix},
\]
noting that that above is a \((q - 1)\)-rank matrix. Therefore applying the same arguments that were used in the proof of cum\(_3[\tilde{Q}_{a,\lambda}(1, 0)]\) and also Lemma C.1(iii) we can show that \(A^j_{2,\ldots,2}(\pi(2,\ldots,2),1) = O\left(\frac{\log^2(q-2)(a)}{\lambda^{q-1}}\right)\). Thus for \(\tilde{j} \in D_{2q}\) we have \(\frac{1}{n^{2q}} \sum_{\tilde{j} \in D_{2q}} A^j_{2,\ldots,2}(\pi(2,\ldots,2),1) = O\left(\frac{\log^2(q-2)(a)}{\lambda^{q-1}}\right)\).

In the case that \(\tilde{j} \in D_{2q-1}\) \((2q-1)\)-terms in \(\tilde{j}\) are different) by using the same arguments as those used to bound \(A_{2,2,2}\) (in the proof of cum\([Q_{a,\lambda}(1, 0)]\)) we have \(\frac{1}{n^{2q}} \sum_{\tilde{j} \in D_{2q-1}} A^j_{2,\ldots,2}(\pi(2,\ldots,2),1) = O\left(\frac{\log^2(2)(a)}{\lambda^{q-2}}\right)\), similarly if \((2q-2)\)-terms in \(\tilde{j}\) are different, then \(\frac{1}{n^{2q}} \sum_{\tilde{j} \in D_{2q-2}} A^j_{2,\ldots,2}(\pi(2,\ldots,2),1) = O\left(\frac{\log^2(2)(a)}{\lambda^{q-3}}\right)\) and so forth. Therefore, since \(|D_{2q-1}| = O(n^{2q-i})\) we have
\[
\frac{1}{n^{2q}} \sum_{\tilde{j} \in D_{2q-1}} |A^j_{2,\ldots,2}(\pi(2,\ldots,2),1)| \leq C \sum_{i=0}^q \frac{\log^2(q-2-i)(a)}{\lambda^{q-1-i} n^i}.
\]

Now by using that \(n >> \lambda/\log^2(a)\) we have
\[
\frac{1}{n^{2q}} \sum_{\tilde{j} \in D_{2q-1}} |A^j_{2,\ldots,2}(\pi(2,\ldots,2),1)| = O\left(\frac{\log^2(q-2)(a)}{\lambda^{q-1}}\right).
\]

The same argument holds for every other second order cumulant indecomposable partition, because the corresponding matrix will always have rank \((q - 1)\) in the case that \(\tilde{j} \in D_{2q}\) or for \(\tilde{j} \in D_{2q-1}\) and the dependent \(s_j\)’s lie in different cumulants (see Deo and Chen (2000)), thus \(B_{2,\ldots,2} = O\left(\frac{\log^2(q-1)(a)}{\lambda^{q-2}}\right)\).

Now, we bound the other extreme \(B_{2q}\). Using the conditional cumulant expansion (56) and noting that cum\(_{2q}[Z(s_{j_1})^e_{i \omega_{k_1}}, \ldots, Z(s_{j_{2q}}) e^{-i\omega_{k_q}}]\) is non-zero, only when at most \((q + 1)\) elements of \(\tilde{j}\) are different we have
\[
B_{2q} = \frac{1}{n^{2q}} \sum_{\tilde{j} \in D_{2q}} \sum_{0 \leq a \leq a} \text{cum}_{2q}[Z(s_{j_1}) e^{i \omega_{k_1}}, \ldots, Z(s_{j_{2q}}) e^{-i\omega_{k_q}}]
\]
\[
= \frac{1}{n^{2q}} \sum_{\tilde{j} \in D_{2q-1}} \sum_{\Omega \in R_{2q}} A^j_{2q}(\Omega).
\]

where \(R_{2q}\) is the set of all pairwise partitions of \{1, 2, 1, 3, \ldots, 1, q\}. We consider a ‘typical’ partition \(\Omega_1 \in R_{2q}\)
\[
A^j_{2q}(\Omega_1) = \sum_{0 \leq a \leq a} \text{cum} \left[c(s_1 - s_2)e^{i(s_1 - s_2)\omega_{k_1}}, \ldots, c(s_1 - s_{q+1})e^{i(s_1 - s_{q+1})\omega_{k_q}}\right].
\]
By expanding the above the cumulant as the sum of the product of expectations we have

$$A_{2q}^j(\Omega_1) = \frac{1}{n^{2q}} \sum_{j \in D_{q+1}} \sum_{\Omega \in \mathcal{R}_{2q}} \sum_{\Pi \in \mathcal{S}_q} A_{2q}^j(\Omega_1, \Pi),$$

where $\mathcal{S}_q$ is the set of all partitions of $\{1, \ldots, q\}$. As we have seen in both the var[$\tilde{Q}_{a,\lambda}(1, 0)$] and cum3[$\tilde{Q}_{a,\lambda}(1, 0)$] calculations, the leading term in the cumulant expansion is the expectation over all the covariance terms. The same result holds true for higher order cumulants, the expectation over all the covariance terms. The same result holds true for higher order cumulants, the expectation over all the covariances in that cumulant is the leading term because the it gives the linear equation of the $k$s in the sinc function with the lowest order rank (we recall the lower the rank the less ‘free’ $\lambda$s). Based on this we will only derive bounds for the expectation over all the covariances. Let $\Pi_1 \in \mathcal{S}_q$, where

$$A_{2q}^j(\Omega_1, \Pi_1) = \sum_{k_1, \ldots, k_q = -a}^a E \left[ \prod_{c=1}^q c(s_1 - s_{c+1}) e^{j(s_1-s_{c+1})\omega_{k_c}} \right].$$

Representing the above expectation as an integral and using the spectral representation theorem and a change of variables gives

$$A_{2q}^j(\Omega_1, \Pi_1) = \sum_{k_1, \ldots, k_q = -a}^a E \left[ \prod_{c=1}^q c(s_1 - s_{c+1}) e^{j(s_1-s_{c+1})\omega_{k_c}} \right]$$

$$= \frac{1}{(2\pi)^q} \sum_{k_1, k_2, k_3 = -a}^a \int_{\mathbb{R}^q} \text{sinc} \left( \frac{\lambda(\sum_{c=1}^q x_c)}{2} + \pi \sum_{c=1}^q k_c \right) \prod_{c=1}^q f(x_c) \text{sinc}(x_c + k_c\pi) \prod_{c=1}^q dx_c$$

$$= \frac{2^q}{(2\pi)^q \lambda^q} \sum_{k_1, \ldots, k_q = -a}^a \int_{\mathbb{R}^q} \text{sinc} \left( \sum_{c=1}^q u_c \right) \prod_{c=1}^q f \left( \frac{2u_c}{\lambda} - \omega_{k_c} \right) \text{sinc}(u_c) du_c = O(1),$$

where the last line follows from Lemma C.1, equation (44). Therefore, $A_{2q}^j(\Omega_1) = O(1)$. By using the same method on every partition $\Omega \in \mathcal{R}_{q+1}$ and $j \in D_{q+1}$ and $|D_{q+1}| = O(n^{q+1})$, we have

$$B_{2q} = \frac{1}{n^{2q}} \sum_{j \in D_{q+1}} \sum_{\Omega \in \mathcal{R}_{2q}} \sum_{\Pi \in \mathcal{S}_q} A_{2q}^j(\Omega_1, \Pi_1) = O\left( \frac{1}{n^{q-1}} \right).$$

Finally, we briefly discuss the terms $B_{2q}$ which lie between the two extremes $B_{2,\ldots,2}$ and $B_{2q}$. Since $B_{2q}$ is the product of $2b_1, \ldots, 2b_m$ cumulants, by Lemma D.2(ii) at most $\sum_{j=1}^m (b_j + 1) = q + m$ elements of $j$ can be different. Thus

$$B_{2q} = \frac{1}{n^{2q}} \sum_{i=q}^{q+m} \sum_{j \in D_{i}} \sum_{\pi_{2b} \in \mathcal{F}_{2q}} A_{2q}^j(\tau_{2q}).$$

By expanding the cumulants in terms of the cumulants of covariances conditioned on the location (which is due to Gaussianity of the random field, see for example, (76)) we have

$$B_{2q} = \frac{1}{n^{2q}} \sum_{i=q}^{q+m} \sum_{j \in D_{i}} \sum_{\pi_{2b} \in \mathcal{F}_{2q}} \sum_{\Omega \in \mathcal{R}_{2q}} A_{2q}^j(\tau_{2q}, \Omega),$$

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where $\mathcal{R}_{2\lambda}$ is the set of all paired partitions of \{(1, \ldots, 2b_1), \ldots, (2b_{m-1}+1, \ldots, 2b_m)\}. The leading terms are the highest order expectations. This term leads to a matrix equation for the $k$'s within the sinc functions, where the rank of the corresponding matrix is at least $(m-1)$ (we do not give a formal proof of this). Therefore, $B_{2\lambda} = O\left(\frac{\log^{2(q-2)}(a)}{\lambda^{q-1}}\right) = O\left(\frac{\log^{2(q-2)}(a)}{\lambda^{q-1}}\right)$ (since $n \gg \lambda/\log^2(a)$). This concludes the proof of (72).

The proof of (73) is identical and we omit the details.

To prove the result for $d > 1$, (74) we use the same method, the main difference is that the spectral density function in (75) is a multivariate function of dimension $d$, there are $2dp$ sinc functions and the integral is over $\mathbb{R}^{dp}$, however the analysis is identical. $\square$

PROOF of Theorem 3.7 By using the well known identities

\[
\text{cov}(\Re A, \Re B) = \frac{1}{2} (\Re \text{cov}(A, B) + \Re \text{cov}(A, \bar{B}))
\]
\[
\text{cov}(\Im A, \Im B) = \frac{1}{2} (\Re \text{cov}(A, B) - \Re \text{cov}(A, \bar{B}))
\]
\[
\text{cov}(\Re A, \Im B) = -\frac{1}{2} (\Im \text{cov}(A, B) - \Im \text{cov}(A, \bar{B}))
\]

and equation (22), we immediately obtain

\[
\lambda^d \text{var} [\Re Q_{a,\lambda}(g; 0)] = \frac{1}{2} (C_1(0) + \Re C_2(0)) + O(\ell_{a,n}),
\]

\[
\lambda^d \text{cov} [\Re Q_{a,\lambda}(g; r_1), \Re Q_{a,\lambda}(g; r_2)] = \begin{cases} 
\Re \frac{1}{2} C_1(\omega_r) + O(\ell_{a,n}) & r_1 = r_2 (= r) \\
\Re \frac{1}{2} C_2(\omega_r) + O(\ell_{a,n}) & r_1 = -r_2 (= r) \\
\Re O(\ell_{a,n}) & \text{otherwise}
\end{cases}
\]

\[
\lambda^d \text{cov} [\Im Q_{a,\lambda}(g; r_1), \Im Q_{a,\lambda}(g; r_2)] = \begin{cases} 
\Re \frac{1}{2} C_1(\omega_r) + O(\ell_{a,n}) & r_1 = r_2 (= r) \\
\Re \frac{1}{2} C_2(\omega_r) + O(\ell_{a,n}) & r_1 = -r_2 (= r) \\
\Re O(\ell_{a,n}) & \text{otherwise}
\end{cases}
\]

and

\[
\lambda^d \text{cov} [\Re Q_{a,\lambda}(g; r_1), \Im Q_{a,\lambda}(g; r_2)] = \begin{cases} 
\Re O(\ell_{a,n}) & r_1 \neq -r_2 \\
\Re \frac{1}{2} C_2(\omega_r) + O(\ell_{a,n}) & r_1 = -r_2 (= r)
\end{cases}
\]

Similar expressions for the covariances of $\Re Q_{a,\lambda}(g; r)$ and $\Im Q_{a,\lambda}(g; r)$ can also be derived.

Finally, asymptotic normality of $\Re Q_{a,\lambda}(g; r)$ and $\Im Q_{a,\lambda}(g; r)$ follows from Lemma D.4. Thus giving (23). $\square$

PROOF of Theorem 3.4 The proof is identical to the proof Lemma D.4, we omit the details. $\square$

PROOF of Theorem 3.5 The proof is similar to the proof of Theorem 3.7, we omit the details.
E Additional proofs

In this section we prove the remaining results required in this paper.

For example, Theorems 3.1, 3.2, 3.3(i,iii), 3.6 and A.2, involve replacing sums with integrals. In the case that the frequency domain is increasing stronger assumptions are required than in the case of fixed frequency domain. We state the required result in the following lemma.

**Lemma E.1** Let us suppose the function \( g_1, g_2 \) are bounded (\( \sup_{\omega \in \mathbb{R}^d} |g_1(\omega)| < \infty \) and \( \sup_{\omega \in \mathbb{R}^d} |g_2(\omega)| < \infty \)) and for all \( 1 \leq j \leq d \), \( \sup_{\omega \in \mathbb{R}^d} |\partial g_j(\omega)| < \infty \) and \( \sup_{\omega \in \mathbb{R}^d} |\partial g_j(\omega)| < \infty \).

(i) Suppose \( a/\lambda = C \) (where \( C \) is a fixed finite constant) and \( h \) is a bounded function whose first partial derivative \( 1 \leq j \leq d \), \( \sup_{\omega \in \mathbb{R}^d} |\partial h(\omega)| < \infty \). Then we have

\[
\left| \frac{1}{\lambda} \sum_{k=-a}^{a} g_1(\omega_k)h(\omega_k) - \frac{1}{(2\pi)^d} \int_{2\pi[-C,C]^d} g_1(\omega)h(\omega)d\omega \right| \leq K\lambda^{-1},
\]

where \( K \) is a finite constant independent of \( \lambda \).

(ii) Suppose \( a/\lambda \to \infty \) as \( \lambda \to \infty \). Furthermore, \( h(\omega) \leq \beta_3(\omega) \) and for all \( 1 \leq j \leq d \) the partial derivatives satisfy \( \left| \frac{\partial h(\omega)}{\partial \omega_j} \right| \leq \beta_3(\omega) \). Then uniformly over \( a \) we have

\[
\left| \frac{1}{\lambda} \sum_{k=-a}^{a} g_1(\omega_k)h(\omega_k) - \frac{1}{(2\pi)^d} \int_{2\pi[-a/\lambda,a/\lambda]^d} g_1(\omega)h(\omega)d\omega \right| \leq K\lambda^{-1}
\]

(iii) Suppose \( a/\lambda \to \infty \) as \( \lambda \to \infty \). Furthermore, \( f_4(\omega_1, \omega_2, \omega_3) \leq \beta_3(\omega_1)\beta_3(\omega_2)\beta_3(\omega_3) \) and for all \( 1 \leq j \leq 3d \) the partial derivatives satisfy \( \left| \frac{\partial f_4(\omega_1, \omega_2, \omega_3)}{\partial \omega_j} \right| \leq \beta_3(\omega) \).

\[
\left| \frac{1}{\lambda^{2d}} \sum_{k_1,k_2=-a}^{a} g_1(\omega_{k_1})g_2(\omega_{k_2})f_4(\omega_{k_1+r_1}, \omega_{k_2+r_2}) - \frac{1}{(2\pi)^d} \int_{2\pi[-a/\lambda,a/\lambda]^d} \int_{2\pi[-a/\lambda,a/\lambda]^d} g_1(\omega_1)g_2(\omega_2)f_4(\omega_1+\omega_{r_1}, \omega_2+\omega_{r_2}) \right| \leq K\lambda^{-1}.
\]

PROOF. We first prove the result in the univariate case. We expand the difference between sum and integral

\[
\frac{1}{\lambda} \sum_{k=-a}^{a} g_1(\omega_k)h(\omega_k) - \frac{1}{2\pi} \int_{-2\pi/a/\lambda}^{2\pi/a/\lambda} g_1(\omega)h(\omega)d\omega
\]

\[
= \frac{1}{\lambda} \sum_{k=-a}^{a} g_1(\omega_k)h(\omega_k) - \frac{1}{2\pi} \sum_{k=-a}^{a-1} \int_{0}^{2\pi/\lambda} g_1(\omega + \omega_k)h(\omega_k + \omega)d\omega.
\]

By applying the mean value theorem for integrals to the integral above we have

\[
= \frac{1}{\lambda} \sum_{k=-a}^{a} [g_1(\omega_k)h(\omega_k) - g_1(\omega_k + \omega_k)h(\omega_k + \omega_k)]
\]

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where \( \omega_k \in [0, \frac{2\pi}{\lambda}] \). Next, by applying the mean value theorem to the difference above we have

\[
\left| \frac{1}{\lambda} \sum_{k=-a}^{a} [g_1(\omega_k)h(\omega_k) - g_1(\omega_k + \omega)h(\omega_k + \omega)] \right| \leq \frac{1}{\lambda^2} \sum_{k=-a}^{a} \left| g'_1(\omega_k)h(\omega_k) + g_1(\omega_k)h'(\omega_k) \right|
\]

(78)

where \( \omega_k \in [\omega_k, \omega_k + \omega] \) (note this is analogous to the expression given in Brillinger (1981), Exercise 1.7.14).

Under the condition that \( a = C\lambda, g_1(\omega) \) and \( h(\omega) \) are bounded and using (78) it is clear that

\[
\frac{1}{\lambda} \sum_{k=-a}^{a} g_1(\omega_k)h(\omega_k) = \frac{1}{2\pi} \int_{-2\pi/\lambda}^{2\pi/\lambda} g_1(\omega)h(\omega)d\omega
\]

\[
\leq \frac{\sup_\omega |h'(\omega)g_1(\omega)| + \sup_\omega |h(\omega)g'_1(\omega)|}{\lambda^2} \sum_{k=-a}^{a} 1 = C\lambda^{-1}.
\]

For \( d = 1 \), this proves (i).

In the case that \( a/\lambda \to \infty \) as \( \lambda \to \infty \), we use that \( h \) and \( h' \) are dominated by a monotonic function and that \( g_1 \) is bounded. Thus by using (78) we have

\[
\frac{1}{\lambda} \sum_{k=-a}^{a} g_1(\omega_k)h(\omega_k) = \frac{1}{2\pi} \int_{-2\pi/\lambda}^{2\pi/\lambda} g_1(\omega)h(\omega)d\omega
\]

\[
\leq \frac{\sup_\omega |g'_1(\omega)h(\omega)| + \sup_\omega |g_1(\omega)h'(\omega)|}{\lambda^2} \sum_{k=0}^{a} \beta_6(\omega_k) \leq \frac{C}{\lambda} \int_{0}^{\infty} \beta_6(\omega)d\omega = O(\lambda^{-1}).
\]

For \( d = 1 \), this proves (ii).

To prove the result for \( d = 2 \) we take differences, that is

\[
\frac{1}{\lambda^2} \sum_{k_1=-a}^{a} \sum_{k_2=-a}^{a} g(\omega_{k_1}, \omega_{k_2})h(\omega_{k_1}, \omega_{k_2}) = \frac{1}{(2\pi)^2} \int_{-\lambda}^{\lambda} \int_{-\lambda}^{\lambda} g(\omega_1, \omega_2)h(\omega_1, \omega_2)
\]

\[
= \frac{1}{\lambda} \sum_{k_1=-a}^{a} \left( \frac{1}{\lambda} \sum_{k_2=-a}^{a} g(\omega_{k_1}, \omega_{k_2})h(\omega_{k_1}, \omega_{k_2}) \right) - \frac{1}{(2\pi)^2} \int_{-\lambda}^{\lambda} g(\omega_1, \omega_2)h(\omega_k, \omega_2)
\]

\[
+ \frac{1}{\lambda^2} \sum_{k_1=-a}^{a} g(\omega_{k_1}, \omega_2)h(\omega_{k_1}, \omega_2) - \int_{-\lambda}^{\lambda} g(\omega_1, \omega_2)h(\omega_1, \omega_2)
\]

For each of the terms above we apply the method described for the case \( d = 1 \); for the first term we take the partial derivative over \( \omega_2 \) and for the second term we take the partial derivative over \( \omega_1 \). This method can easily be generalized to the case \( d > 2 \). The proof of (iii) is identical to the proof of (ii).

We mention that the assumptions on the derivatives (used replace sum with integral) can be relaxed to that of bounded variation of the function. However, since we require the bounded derivatives to decay at certain rates (to prove other results) we do not relax the assumption here.

We now prove the claims at the start of Appendix A.2.
Lemma E.2 Suppose Assumptions 2.1, 2.2 and

(i) Assumptions 2.4(i) and 2.4(a,c) hold. Then we have

\[
\lambda^d \text{cov} \left[ Q_{a,\lambda}(g; r_1), Q_{a,\lambda}(g; r_2) \right] = \begin{cases} 
C_1(\omega_r) + O\left( \frac{1}{\lambda} + \frac{\lambda^d}{n} \right) & r_1 = r_2(= r) \\
O\left( \frac{1}{\lambda} + \frac{\lambda^d}{n} \right) & r_1 \neq r_2
\end{cases}
\]

and

\[
\lambda^d \text{cov} \left[ \bar{Q}_{a,\lambda}(g; r_1), \bar{Q}_{a,\lambda}(g; r_2) \right] = \begin{cases} 
C_2(\omega_r) + O\left( \frac{1}{\lambda} + \frac{\lambda^d}{n} \right) & r_1 = -r_2(= r) \\
O\left( \frac{1}{\lambda} + \frac{\lambda^d}{n} \right) & r_1 \neq -r_2
\end{cases}
\]

where

\[
C_1(\omega_r) = C_{1,1}(\omega_r) + C_{1,2}(\omega_r) \quad \text{and} \quad C_2(\omega_r) = C_{2,1}(\omega_r) + C_{2,2}(\omega_r),
\]

with

\[
C_{1,1}(\omega_r) = \frac{1}{(2\pi)^d} \int_{2\pi[-a'/\lambda, a'/\lambda]^d} f(\omega)f(\omega + \omega_r)|g(\omega)|^2d\omega,
\]

\[
C_{1,2}(\omega_r) = \frac{1}{(2\pi)^d} \int_{D_r} f(\omega)f(\omega + \omega_r)g(\omega)\overline{g(\omega - \omega_r)}d\omega,
\]

\[
C_{2,1}(\omega_r) = \frac{1}{(2\pi)^d} \int_{2\pi[-a'/\lambda, a'/\lambda]^d} f(\omega)f(\omega + \omega_r)g(\omega)g(-\omega)d\omega,
\]

\[
C_{2,2}(\omega_r) = \frac{1}{(2\pi)^d} \int_{D_r} f(\omega)f(\omega + \omega_r)g(\omega)g(\omega + \omega_r)d\omega,
\]

where the integral is defined as

\[
\int_{D_r} = \int_{2\pi \min(a-a_r_1)/\lambda}^{2\pi \min(a,a-r_d)/\lambda} \cdots \int_{2\pi \max(-a-a_1)/\lambda}^{2\pi \max(-a,-a-r_d)/\lambda} (\text{note that } C_{1,1}(\omega_r) \text{ and } C_{1,2}(\omega_r) \text{ are real}).
\]

(ii) Assumptions 2.4(ii) and 2.5(b) hold. Then

\[
\lambda^d \text{cov} \left[ \bar{Q}_{a,\lambda}(g; r_1), \bar{Q}_{a,\lambda}(g; r_2) \right] = A_1(r_1, r_2) + A_2(r_1, r_2) + O\left( \frac{\lambda^d}{n} \right),
\]

where

\[
A_1(r_1, r_2) = \frac{1}{\pi^{2d}\lambda^d} \sum_{m=-2a}^{2a} \min(a,a+m) \sum_{k=\max(-a,-a+m)} f\left( \frac{2u}{\lambda} - \omega_k \right)f\left( \frac{2v}{\lambda} + \omega_k + \omega_r \right) g(\omega_k)g(\omega_k - \omega_m)\text{Sinc}(u - m\pi)\text{Sinc}(v + (m + r_1 - r_2)\pi)\text{Sinc}(u)\text{Sinc}(v)dudv
\]

\[
A_2(r_1, r_2) = \frac{1}{\pi^{2d}\lambda^d} \sum_{m=-2a}^{2a} \min(a,a+m) \sum_{k=\max(-a,-a+m)} f\left( \frac{2u}{\lambda} - \omega_k \right)f\left( \frac{2v}{\lambda} + \omega_k + \omega_r \right) g(\omega_k)g(\omega_k - \omega_m)\text{Sinc}(u - (m + r_2)\pi)\text{Sinc}(v + (m + r_1)\pi)\text{Sinc}(u)\text{Sinc}(v)dudv
\]
A_3(r_1, r_2) = \frac{1}{\pi^{2d} \lambda^d} \sum_{m=-2a}^{2a} \sum_{k = \max(-a, -a + m)}^{\min(a, a + m)} \int_{\mathbb{R}^{2d}} f \left( \frac{2u}{\lambda} - \omega_k \right) f \left( \frac{2v}{\lambda} + \omega_k + \omega_r \right) g(\omega_k) g(\omega_m - \omega_k) \text{Sinc}(u + m\pi) \text{Sinc}(v + (m + r_2 + r_1)\pi) \text{Sinc}(u) \text{Sinc}(v) du dv

A_4(r_1, r_2) = \frac{1}{\pi^{2d} \lambda^d} \sum_{m=-2a}^{2a} \sum_{k = \max(-a, -a + m)}^{\min(a, a + m)} \int_{\mathbb{R}^{2d}} f \left( \frac{2u}{\lambda} - \omega_k \right) f \left( \frac{2v}{\lambda} + \omega_k + \omega_r \right) g(\omega_k) g(\omega_m - \omega_k) \text{Sinc}(u - (m - r_2)\pi) \text{Sinc}(v + (m + r_1)\pi) \text{Sinc}(u) \text{Sinc}(v) du dv

where \( k = \max(-a, -a - m) \) = \{ k_1 = \max(-a, -a - m_1), \ldots, k_d = \max(-a, -a - m_d) \},

\( k = \min(-a + m) \) = \{ k_1 = \max(-a, -a + m_1), \ldots, k_d = \min(-a + m_d) \}.

(iii) Assumptions 2.4(ii) and 2.5(b) hold. Then we have
\[
\lambda^d \sup_a \left| \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] \right| < \infty \quad \text{and} \quad \lambda^d \sup_a \left| \text{cov} \left[ \overline{Q}_{a,\lambda}(g; r_1), \overline{Q}_{a,\lambda}(g; r_2) \right] \right| < \infty,
\]
if \( \lambda^d / n \to c \) (0 \( c < \infty \)) as \( \lambda \to \infty \) and \( n \to \infty \).

**PROOF** We prove the result in the case \( d = 1 \) (the proof for \( d > 1 \) is identical). We first prove (i).

By using indecomposable partitions, Theorem 2.1 and Lemma D.1 in Subba Rao (2015b), noting that the fourth order cumulant partitions is of order \( O(1/n) \), it is straightforward to show that
\[
\lambda \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] = \frac{1}{\lambda} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \left[ \text{cov} \left( J_n(\omega_{k_1}), J_n(\omega_{k_1+r_1}) \right) - \frac{\lambda^d}{n} \sum_{j=1}^{n} Z(s_j)^2 e^{-is_j\omega_r}, J_n(\omega_{k_2}) J_n(\omega_{k_2+r_2}) - \frac{\lambda^d}{n} \sum_{j=1}^{n} Z(s_j)^2 e^{-is_j\omega_r} \right] \right]
\]
\[
= \left\{ \begin{array}{ll}
\frac{1}{\lambda} \sum_{k = -a}^{a} f(\omega_k) f(\omega_k + \omega_r) g(\omega_k) g(\omega_k) + O \left( \frac{a^2 + \frac{1}{\lambda}}{n} \right) & r_1 = r_2 \\
O \left( \frac{a^2 + \frac{1}{\lambda}}{n} \right) & r_1 \neq r_2
\end{array} \right.
\]

Since \( a = C \lambda \) we have \( O \left( \frac{a^2}{n} \right) = O \left( \frac{1}{\lambda} \right) \) and by replacing sum with integral (using Lemma E.1(i)) we have
\[
\lambda \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] = \left\{ \begin{array}{ll}
C_1(\omega_r) + O \left( \frac{1}{\lambda} + \frac{1}{n} \right) & r_1 = r_2 = r \\
O \left( \frac{1}{\lambda} + \frac{1}{n} \right) & r_1 \neq r_2
\end{array} \right.
\]

where
\[
C_1(\omega_r) = C_{11}(\omega_r) + C_{12}(\omega_r) + O \left( \frac{1}{\lambda} \right),
\]
with
\[
C_{11}(\omega_r) = \frac{1}{2\pi} \int_{-2\pi \omega_r / \lambda}^{2\pi \omega_r / \lambda} f(\omega) f(\omega + \omega_r) g(\omega) \text{Sinc}(\omega) d\omega
\]
\[
C_{12}(\omega_r) = \frac{1}{2\pi} \int_{2\pi \min(-a, -a - r) / \lambda}^{2\pi \max(-a, -a - r) / \lambda} f(\omega) f(\omega + \omega_r) g(\omega) \text{Sinc}(\omega) d\omega.
\]
We now show that \( C_1(\omega_r) = C_{11}(\omega_r) + C_{12}(\omega_r) \). It is clear that \( C_{11}(\omega_r) \) is real. Thus we focus on \( C_{12}(\omega_r) \). To do this, we write \( g(\omega) = g_1(\omega) + ig_2(\omega) \), therefore

\[
\Im[g(\omega)g(-\omega - \omega_r)] = [g_2(\omega)g_1(-\omega - \omega_r) - g_1(\omega)g_2(-\omega - \omega_r)].
\]

Substituting the above into \( \Im C_{12}(\omega_r) \) gives us \( \Im C_{12}(r) = [C_{121}(r) + C_{122}(r)] \) where

\[
C_{121}(r) = \frac{1}{2\pi} \int_{2\pi \max(-a,-a-r)/\lambda}^{2\pi \min(a,a-r)/\lambda} f(\omega)f(\omega + \omega_r)g(\omega)g(\omega - \omega_r) d\omega,
\]

\[
C_{122}(r) = -\frac{1}{2\pi} \int_{2\pi \max(-a,-a-r)/\lambda}^{2\pi \min(a,a-r)/\lambda} f(\omega)f(\omega + \omega_r)g(\omega)g(\omega - \omega_r) d\omega.
\]

We will show that \( C_{122}(r) = -C_{121}(r) \). Focusing on \( C_{122} \) and making the change of variables \( u = -\omega - \omega_r \) gives us

\[
C_{122} = \frac{1}{2\pi} \int_{2\pi \min(a,a-r)/\lambda}^{2\pi \max(-a,-a-r)/\lambda} f(u + \omega_r)f(-u)g(1 - u - \omega_r)g(2u) du,
\]

noting that the spectral density function is symmetric with \( f(-u) = f(u) \), and that \( \int_{2\pi \min(a,a-r)/\lambda}^{2\pi \max(-a,-a-r)/\lambda} f(\omega)d\omega = -\int_{2\pi \max(-a,-a-r)/\lambda}^{2\pi \min(a,a-r)/\lambda} f(\omega)d\omega \). Thus we have \( \Im C_{12}(r) = 0 \), which shows that \( C_1(\omega_r) \) is real. The proof of \( \lambda^d \text{cov}[\bar{Q}_{a,\lambda}(g; r_1), \bar{Q}_{a,\lambda}(g; r_2)] \) is the same. Thus, we have proven (i).

To prove (ii) we first expand \( \text{cov}[\bar{Q}_{a,\lambda}(g; r_1), \bar{Q}_{a,\lambda}(g; r_2)] \) to give

\[
\lambda \text{cov}[\bar{Q}_{a,\lambda}(r_1), \bar{Q}_{a,\lambda}(g; r_2)] = \lambda \sum_{k_1, k_2 = -a}^{a} \sum_{j_1, j_2 = 1}^{n} \left( E[c(s_{j_1} - s_{j_2})e^{i\delta j_1\omega_{k_1}}e^{-i\delta j_2\omega_{k_2}}] E[c(s_{j_2} - s_{j_3})e^{-i\delta j_2\omega_{k_1}}e^{i\delta j_3\omega_{k_2}}] + \right.
\]

\[
E[c(s_{j_1} - s_{j_2})e^{i\delta j_1\omega_{k_1}}e^{i\delta j_2\omega_{k_2}}] E[c(s_{j_2} - s_{j_3})e^{-i\delta j_2\omega_{k_1}}e^{-i\delta j_3\omega_{k_2}}] +
\]

\[
\text{cum}[Z(s_{j_1})e^{i\delta k_1 s_{j_1}}, Z(s_{j_2})e^{-i\delta k_2 s_{j_2}}, Z(s_{j_3})e^{-i\delta k_3 s_{j_3}}, Z(s_{j_4})e^{i\delta k_4 (\omega_{k_2} + \omega_{k_2})}].
\]

By applying Lemma D.1, (59) in Subba Rao (2015b), to the fourth order cumulant we can show that

\[
\lambda \text{cov}[\bar{Q}_{a,\lambda}(r_1), \bar{Q}_{a,\lambda}(g; r_2)] = A_1(r_1, r_2) + A_2(r_1, r_2) + O(\frac{\lambda}{n}) \quad (81)
\]

where

\[
A_1(r_1, r_2) = \lambda \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1})g(\omega_{k_2}) E[c(s_1 - s_3) \exp(\omega_{k_1} - \omega_{k_2})] \times
\]

\[
E[c(s_2 - s_4) \exp(\omega_{k_1} + \omega_{k_2})] \ni
\]

\[
A_2(r_1, r_2) = \lambda \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1})g(\omega_{k_2}) E[c(s_1 - s_4) \exp(\omega_{k_1} + \omega_{k_2})] \times
\]

\[
E[c(s_2 - s_3) \exp(\omega_{k_1} + \omega_{k_2})].
\]

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Note that the $O(\lambda/n)$ term include the error $n^{-1}[A_1(r_1, r_2) + A_2(r_1, r_2)]$ (we show below that $A_1(r_1, r_2)$ and $A_2(r_1, r_2)$ are both bounded over $\lambda$ and $\alpha$). To write $A_1(r_1, r_2)$ in the form stated in the lemma we integrate over $s_1, s_2, s_3$ and $s_4$ to give

$$A_1(r_1, r_2) = \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \frac{1}{\lambda^4} c(s_1 - s_3)c(s_2 - s_4) e^{i \omega_{k_1} - i \omega_{k_2}} e^{-i \omega_{k_1} + r_1 + i \omega_{k_2} + r_2} ds_1 \ldots ds_4.$$ 

By using the spectral representation theorem and integrating out $s_1, \ldots, s_4$ we can write the above as

$$A_1(r_1, r_2) = \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) f(y) \text{sinc} \left( \frac{\lambda x}{2} + k_1 \pi \right) \text{sinc} \left( \frac{\lambda y}{2} - (r_1 + k_1) \pi \right) \text{sinc} \left( \frac{\lambda y}{2} + k_2 \pi \right) \text{sinc} \left( \frac{\lambda y}{2} - (r_2 + k_2) \pi \right) dxdy$$

$$= \frac{1}{\pi^2 \lambda} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left( \frac{2u}{\lambda} - \omega_{k_1} \right) f\left( \frac{2v}{\lambda} + \omega_{k_1} + \omega_{k_2} \right) \text{sinc}(u) \text{sinc}(u + (k_2 - k_1) \pi) \text{sinc}(v) \text{sinc}(v + (k_1 - k_2 + r_2 - r_1) \pi) dudv,$$

where the second equality is due to the change of variables $u = \frac{\lambda x}{2} + k_1 \pi$ and $v = \frac{\lambda y}{2} - (r_1 + k_1) \pi$.

Finally, by making a change of variables $k = k_1$ and $m = k_1 - k_2$ ($k_1 = -k_2 + m$) we obtain the expression for $A_1(r_1, r_2)$ given in Lemma E.2.

A similar method can be used to obtain the expression for $A_2(r_1, r_2)$

$$A_2(r_1, r_2) = \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) f(y) \text{sinc} \left( \frac{\lambda x}{2} + k_1 \pi \right) \text{sinc} \left( \frac{\lambda y}{2} - (r_1 + k_1) \pi \right) \text{sinc} \left( \frac{\lambda y}{2} + k_2 \pi \right) \text{sinc} \left( \frac{\lambda y}{2} - (r_2 + k_2) \pi \right) dxdy$$

$$= \frac{1}{\pi^2 \lambda} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left( \frac{2u}{\lambda} - \omega_{k_1} \right) f\left( \frac{2v}{\lambda} + \omega_{k_1} + \omega_{k_2} \right) \text{sinc}(u) \text{sinc}(u - (k_2 + r_2 + k_1) \pi) \text{sinc}(v) \text{sinc}(v + (k_2 + k_1 + r_1) \pi) dudv.$$

By making the change of variables $k = k_1$ and $m = k_1 + k_2$ ($k_1 = -k_2 + m$) we obtain the stated expression for $A_2(r_1, r_2)$.

Finally following the same steps as those above we obtain

$$\lambda \text{cov}(\tilde{Q}_{a, \lambda}(g; r_1), \tilde{Q}_{a, \lambda}(g; r_2)) = A_3(r_1, r_2) + A_4(r_1, r_2) + O\left( \frac{\lambda}{n} \right),$$

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where

\[
A_3(r_1, r_2) = \frac{1}{\lambda^3} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \int_{[-\lambda/2, \lambda/2]^4} e^{i(s_1 \omega_{k_1} + s_3 \omega_{k_2})} e^{-i(s_2 \omega_{k_1} + r_1 - i8 \omega_{k_2} + r_2} \times c(s_1 - s_3) c(s_2 - s_4) ds_1 ds_2 ds_3 ds_4
\]

\[
A_4(r_1, r_2) = \frac{1}{\lambda^3} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \int_{[-\lambda/2, \lambda/2]^4} e^{i(s_1 \omega_{k_1} + s_3 \omega_{k_2})} e^{-i(s_2 \omega_{k_1} + r_1 - i8 \omega_{k_2} + r_2} \times c(s_1 - s_4) c(s_2 - s_3) ds_1 ds_2 ds_3 ds_4.
\]

Again by replacing the covariances in \(A_3(r_1, r_2)\) and \(A_4(r_1, r_4)\) with their spectral representation gives (ii) for \(d = 1\). The result for \(d > 1\) is identical.

It is clear that (iii) is true under Assumption 2.4(a,b). To prove (iii) under Assumption 2.5(a,b) we will show that for \(1 \leq j \leq 4\), \(\sup_a |A_j(r_1, r_2)| < \infty\). To do this, we first note that by the Cauchy Schwarz inequality we have

\[
\sup_{a, \lambda} \frac{1}{\pi^{2d}} \frac{1}{\lambda^d} \sum_{k = \max(-a, -a + m)} \min(-a - m) g(\omega_k) g(\omega_k - \omega_m) f(\frac{2u}{\lambda} - \omega_k) f(\frac{2v}{\lambda} + \omega_k + \omega_r) \leq C \sup_{\omega} |g(\omega)|^2 \|f\|_2^2,
\]

where \(\|f\|_2\) is the \(L_2\) norm of the spectral density function and \(C\) is a finite constant. Thus by taking absolutes of \(A_1(r_1, r_2)\) we have

\[
|A_1(r_1, r_2)| \leq C \sup_{\omega} |g(\omega)|^2 \|f\|_2^2 \sum_{m = -\infty}^{\infty} \int_{\mathbb{R}^{2d}} |\text{Sinc}(u - m \pi) \text{Sinc}(v + (m + r_1 - r_2) \pi) \text{Sinc}(u) \text{Sinc}(v)| du dv.
\]

Finally, by using Lemma C.1(iii), Subba Rao (2015b), we have that \(\sup_a |A_1(r_1, r_2)| < \infty\). By using the same method we can show that \(\sup_a |A_1(r_1, r_2)|, \ldots, \sup_a |A_4(r_1, r_4)| < \infty\). This completes the proof. \(\square\)

PROOF of Theorem 5.1 Making the classical variance-bias decomposition we have

\[
E \left( \bar{V} - \lambda^d \text{var} [\bar{Q}_{a, \lambda}(g; 0)] \right)^2 = \text{var} [\bar{V}] + \left( E[\bar{V}] - \lambda^d \text{var} [\bar{Q}_{a, \lambda}(g; 0)] \right)^2.
\]

We first analysis the bias term, in particular \(E[\bar{V}]\). We note that by using the expectation and variance result in Theorem 3.1 and equation (22), respectively, we have

\[
E[\bar{V}] = \frac{\lambda^d}{|S|} \sum_{r \in S} \text{var} [\bar{Q}_{a, \lambda}(g; r)] + \frac{\lambda^d}{|S|} \sum_{r \in S} \left[ E[\bar{Q}_{a, \lambda}(g; r)] \right]^2
\]

\[
- O(\lambda^{-2d} \prod_{j=1}^{d} (\log \lambda + \log |r_j|)^2)
\]

\[
= \frac{1}{|S|} \sum_{r \in S} C_1(\omega_r) + O \left( \ell_{\lambda, a, n} + \frac{[\log \lambda + \log M]^d}{\lambda^d} \right)
\]

\[
= C_1 + O \left( \ell_{\lambda, a, n} + \frac{[\log \lambda + \log M]^d}{\lambda^d} + \frac{|M|}{\lambda} \right).
\]
Next we consider \( \text{var} [\widetilde{V}] \), by using the classical cumulant decomposition we have

\[
\text{var} [\widetilde{V}] = \frac{\lambda^{2d}}{|S|^2} \sum_{\mathbf{r}_1, \mathbf{r}_2 \in S} \left( |\text{cov} [\widetilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \widetilde{Q}_{a,\lambda}(g; \mathbf{r}_2)]|^2 + |\text{cov} [\widetilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \overline{\widetilde{Q}_{a,\lambda}(g; \mathbf{r}_2)}]|^2 \right)
+ \frac{\lambda^{2d}}{|S|^2} \sum_{\mathbf{r}_1, \mathbf{r}_2 \in S} \text{cum} \left( \widetilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \overline{\widetilde{Q}_{a,\lambda}(g; \mathbf{r}_1)}, \overline{\widetilde{Q}_{a,\lambda}(g; \mathbf{r}_2)}, \widetilde{Q}_{a,\lambda}(g; \mathbf{r}_2) \right)
+ O \left( \frac{\lambda^{2d}}{|S|^2} \sum_{\mathbf{r}_1, \mathbf{r}_2 \in S} \text{cum} \left( \widetilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \overline{\widetilde{Q}_{a,\lambda}(g; \mathbf{r}_1)}, \overline{\widetilde{Q}_{a,\lambda}(g; \mathbf{r}_2)} \right) E \left( \overline{\widetilde{Q}_{a,\lambda}(g; \mathbf{r}_2)} \right) \right).
\]

By substituting the variance/covariance results for \( \widetilde{Q}_{a,\lambda}(\cdot) \) based on uniformly sampled locations in (22) and the cumulant bounds in Lemma D.4 into the above we have

\[
\text{var} [\widetilde{V}] = \frac{\lambda^{2d}}{|S|^2} \sum_{\mathbf{r} \in S} \left| \text{var} [\widetilde{Q}_{a,\lambda}(g; \mathbf{r})] \right|^2 + O \left( \ell_{\lambda,a,n} + \frac{\log^4 d(a)}{\lambda^d} \right) = O \left( \frac{1}{|S|} + \ell_{\lambda,a,n} + \frac{\log^4 d(a)}{\lambda^d} \right).
\]

Thus altogether we have the result. \( \square \)

## F Sample properties of \( Q_{a,\lambda}(g; \mathbf{r}) \)

In this section we summarize the result for \( Q_{a,\lambda}(g; \mathbf{r}) \). To simplify notation we state the results only for the case that the locations are uniformly distributed.

We recall that

\[
Q_{a,\lambda}(g; \mathbf{r}) = \widetilde{Q}_{a,\lambda}(g; \mathbf{r}) + G_{\lambda} V_{\mathbf{r}}
\]

where \( V_{\mathbf{r}} = \frac{1}{n} \sum_{j=1}^{n} Z(s_j) \exp(-i \omega_{\mathbf{r}} s_j) \).

**Theorem F.1** Suppose Assumptions 2.1(i), 2.2, \( b = b(\mathbf{r}) \) denotes the number of zero elements in the vector \( \mathbf{r} \in \mathbb{Z}^d \) and

(i) Assumptions 2.4(i) and 2.5(a,c) hold. Then we have

\[
\text{E} [Q_{a,\lambda}(g; \mathbf{r})]
= \left\{ \begin{array}{ll}
O(1) & \mathbf{r} \in \mathbb{Z}^d/\{0\} \\
\frac{1}{(2\pi)^d} \int_{\omega \in 2\pi [-C,C]^d} f(\omega) g(\omega) d\omega + O(1 + \frac{\lambda^d}{n}) & \mathbf{r} = 0
\end{array} \right.
\]

(ii) Suppose the Assumptions 2.4(ii) and Assumption 2.5(b,c) hold and \( \{m_1, \ldots, m_{d-b}\} \) is the subset of non-zero values in \( \mathbf{r} = (r_1, \ldots, r_d) \), then we have

\[
\text{E} [Q_{a,\lambda}(g; \mathbf{r})]
= \left\{ \begin{array}{ll}
O \left( \frac{1}{\lambda^d} \prod_{j=1}^{d-b} (\log \lambda + \log |m_j|) \right) & \mathbf{r} \in \mathbb{Z}^d/\{0\} \\
\frac{1}{(2\pi)^d} \int_{\omega \in \mathbb{R}^d} f(\omega) g(\omega) d\omega + \frac{a(0)}{n} \sum_{k=-a}^{a} g(\omega_k) + O \left( \frac{\log \lambda + \log \|\mathbf{r}\|_1}{\lambda} + \frac{1}{n} \right) & \mathbf{r} = 0
\end{array} \right.
\]

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PROOF The proof of (i) immediately follows from Theorem 2.1.

The proof of (ii) follows from writing \( Q_{a,\lambda}(g; r) \) as a quadratic form and taking expectations

\[
E [Q_{a,\lambda}(g; r)] = c_2 \sum_{k=-a}^{a} g(\omega_k) \frac{1}{\lambda^d} \int_{[-\lambda/2, \lambda/2]^d} c(s_1 - s_2) \exp(i\omega_k^2(s_1 - s_2) - i\omega_k \omega_r) ds_1 ds_2 + W_r,
\]

where \( c_2 = n(n-1)/n^2 \) and \( W_r = \frac{c(0)I(r=0)}{n} \sum_{k=-a}^{a} g(\omega_k) \) (\( I(r = 0) \) denotes the indicator variable).

We then follow the same proof used to prove Theorem 3.1.

\[
\text{Theorem F.2} \quad \text{[Asymptotic expression for variance]} \quad \text{Suppose Assumptions 2.1, 2.2, 2.4 2.5(b,c) hold. Then we have}
\]

\[
\lambda^d \text{cov} [Q_{a,\lambda}(g; r_1), Q_{a,\lambda}(g; r_2)] = \begin{cases} 
C_1(\omega_r) + 2\Re[C_3(\omega_r)G_\lambda] + 2f_2(\omega_r)|G_\lambda|^2 + O(\ell_{a,n}) & r_1 = r_2 (= r) \\
O(\ell_{a,n}) & r_1 \neq r_2
\end{cases}
\]

\[
\lambda^d \text{cov} \left[ Q_{a,\lambda}(g; r_1), \overline{Q_{a,\lambda}(g; r_2)} \right] = \begin{cases} 
C_2(\omega_r) + 2G_\lambda C_3(\omega_r) + 2f_2(\omega_r)G_\lambda^2 + O(\ell_{a,n}) & r_1 = -r_2 (= r) \\
O(\ell_{a,n}) & r_1 \neq -r_2
\end{cases}
\]

where \( C_1(\omega_r) \) and \( C_2(\omega_r) \) are defined in (79), \( G_\lambda = \frac{1}{\pi} \sum_{k=-a}^{a} g(\omega_k), a^d = O(n), \)

\[
C_3(\omega_r) = \frac{2}{(2\pi)^d} \int_{2\pi[-a, a/\lambda]^d} g(\omega) f(-\omega) f(\omega + \omega_r) d\omega \quad \text{and} \quad f_2(\omega_r) = \int_{\mathbb{R}^d} f(\lambda) f(\omega_r - \lambda) d\lambda. \quad (83)
\]

PROOF To prove the result we use (82). Using this expansion we rewrite \( \text{cov}[Q_{a,\lambda}(g; r_1), Q_{a,\lambda}(g; r_2)] \) in terms of \( \tilde{Q}_{a,\lambda}(g; r) \) and \( V_r \)

\[
\text{cov} [Q_{a,\lambda}(g; r_1), Q_{a,\lambda}(g; r_2)] = \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] + |G_\lambda|^2 \text{cov} [V_{r_1}, V_{r_2}] + \gamma_{\lambda} \text{cov} \left[ Q_{a,\lambda}(g; r_1), V_{r_2} \right] + G_\lambda \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_2), V_{r_1} \right].
\]

In Theorem A.1 we have evaluated an asymptotic expression for \( \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] \). We now evaluate similar expressions for \( \text{cov} [V_{r_1}, V_{r_2}] \) and \( \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_1), V_{r_2} \right] \). It is straightforward to show that

\[
\lambda^d \text{cov} [V_{r_1}, V_{r_2}] = \begin{cases} 
O\left(\frac{1}{\lambda}\right) & r_1 \neq r_2 \\
2f_2(\omega_{r_1}) + O\left(\frac{1}{\lambda} + \frac{\lambda^d}{n}\right) & r_1 = r_2 \end{cases} \quad (84)
\]

To evaluate an expression for \( \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_1), V_{r_2} \right] \) we consider the case \( d = 1 \). Using similar
arguments to those in the proof of Lemma E.2 we can show that

\[ \lambda \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_1), V_{r_2} \right] \]

\[ = 2\lambda c_3 \sum_{k=-a}^{a} g(\omega_k) \mathbb{E} \left[ \left( c(s_1 - s_3)c(s_2 - s_3) e^{i\omega_k(x - y)} e^{i\omega_k(2\pi - (x - y))} \right) + O \left( \frac{\lambda}{n^2} \right) \right] \]

\[ = \frac{2c_3\lambda}{(2\pi)^2} \sum_{k=-a}^{a} g(\omega_k) \int_{\mathbb{R}^2} f(x) f(y) \text{sinc} \left( \frac{\lambda x}{2} + k\pi \right) \text{sinc} \left( \frac{\lambda x}{2} - (k + 1)\pi \right) \text{sinc} \left( \frac{\lambda(x + y)}{2} - r_2\pi \right) dxdy + O \left( \frac{\lambda}{n^2} \right) \]

\[ = \frac{2c_3}{\lambda\pi^2} \sum_{k=-a}^{a} g(\omega_k) \int_{\mathbb{R}^2} f \left( \frac{2u}{\lambda} - \omega_k \right) f \left( \frac{2v}{\lambda} + \omega_k \right) \text{sinc}(u) \text{sinc}(v) (u + v + (r_1 - r_2)^2) dudv + O \left( \frac{\lambda}{n^2} \right) \]

where \( c_3 = n(n-1)(n-2)/n^3 \). Finally by replacing sum with an integral, using Lemma E.1 and replacing \( f \left( \frac{2u}{\lambda} - \omega \right) f \left( \frac{2v}{\lambda} + \omega + \omega_k \right) \) with \( f(-\omega) f(\omega + \omega_k) \) and by using (54) we have

\[ \lambda \text{cov} \left[ \tilde{Q}_{a,\lambda}(g; r_1), V_{r_2} \right] \]

\[ = \frac{2c_3}{2\pi^2} \int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(\omega) f(-\omega) f(\omega + \omega_k) d\omega \int_{\mathbb{R}^2} \text{sinc}(u) \text{sinc}(v) (u + v + (r_1 - r_2)^2) dudv + O \left( \frac{\lambda}{n^2} + \frac{\log^2 \lambda}{\lambda} \right) \]

\[ = \begin{cases} 
C_3(\omega_{r_1}) + O \left( \frac{\lambda}{n^2} + \frac{\log^2 \lambda}{\lambda} + \frac{1}{n} \right) & r_1 = r_2 \\
O \left( \frac{\lambda^d}{n^2} + \frac{\log^2 \lambda}{\lambda} + \frac{1}{n} \right) & r_1 \neq r_2 
\end{cases} \] (85)

Altogether, by using Theorem A.1, (85) and (84) we obtain the result. \( \square \)

In the following lemma we show that we further simplify the expression for the asymptotic variance if we keep \( r \) fixed.

**Corollary F.1** Suppose Assumption 2.4, 2.5(a,c) or 2.5(b,c) holds, and \( r \) is fixed. Let \( C_3(\omega) \) and \( f_2(\omega) \) be defined in (83). Then we have

\[ C_3(\omega_r) = C_3 + O \left( \frac{\|r\|_1}{\lambda} \right), \]

and \( f_2(\omega_r) = f_2 + O \left( \frac{\|r\|_1}{\lambda} \right) \), where

\[ C_3 = \frac{2}{(2\pi)^d} \int_{2\pi [-a/\lambda, a/\lambda]^d} g(\omega) f(\omega)^2 d\omega \]

and \( f_2 = f_2(0) \) (note, if \( g(\omega) = g(-\omega) \), then \( C_1 = C_2 \)).

**Proof** The proof is the same as the proof of Corollary 3.1. \( \square \)
Theorem F.3 [CLT on real and imaginary parts] Suppose Assumptions 2.1, 2.2, 2.4 and 2.5(b,c) hold. Let $C_1$, $C_2$, $C_3$ and $f_2$ be defined as in Corollary F.1. We define the $m$-dimension complex random vectors $Q_m = (Q_{a,\lambda}(g, r_1), \ldots, Q_{a,\lambda}(g, r_m))$, where $r_1, \ldots, r_m$ are such that $r_i \neq -r_j$ and $r_i \neq 0$. Under these conditions we have

$$
\frac{2\lambda^{d/2}}{E_1} \left( \frac{E_1}{E_1 + \Re E_2} \mathbb{R}Q_{a,\lambda}(g, 0), \mathbb{R}Q_m, \Im Q_m \right) \overset{D}{\rightarrow} \mathcal{N}(0, I_{2m+1}),
$$

where $E_1 = C_1 + 2\Re[G_\lambda C_3] + 2f_2|G_\lambda|^2$ and $E_2 = C_2 + 2G_\lambda C_3 + 2f_2G_\lambda^2$ with $\frac{\lambda^d}{n \log^2(a)} \rightarrow 0$ and $\frac{\log^2(a)}{\lambda^d} \rightarrow 0$ as $\lambda \rightarrow \infty$, $n \rightarrow \infty$ and $a \rightarrow \infty$.

**PROOF** Using the same method used to prove Lemma D.4, and analogous results can be derived for the cumulants of $Q_{a,\lambda}(g; r)$. Asymptotic normality follows from this. We omit the details. \(\square\)

**Application to nonparametric covariance estimator without bias correction**

In this section we apply the results to the nonparametric estimator considered in Example 2.2(ii); the estimator without bias correction and is a non-negative definite sequence. We recall that $\tilde{c}_n(v) = T \left( \frac{2v}{\lambda} \right) \bar{c}_n(v)$ where $\bar{c}_n(v) = \left( \frac{1}{\lambda^d} \sum_{k=-a}^a |J_n(\omega_k)|^2 \exp(i\omega^\top \omega_k) \right)$, and $T$ is the $d$-dimensional triangle kernel. It is clear the asymptotic sampling properties of $\tilde{c}_n(v)$ are determined by $\bar{c}_n(v)$. Therefore, we first derive the asymptotic sampling properties of $\bar{c}_n(v)$. We observe that $\bar{c}_n(v) = Q_{a,\lambda}(e^{iv^\top}; 0)$, thus we use the results in Section 3 to derive the asymptotic sampling properties of $\bar{c}_n(v)$. By using Theorem 3.1(ii)(a,b) and Assumption 2.4(ii) we have

$$
\mathbb{E}[\bar{c}_n(v)] = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\omega) \exp(i\omega^\top v) d\omega + O \left( \frac{\log \lambda}{\lambda} \right)
$$

and

$$
\mathbb{E}[\bar{c}_n(v)] = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\omega) \exp(i\omega^\top v) d\omega + O \left( \left( \frac{\lambda}{a} \right)^\delta + \frac{\log \lambda}{\lambda} \right),
$$

where $\delta$ is such that it satisfies Assumption 2.4(ii)(a). Therefore, if $\lambda^{d/2}/a^\delta \rightarrow 0$ as $a \rightarrow \infty$ and $\lambda \rightarrow \infty$, then by using Theorem F.3 we have

$$
\lambda^{d/2} (\bar{c}_n(v) - c(v)) \overset{D}{\rightarrow} \mathcal{N}(0, \Sigma)
$$

where

$$
\Sigma = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\omega)^2 \left( 1 + \exp(i2\omega^\top v) \right) d\omega + \frac{2G_\lambda}{(2\pi^d)} \int_{\mathbb{R}^d} f(\omega)^2 \exp(i\omega^\top v) d\omega + 2G_\lambda^2 f_2,
$$

and $G_\lambda = \frac{1}{n} \sum_{k=-a}^a \exp(i\omega^\top \omega_k)$, which is real. We now derive the sampling properties of $\tilde{c}_n(v)$. By using (87), (88) and properties of the triangle kernel we have

$$
\mathbb{E}[\tilde{c}_n(v)] = c(v) + O \left( \frac{\lambda^\delta}{a^\delta} + \frac{\log \lambda}{\lambda} \right) \left( 1 - \frac{\min_{1 \leq i \leq d} |v_i|}{\lambda} \right) + \frac{\max_{1 \leq i \leq d} |v_i|}{\lambda}
$$
and

\[ \lambda^{d/2} \left( \bar{c}_n(v) - T \left( \frac{2v}{\lambda} \right) c(v) \right) \overset{D}{\to} \mathcal{N} \left( 0, T \left( \frac{2v}{\lambda} \right)^2 \Sigma \right), \]

with \( \lambda^{d+2/2} / a^2 \to 0 \), \( \frac{\lambda^d}{n \log^{2d} (a)} \to 0 \) and \( \frac{\log^2 (a)}{\lambda^{1/2}} \to 0 \) as \( a \to \infty \) and \( \lambda \to \infty \).

**Application to parameter estimation using an L_2 criterion**

In this section we consider the asymptotic sampling properties of \( \hat{\theta}_n = \arg \min_{\theta \in \Theta} L_n(\theta) \), where \( L_n(\cdot) \) is defined in (10) and \( \Theta \) is a compact set. We will assume that there exists a \( \theta_0 \in \Theta \), such that for all \( \omega \in \mathbb{R}^d \), \( f_{\theta_0}(\omega) = f(\omega) \) and there does not exist another \( \theta \in \Theta \) such that for all \( \omega \in \mathbb{R}^d \) \( f_{\theta_0}(\omega) = f_{\theta}(\omega) \) and in addition \( \int_{\mathbb{R}^d} \| \nabla_{\theta} f(\omega; \theta_0) \|^2_1 d\omega < \infty \). Furthermore, we will assume that \( \hat{\theta}_n \to \theta_0 \).

Making the usual Taylor expansion we have \( \lambda^{d/2}(\hat{\theta}_n - \theta_0) = A^{-1} \lambda^{d/2} \nabla L_n(\theta_0) + o_p(1) \), where

\[ A = \frac{2}{(2\pi)^d} \int_{\mathbb{R}^d} [\nabla_{\theta} f(\omega; \theta_0)] [\nabla_{\theta} f(\omega; \theta_0)]' d\omega, \]  

and it is clear the asymptotic sampling properties of \( \hat{\theta}_n \) are determined by \( \nabla_{\theta} L_n(\theta_0) \), which we see from (11) can be written as \( \nabla_{\theta} L_n(\theta_0) = Q_{a,\lambda}(-2\nabla_{\theta} f_{\theta_0}(\cdot); 0) + \frac{1}{\lambda} \sum_{k=-a}^{a} \nabla_{\theta} f_{\theta_0}(\omega_k)^2 \).

Thus by using Theorem F.3 we have \( \lambda^{d/2} \nabla_{\theta} L_n(\theta_0) \overset{D}{\to} \mathcal{N}(0, B) \), where

\[ B = \frac{4}{(2\pi)^d} \int_{\mathbb{R}^d} f(\omega)^2 \left[ \nabla_{\theta} f_{\theta_0}(\omega) \right] \left[ \nabla_{\theta} f_{\theta_0}(\omega) \right]' \bigg|_{\theta = \theta_0} d\omega \]

\[ - 4RG_{\lambda} \left( \frac{2}{(2\pi)^d} \int_{\mathbb{R}^d} f(\omega)^2 \nabla_{\theta} f_{\theta_0}(\omega) \bigg|_{\theta = \theta_0} d\omega + 2 |G_{\lambda}|^2 \int_{\mathbb{R}^d} f(\omega)^2 d\omega \right) \]

and \( G_{\lambda} = \frac{2}{\lambda} \sum_{k=-a}^{a} \nabla_{\theta} f_{\theta_0}(\omega_k) \bigg|_{\theta = \theta_0} \). Therefore, by using the above we have

\[ \lambda^{d/2}(\hat{\theta}_n - \theta_0) \overset{D}{\to} \mathcal{N}(0, A^{-1}BA^{-1}) \]

with \( \frac{\lambda^d}{n} \to 0 \) and \( \frac{\log^2 (a)}{\lambda^{1/2}} \to 0 \) as \( a \to \infty \) and \( \lambda \to \infty \).