Reconciling the Gaussian and Whittle Likelihood with an application to estimation in the frequency domain

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Abstract

In time series analysis there is an apparent dichotomy between time and frequency domain methods. The aim of this paper is to draw connections between frequency and time domain methods. Our focus will be on reconciling the Gaussian likelihood and the Whittle likelihood. We derive an exact, interpretable, bound between the Gaussian and Whittle likelihood of a second order stationary time series. The derivation is based on obtaining the transformation which is biorthogonal to the discrete Fourier transform of the time series. Such a transformation yields a new decomposition for the inverse of a Toeplitz matrix and enables the representation of the Gaussian likelihood within the frequency domain. We show that the difference between the Gaussian and Whittle likelihood is due to the omission of the best linear predictions outside the domain of observation in the periodogram associated with the Whittle likelihood. And obtain an approximation for the difference between the Gaussian and Whittle likelihoods in terms of the best fitting, finite order autoregressive parameters. These approximations are used to define two new frequency domain quasi-likelihoods criteria. We show these new criteria yield a higher order approximation of the spectral divergence criterion, as compared to both the Gaussian and Whittle likelihoods. In simulations, we show that the proposed estimators have satisfactory finite sample properties.

Keywords and phrases: Biorthogonal transforms, discrete Fourier transform, periodogram, quasi-likelihoods and second order stationary time series.

1 Introduction

In his seminal work, Whittle (1951 and 1952) introduces the Whittle likelihood as an approximation of the Gaussian likelihood. A decade later, the asymptotic sampling properties of moving average models fitted using the Whittle likelihood were derived in Walker (1964). Subsequently,
the Whittle likelihood has become a popular method for parameter estimation of various stationary time series (both long and short memory) and spatial models. The Whittle likelihood is computationally a very attractive method for estimation. Despite the considerable improvements in technology, interest in the Whittle likelihood has not abated. The Whittle likelihood has gained further traction as a quasi-likelihood (or as an information criterion, see Parzen (1983)) between the periodogram and the spectral density. Several diverse applications of the Whittle likelihood can be found in Dahlhaus and Künsch (1987), Dahlhaus (2000), Giraitis and Robinson (2001), Choudhuri et al. (2004), Abadir et al. (2007), Shao and Wu (2007), Panaretos and Tavakoli (2013), Kirch et al. (2019) and van Delft and Eichler (2019) (to name but a few).

Despite its advantages, it is well known that for small samples the Whittle likelihood can give rise to estimators with a substantial bias (see Priestley (1981) and Dahlhaus (1988)). Dahlhaus (1988) shows that the finite sample bias in the periodogram impacts the performance of the Whittle likelihood. Motivated by this discrepancy, Sykulski et al. (2019) proposes the debiased Whittle likelihood, which fits directly to the expectation of the periodogram rather than the limiting spectral density. Alternatively, Dahlhaus (1988) shows that the tapered periodogram is better at capturing the features in the spectral density, such as peaks, than the regular periodogram. He uses this as the basis of the tapered Whittle likelihood. Empirical studies show that the tapered Whittle likelihood yields a smaller bias than the regular Whittle likelihood. As a theoretical justification, Dahlhaus (1988, 1990) uses an alternative asymptotic framework to show that tapering yields a good approximation to the inverse of the Toeplitz matrix. It is worth mentioning that within the time domain, several authors, including Shaman (1975, 1976), Bhansali (1982) and Coursol and Dacunha-Castelle (1982), have studied approximations to the inverse of the Toeplitz matrix. These results can be used to approximate the Gaussian likelihood.

However, as far as we are aware, there are no results which explain why the Gaussian likelihood generally performs better than the Whittle likelihood for small sample sizes. Nor what is lost when using the Whittle likelihood rather than the Gaussian likelihood. The objective of this paper is to address some of these issues. The benefits of such insights are not only of theoretical interest but also lead to the development of frequency domain methods which are comparable and may even outperform the Gaussian likelihood in certain situations.

We first recall the definition of the Gaussian and Whittle likelihood. Our aim is to fit a parametric second order stationary model with spectral density $f_\theta(\omega)$ and corresponding autocovariance function $\{c_{f_\theta}(r)\}_{r \in \mathbb{Z}}$ to the observed time series $\{X_t\}_{t=1}^n$. The (quasi) log-Gaussian likelihood is proportional to

$$
L_n(\theta; X_n) = n^{-1} \left( X_n' \Gamma_n(f_\theta)^{-1} X_n + \log |\Gamma_n(f_\theta)| \right)
$$

(1.1)

where $\Gamma_n(f_\theta)_{s,t} = c_{f_\theta}(s-t)$, $|A|$ denotes the determinant of the matrix $A$ and $X_n' = (X_1, \ldots, X_n)$. In contrast, the Whittle likelihood is a “spectral divergence” between the periodogram and the conjectured spectral density. There are two subtly different methods for defining this contrast,
one is with an integral the other is to use the Riemann sum. In this paper, we focus on the Whittle likelihood defined in terms of the Riemann sum over the fundamental frequencies

\[ K_n(\theta; X_n) = n^{-1} \sum_{k=1}^{n} \left( \frac{|J_n(\omega_{k,n})|^2}{f_{\theta}(\omega_{k,n})} + \log f_{\theta}(\omega_{k,n}) \right) \quad \omega_{k,n} = \frac{2\pi k}{n}, \]  

(1.2)

where \( J_n(\omega_{k,n}) = n^{-1/2} \sum_{t=1}^{n} X_t \exp(it\omega_{k,n}) \) is the discrete Fourier transform (DFT) of the observed time series. To compare the Gaussian and Whittle likelihood, we rewrite the Whittle likelihood in matrix form. We define the \( n \times n \) circulant matrix \( C_n(f_{\theta}) \) with entries \( (C_n(f_{\theta}))_{s,t} = n^{-1} \sum_{k=1}^{n} f_{\theta}(\omega_{k,n}) \exp(-is(t)) \omega_{k,n}) \). The Whittle likelihood \( K_n(\theta; X_n) \) can be written as

\[ K_n(\theta; X_n) = n^{-1} \left( X_n'C_n(f^{-1}_{\theta})X_n + \sum_{k=1}^{n} \log f_{\theta}(\omega_{k,n}) \right). \]  

(1.3)

To obtain an exact expression for \( \Gamma_n(f_{\theta}^{-1}) - C_n(f_{\theta}^{-1}) \) and \( X_n'[\Gamma_n(f_{\theta})^{-1} - C_n(f_{\theta}^{-1})]X_n \), we focus on the DFT of the time series. The idea is to obtain the linear transformation of the observed time series \( \{X_t\}_{t=1}^{n} \) which is biorthogonal to the regular DFT, \( \{J_n(\omega_{k,n})\}_{k=1}^{n} \). The biorthogonal transform when coupled with the regular DFT exactly decorrelates the time series. In Section 2.3, we show that the biorthogonal transform corresponding to the DFT contains the regular DFT plus the Fourier transform of the best linear predictors of the time series outside the domain of observation. Since this transformation completes the information not found in the regular DFT, we call it the complete DFT. It is common to use the Cholesky decomposition to decompose the inverse of a Toeplitz matrix. An interesting aspect of the biorthogonal transformation is that it provides an alternative decomposition of the inverse of a Toeplitz matrix.

In Section 2.4 we show that the complete DFT, together with the regular DFT, allows us to rewrite the Gaussian likelihood within the frequency domain (which, as far as we are aware, is new). Further, it is well known that the Whittle likelihood is biased due to the boundary effect. By rewriting the Gaussian likelihood within the frequency domain we show that the Gaussian likelihood avoids the boundary effect problem by predicting the time series outside the domain of observation. Precisely, the approximation error between the Gaussian and Whittle likelihood is due to the omission of these linear predictors in the regular DFT. From this result, we observe that the greater the persistence in the time series model (which corresponds to a more peaked spectral density) the larger the loss in approximating the complete DFT with the regular DFT. In order to obtain a better approximation of the Gaussian likelihood in the frequency domain, it is of interest to approximate the difference of the two likelihoods \( L_n(\theta; X_n) - K_n(\theta; X_n) \). For autoregressive processes of finite order, one can obtain an analytic expression for the difference in the two likelihoods in terms of the AR parameters (see equation (2.17)). Such an expression does not exist for general second order stationary models. However, in Section 3 we obtain approximations for \( L_n(\theta; X_n) - K_n(\theta; X_n) \) in terms of infinite order autoregressive representations and finite order.
autoregressive approximations. Using the finite order autoregressive approximation, in Section 4 we define two spectral divergence criteria which are “almost” unbiased estimators of the spectral divergence between the true (underlying spectral) density and the parametric spectral density. We use these criteria to define two new frequency domain estimators. Finally, in Section 5 we illustrate and compare the proposed frequency domain estimators through some simulations. We study the performance of the estimation scheme when the parametric model is both correctly specified and misspecified.

Most proofs can be found in the Supplementary Appendix A and B. In Supplementary Appendix C we derive an expression for the asymptotic bias of the Gaussian and Whittle likelihoods.

2 The Gaussian likelihood in the frequency domain

2.1 Preliminaries

In this section we introduce most of the notation used in the paper, it can be skipped on first reading. To reduce notation, we omit the symbol $X_n$ in the Gaussian and Whittle likelihood. Moreover, since the focus in this paper would be on the first terms in the Gaussian and Whittle likelihoods we use $L_n(\theta)$ and $K_n(\theta)$ to denote only these terms:

$$L_n(\theta) = n^{-1}X_n^\prime\Gamma_n(f_{\theta}^{-1})^{-1}X_n \quad \text{and} \quad K_n(\theta) = n^{-1}X_n^\prime C_n(f_{\theta}^{-1})X_n.$$ (2.1)

Let $A^*$ denote the conjugate transpose of the matrix $A$. We recall that the circulant matrix $C_n(g)$ can be written as $C_n(g) = F_n^*\Delta_n(g)F_n$, where $\Delta_n(g)$ is a diagonal matrix, $\Delta_n(g) = \text{diag}(g(\omega_{1,n}), \ldots, g(\omega_{n,n}))$ and $F_n$ is the DFT matrix with entries $(F_n)_{s,t} = n^{-1/2}\exp(is\omega_{t,n})$. We recall that the eigenvalues and eigenvectors of any circulant matrix $C_n(g)$ are $\{g(\omega_{k,n})\}_{k=1}^n$ and $\{e^{ik\omega_{k,n}}, \ldots, e^{ik\omega_{n,n}}\}_{k=1}^n$ respectively.

In general, we assume that $\mathbb{E}[X_t] = 0$ (as it makes the derivations cleaner). We use $\{c_f(r)\}_{r \in \mathbb{Z}}$ to denote an autocovariance function and $f(\omega) = \sum_{h \in \mathbb{Z}} c_f(h) \exp(ih\omega)$ its corresponding spectral density. Sometimes, it will be necessary to make explicit the true underlying covariance (or spectral density) of the process. In this case, we use $\text{cov}_f(X_t, X_{t+r}) = \mathbb{E}_f[X_tX_{t+r}] = c_f(r)$.

Next we define the norms we will use. Suppose $A$ is a $n \times n$ square matrix, let $\|A\|_p = (\sum_{i,j=1}^n |a_{i,j}|^p)^{1/p}$ be an entrywise $p$-norm for $p \geq 1$, and $\|A\|_{\text{spec}}$ denote the spectral norm. Let $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$, where $X$ is a random variable. For functions, suppose $g \in L_2[0, 2\pi]$ with $g(\omega) = \sum_{r \in \mathbb{Z}} g_r \exp(ir\omega)$, we use the sub-multiplicative norm $\|g\|_K = \sum_{r \in \mathbb{Z}} (2^K + |r|^K)|g_r|$. Note that if $\sum_{j=0}^{K+2} \sup_\omega |g^{(j)}(\omega)| < \infty$ then $\|g\|_K < \infty$, where $g^{(j)}(\cdot)$ denotes the $j$th derivative of $g$. Lastly, we denote $\Re$ and $\Im$ as the real and imaginary part of a complex variable.
2.2 Motivation

In order to motivate our approach, we first study the difference in the bias of the AR(1) parameter estimator using the both the Gaussian and Whittle likelihood. In Figure 1 we plot the bias in the estimator of $\phi$ in the AR(1) model $X_t = \phi X_{t-1} + \varepsilon_t$ for different values of $\phi$ (based on sample size $n = 50$). We observe that the difference between the bias of the two estimators increases as $|\phi|$ approaches one. Further, the Gaussian likelihood clearly has a smaller bias than the Whittle likelihood (which is more pronounced when $|\phi|$ is “far” from zero). Let $\{X_t\}_{t=1}^n$ denote

\[ X_t = \phi X_{t-1} + \varepsilon_t \]

the observed time series. Straightforward calculations (based on expressions for $\Gamma_n(f_{\phi})^{-1}$ and $C_n(f^{-1}_{\phi})$) show that the difference between the Gaussian and Whittle likelihoods for an AR(1) model is

\[ L_n(\phi) - K_n(\phi) = n^{-1} \left[ 2\phi X_1 X_n - \phi^2 (X_1^2 + X_n^2) \right] \]  

(2.2)

Thus we observe that the closer $|\phi|$ is to one, the larger the expected difference between the likelihoods. Using (2.2) and the Bartlett correction (see Bartlett (1952) and Cox and Snell (1968)) it is possible to obtain an asymptotic expression for the difference in the biases, see Supplementary C.1 (this expression closely matches the simulations in Figure 1). Generalisations of this result to higher order AR($p$) models may also be possible using the analytic expression for the inverse of the Toeplitz matrix corresponding to an AR($p$) model derived in Siddiqui (1958) and Galbraith and Galbraith (1974).

However, for more general models, such as the MA($q$) or ARMA($p,q$) models, using brute force calculations for deriving the difference is extremely difficult. Furthermore, such results

Figure 1: The model $X_t = \phi X_{t-1} + \varepsilon_t$ with independent standard normal errors is simulated. The bias of the estimator of $\phi$ is estimated using 500 replications.
do not offer any insight on how the Gaussian and Whittle likelihood are related, nor what is “lost” when going from the Gaussian likelihood to the Whittle likelihood. In the remainder of this section, we derive an exact expression for the Gaussian likelihood in the frequency domain. Using these derivations, we obtain a simple expression for the difference between the Whittle and Gaussian likelihood for AR($p$) models. In subsequent sections, we obtain approximations for this difference for general time series models.

### 2.3 The biorthogonal transform to the discrete Fourier transform

In order to obtain an exact bound, we start with the Whittle likelihood and recall that the DFT of the time series plays a fundamental role in its formulation. With this in mind, our approach is based on deriving the transformation \( \{Z_{k,n}\}_{k=1}^{n} \subset \text{sp}(X_n) \) (where \( \text{sp}(X_n) \) denotes the linear space over a complex field spanned by \( X_n = \{X_t\}_{t=1}^{n} \)), which is biorthogonal to \( \{J_n(\omega_k)\}_{k=1}^{n} \).

That is, we derive a transformation \( \{Z_{k,n}\}_{k=1}^{n} \) which when coupled with \( \{J_n(\omega_k,n)\}_{k=1}^{n} \) satisfies the following condition
\[
\text{cov}_f(Z_{k_1,n}, J_n(\omega_{k_2,n})) = f(\omega_{k_1})\delta_{k_1,k_2}
\]
where \( \delta_{k_1,k_2} = 1 \) if \( k_1 = k_2 \) (and zero otherwise). Since \( Z_n' = (Z_{1,n}, \ldots, Z_{n,n}) \in \text{sp}(X_n)^n \), there exists an \( n \times n \) complex matrix \( U_n \), such that \( Z_n = U_n X_n \). Since \( (J_n(\omega_{k,1}), \ldots, J_n(\omega_{n,n}))' = F_n X_n \), Thus biorthogonality of \( U_n X_n \) and \( F_n X_n \) gives \( \text{cov}_f[U_n X_n, F_n X_n] = \Delta_n(f) \). The benefit of biorthogonality is that it leads to the following simple identity on the inverse of the variance matrix

**Lemma 2.1** Suppose that \( U_n \) and \( V_n \) are invertible, biorthogonal transformation matrices in the sense that \( \text{cov}(U_n X_n, V_n X_n) = \Delta_n \), where \( \Delta_n \) is a diagonal matrix. Then
\[
\text{var}(X_n)^{-1} = V_n^* \Delta_n^{-1} U_n.
\]

PROOF. Follows immediately from \( \text{cov}(U_n X_n, V_n X_n) = U_n \text{var}(X_n) V_n^* = \Delta_n \) and \( \text{var}(X_n) = U_n^{-1} \Delta_n (V_n^*)^{-1} \).

To understand how \( U_n X_n \) is related to \( F_n X_n \) we rewrite \( U_n = F_n + D_n(f) \). We show in the following theorem that \( D_n(f) \) has a specific form with an intuitive interpretation. In order to develop these ideas we use methods from linear prediction. In particular, we define the best linear predictor of \( X_\tau \) for \( \tau \leq 0 \) and \( \tau > n \) given \( \{X_t\}_{t=1}^{n} \) as
\[
\hat{X}_{\tau,n} = \sum_{t=1}^{n} \phi_{t,n}(\tau; f) X_t,
\]
where \( \{\phi_{t,n}(\tau; f)\}_{t=1}^{n} \) are the coefficients which minimize the \( L_2 \)-distance \( \mathbb{E}_f(X_\tau - \sum_{t=1}^{n} \phi_{t,n}(\tau; f) X_t)^2 \).
Using this notation we obtain the following theorem.

**Theorem 2.1 (The biorthogonal transform)** Let \( \{X_t\} \) be a second order stationary, zero mean time series with spectral density \( f \) which is bounded away from zero and whose autocovariance satisfies \( \sum_{r \in \mathbb{Z}} |rc(r)| < \infty \). Let \( \hat{X}_{\tau,n} \) denote the best linear predictors of \( X_\tau \) as defined in (2.4) and \( \{\phi_s,n(\tau; f)\} \) the corresponding coefficients. Then

\[
\text{cov}_f [(F_n + D_n(f))X_n, F_nX_n] = \Delta_n(f),
\]

(2.5)

where \( D_n(f) \) has entries

\[
D_n(f)_{k,t} = n^{-1/2} \sum_{\tau \leq 0} (\phi_{t,n}(\tau; f)e^{i\tau \omega_{k,n}} + \phi_{n+1-t,n}(\tau; f)e^{-i(\tau-1)\omega_{k,n}}) \quad 1 \leq k, t \leq n.
\]

(2.6)

And, entrywise \( 1 \leq k_1, k_2 \leq n \), we have

\[
\text{cov}_f \left( \tilde{J}_n(\omega_{k_1,n}; f), J_n(\omega_{k_2,n}) \right) = f(\omega_{k_1,n})\delta_{k_1,k_2}
\]

(2.7)

where \( \tilde{J}_n(\omega; f) = J_n(\omega) + \hat{J}_n(\omega; f) \) and

\[
\hat{J}_n(\omega; f) = n^{-1/2} \sum_{\tau \leq 0} \hat{X}_{\tau,n}e^{i\tau \omega} + n^{-1/2} \sum_{\tau > n} \hat{X}_{\tau,n}e^{i\tau \omega}.
\]

PROOF See Appendix A.1 (note that identity (2.7) can be directly verified using results on best linear predictors).

**Remark 2.1 (Inverse Toeplitz identity)** Let \( \Gamma_n(f) \) denote an \( n \times n \) Toeplitz matrix generated by the spectral density \( f \). Then equation (2.3) and (2.5) yields the following identity

\[
\Gamma_n(f)^{-1} = F_n^*\Delta_n(f^{-1})(F_n + D_n(f)),
\]

(2.8)

where \( D_n(f) \) is defined in (2.6). Observe that two spectral density functions \( f_1(\omega) \) and \( f_2(\omega) \) with the same autocovariance up to lag \( n \) \( \{c(r); 0 \leq r \leq (n-1)\} \) can give rise to two different representations \( \Gamma_n(f_1)^{-1} = F_n^*\Delta_n(f_1^{-1})(F_n + D_n(f_1)) = F_n^*\Delta_n(f_2^{-1})(F_n + D_n(f_2)) = \Gamma_n(f_2)^{-1} \).

What we observe is that the biorthogonal transformation \( (F_n + D_n(f))X_n \) extends the domain of observation by predicting outside the boundary. A visualisation of the observations and the predictors that are involved in the construction of \( \tilde{J}_n(\omega; f) \) is given in Figure 2. It is quite surprising that only a small modification of the regular DFT leads to its biorthogonal transformation. Furthermore, the contribution of the predictive DFT is \( \tilde{J}_n(\omega_{k,n}; f) = O_p(n^{-1/2}) \). This is why the
Figure 2: $J_n(\omega; f)$ is the Fourier transform over both the observed time series and its predictors outside this domain.

The regular DFT satisfies the well known "near" orthogonal property

$$\text{cov}_f(J_n(\omega_{k_1,n}), J_n(\omega_{k_2,n})) = f(\omega_{k_1}) \delta_{k_1,k_2} + O(n^{-1}),$$

see Lahiri (2003) and Brillinger (2001). For future reference we will use the following definitions.

**Definition 2.1** We refer to $\hat{J}_n(\omega; f)$ as the predictive DFT (as it is the Fourier transform of all the linear predictors), noting that basic algebra yields the expression

$$\hat{J}_n(\omega; f) = n^{-1/2} \sum_{t=1}^{n} X_t \sum_{\tau \leq 0} (\phi_{t,n}(\tau; f)e^{i\tau\omega} + \phi_{n+1-t,n}(\tau; f)e^{-i(\tau-1)\omega}).$$

(2.9)

Further, we refer to $\tilde{J}_n(\omega; f)$ as the complete DFT (as it contains the classical DFT of the time series together with the predictive DFT). Note that both $\tilde{J}_n(\omega; f)$ and $\hat{J}_n(\omega; f)$ are functions of $f$ since they involve the spectral density $f(\cdot)$, unlike the regular DFT which is model-free.

**Example 2.1 (The AR(1) process)** Suppose that $X_t$ has an AR(1) representation $X_t = \phi X_{t-1} + \varepsilon_t (|\phi| < 1)$. Then the best linear predictors are simply a function of the observations at the two end points. That is for $\tau \leq 0$, $\hat{X}_{\tau,n} = \phi^{|\tau|+1}X_1$ and for $\tau > n$ $\hat{X}_{\tau,n} = \phi^{\tau-n}X_n$. An illustration is given in Figure 3.

Figure 3: The past and future best linear predictors based on a AR(1) model.

Then the predictive DFT for the AR(1) model is

$$\hat{J}_n(\omega; f_\phi) = \frac{\phi}{\sqrt{n}} \left( \frac{1}{\phi(\omega)} X_1 + \frac{e^{i\omega}}{\phi(\omega)} X_n \right) \quad \text{where} \quad \phi(\omega) = 1 - \phi \exp(-i\omega).$$
In other words, a small adjustment at the boundary leads to \( \tilde{J}_n(\omega; f_\phi) \overline{J}_n(\omega) \) being an unbiased estimator of \( f(\omega) = \sigma^2|\phi(\omega)|^{-2} \).

**Remark 2.2** Biorthogonality of random variables is rarely used in statistics. An interesting exception is Kasahara et al. (2009). They apply the notion of biorthogonality to problems in prediction. In particular they consider the biorthogonal transform of \( \tilde{X}_n \), which is the random vector \( \tilde{X}_n = \Gamma_n(f)^{-1} X_n \) (since \( \text{cov}_f(\tilde{X}_n, X_n) = I_n \)). They obtain an expression for the entries of \( \tilde{X}_n \) in terms of the Cholesky decomposition of \( \Gamma_n(f)^{-1} \). However, there is an interesting duality between \( \tilde{X}_n \) and \( \tilde{J}_n = (\tilde{J}_n(\omega_k; f), \ldots, \tilde{J}_n(\omega_n; f)) \). In particular, applying identity (2.8) to the DFT of \( \tilde{X}_n \) gives

\[
F_n \tilde{X}_n = F_n \Gamma_n(f)^{-1} X_n = \Delta_n(f^{-1}) \tilde{J}_n.
\]

This shows that the DFT of the biorthogonal transform of \( X_n \) is the standardized complete DFT. Conversely, the inverse DFT of the standardized complete DFT gives the biorthogonal transform to the original time series, where the entries of \( \tilde{X}_n \) are

\[
\tilde{X}_{j,n} = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{\tilde{J}_n(\omega_{k,n}; f)}{f(\omega_{k,n})} \exp(-ij\omega_{k,n}).
\]

**Remark 2.3** (Connection to the orthogonal increment process) Suppose that \( Z(\omega) \) is the orthogonal increment process associated with the stationary time series \( \{X_t\} \) and \( f \) the corresponding spectral density. If \( \{X_t\} \) is a Gaussian time series, then by using Theorem 4.9.1, Brockwell and Davis (2006), we have

\[
\tilde{X}_{\tau,n} = \mathbb{E}(X_{\tau} | X_n) = \frac{1}{2\pi} \int_0^{2\pi} \exp(-i\omega\tau) \mathbb{E}(Z(d\omega) | X_n) = \frac{n}{2\pi} \int_0^{2\pi} \exp(-i\omega\tau) \tilde{J}_n(\omega; f) d\omega.
\]

### 2.4 The Gaussian likelihood in the frequency domain

In the following theorem, we exploit the biorthogonality between the regular DFT and the complete DFT to yield an exact "frequency domain" representation for the Gaussian likelihood. We use the notation defined in Theorem 2.1.

**Theorem 2.2** (A frequency domain representation of the Gaussian likelihood) Suppose the spectral density \( f_\theta \) is bounded away from zero, and the corresponding autocovariance is such that \( \sum_r |rc_{f_\theta}(r)| < \infty \). Let \( L_n(\theta) \) and \( K_n(\theta) \) be defined as in (2.1). Then we have

\[
L_n(\theta) = \frac{1}{n} X_n' \Gamma_n(f_\theta)^{-1} X_n = \frac{1}{n} \sum_{k=1}^{n} \frac{\tilde{J}_n(\omega_{k,n}; f_\theta) \overline{J}_n(\omega_{k,n})}{f_\theta(\omega_{k,n})}.
\]

(2.10)
Further

\[ \Gamma_n(f_\theta)^{-1} - C_n(f_\theta)^{-1} = F_n^* \Delta_n(f_\theta^{-1})D_n(f_\theta). \]  

(2.11)

This yields the difference

\[ \mathcal{L}_n(\theta) - K_n(\theta) = \frac{1}{n} X_n' \left[ \Gamma_n(f_\theta)^{-1} - C_n(f_\theta)^{-1} \right] X_n = \frac{1}{n} \sum_{k=1}^{n} \frac{\hat{J}_n(\omega_{k,n}; f_\theta)\tilde{J}_n(\omega_{k,n})}{f_\theta(\omega_{k,n})}. \]  

(2.12)

PROOF. By using (2.8) we have \( \Gamma_n(f_\theta)^{-1} = F_n^* \Delta_n(f_\theta^{-1})(F_n + D_n(f_\theta)) \), thus we obtain (2.11). Next, we note that \( F_n X_n = J_n \) and \( (F_n + D_n(f_\theta))X_n = \tilde{J}_n \), thus we immediately obtain equation (2.10), and since \( \tilde{J}_n(\omega_{k,n}; f_\theta) = J_n(\omega_{k,n}) + \hat{J}_n(\omega_{k,n}; f_\theta) \), it proves (2.12).

From the above theorem, we observe that the Gaussian likelihood is the Whittle likelihood plus an additional “correction”

\[ \mathcal{L}_n(\theta) = \frac{1}{n} \sum_{k=1}^{n} \frac{|J_n(\omega_{k,n})|^2}{f_\theta(\omega_{k,n})} + \frac{1}{n} \sum_{k=1}^{n} \frac{\hat{J}_n(\omega_{k,n}; f_\theta)\tilde{J}_n(\omega_{k,n})}{f_\theta(\omega_{k,n})}. \]

\[ = K_n(\theta) \]

To summarize, the Gaussian likelihood compensates for the well known boundary effect in the Whittle likelihood, by predicting outside the domain of observation. The Whittle likelihood estimator selects the spectral density \( f_\theta \) which best fits the periodogram. On the other hand, since \( \mathbb{E}_{f_\theta}[\hat{J}_n(\omega_{k,n}; f_\theta)\tilde{J}_n(\omega_{k,n})] = f_\theta(\omega_{k,n}) \), the Gaussian likelihood estimator selects the spectral density which best fits \( \hat{J}_n(\omega_{k,n}; f_\theta)\tilde{J}_n(\omega_{k,n}) \) by simultaneously predicting and fitting. Therefore, the “larger” the level of “persistence” in the time series, the greater the predictive DFT \( \hat{J}_n(\omega_{k,n}; f_\theta) \), and subsequently the larger the approximation error between the two likelihoods. This fits with the insights of \cite{Dahlhaus1988}, who shows that the more peaked the spectral density the greater the leakage effect in the Whittle likelihood, leading to a large finite sample bias.

In the remainder of this section and the subsequent section, we study the difference between the two likelihoods and corresponding matrices. This will allow us to develop methods that better capture the Gaussian likelihood within the frequency domain. By using Theorem 2.2 we have

\[ \mathcal{L}_n(\theta) - K_n(\theta) = \frac{1}{n} \sum_{k=1}^{n} \frac{\hat{J}_n(\omega_{k,n}; f_\theta)\tilde{J}_n(\omega_{k,n})}{f_\theta(\omega_{k,n})} = X_n' F_n^* \Delta_n(f_\theta^{-1})D_n(f_\theta)X_n, \]

where the entries of \( F_n^* \Delta_n(f_\theta^{-1})D_n(f_\theta) \) are

\[ (F_n^* \Delta_n(f_\theta^{-1})D_n(f_\theta))_{s,t} = \sum_{\tau \leq 0} [\phi_{1,n}(\tau; f_\theta)G_{1,n}(s, \tau; f_\theta) + \phi_{n+1-t,n}(\tau; f_\theta)G_{2,n}(s, \tau; f_\theta)] \]  

(2.13)
with

\[
G_{1,n}(s, \tau; f_\theta) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{f_\theta(\omega_{k,n})} e^{i(\tau-s)\omega_{k,n}} = \sum_{a \in \mathbb{Z}} K_{f_\theta^{-1}}(\tau - s + an)
\]

\[
G_{2,n}(s, \tau; f_\theta) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{f_\theta(\omega_{k,n})} e^{-i(\tau+s-1)\omega_{k,n}} = \sum_{a \in \mathbb{Z}} K_{f_\theta^{-1}}(\tau + s - 1 + an)
\]

and \(K_{f_\theta^{-1}}(r) = \int_0^{2\pi} f_\theta(\omega)^{-1} \exp(i\omega) d\omega\). We observe that for \(1 < t < n\), \(\phi_{t,n}(\tau; f_\theta)\) and \(\phi_{n+1-t,n}(\tau; f_\theta)\) will be “small” as compared with \(t\) close to one or \(n\). The same is true for \(G_{1,n}(s, \tau; f_\theta)\) and \(G_{2,n}(s, \tau; f_\theta)\) when \(1 < s < n\). Thus the the entries of \(F_n^* \Delta_n(f_\theta^{-1}) D_n(f_\theta)\) will be “small” far from the four corners of matrices. In contrast the entries of \(F_n^* \Delta_n(f_\theta^{-1}) D_n(f_\theta)\) will be largest at the four corners at the matrix. This can be clearly seen in the following theorem, where we consider the special case of AR(p) models. We showed in Example 2.1 that for AR(1) processes, the predictive DFT has a simple form. In the following theorem, we obtain an analogous result for AR(p) models (where \(p \leq n\)).

**Theorem 2.3 (Finite order autoregressive models)** Suppose that \(f_\theta(\omega) = \sigma^2 |\phi_p(\omega)|^{-2}\) where \(\phi_p(\omega) = 1 - \sum_{u=1}^{p} \phi_u \exp(-iu\omega)\) (the roots of the corresponding characteristic polynomial lie outside the unit circle) and \(p \leq n\). The predictive DFT has the analytic form

\[
\hat{J}_n(\omega; f_\theta) = \frac{n^{-1/2}}{\phi_p(\omega)} \sum_{\ell=1}^{p} X_\ell \sum_{s=0}^{p-\ell} \phi_{\ell+s} \exp(-is\omega) + \frac{n^{-1/2}}{\phi_p(\omega)} \sum_{\ell=1}^{p} X_{n+1-\ell} \sum_{s=0}^{p-\ell} \phi_{\ell+s} \exp(i(s+1)\omega).
\]

(2.14)

If \(p \leq n/2\), then \(D_n(f_\theta)\) is a rank 2p matrix where

\[
D_n(f_\theta) = n^{-1/2}
\begin{pmatrix}
\phi_{1,1}(\omega_{n,1}) & \ldots & \phi_{p,1}(\omega_{n,1}) & 0 & \ldots & 0 & e^{i\omega_{1,n}} \phi_{p,p}(\omega_{1,n}) & \ldots & e^{i\omega_{1,n}} \phi_{1,p}(\omega_{1,n})
\end{pmatrix}
\]

\[
\begin{pmatrix}
\phi_{1,2}(\omega_{n,2}) & \ldots & \phi_{p,2}(\omega_{n,2}) & 0 & \ldots & 0 & e^{i\omega_{2,n}} \phi_{p,p}(\omega_{2,n}) & \ldots & e^{i\omega_{2,n}} \phi_{1,p}(\omega_{2,n})
\end{pmatrix}
\]

\[
\begin{pmatrix}
\phi_{1,n}(\omega_{n,n}) & \ldots & \phi_{p,n}(\omega_{n,n}) & 0 & \ldots & 0 & e^{i\omega_{n,n}} \phi_{p,p}(\omega_{n,n}) & \ldots & e^{i\omega_{n,n}} \phi_{1,p}(\omega_{n,n})
\end{pmatrix}
\]

(2.15)

and \(\phi_{j,p}(\omega) = \phi_p(\omega)^{-1} \sum_{s=0}^{p-j} \phi_{j+s} \exp(-is\omega)\). Note if \(n/2 < p \leq n\), then the entries of \(D_n(f_\theta)\) will overlap. Let \(\tilde{\omega}_0 = 1\) and for \(1 \leq s \leq p\), \(\tilde{\omega}_s = -\phi_s\) (zero otherwise), then if \(1 \leq p \leq n/2\) we have

\[
(G_n(f_\theta)^{-1} - C_n(f_\theta^{-1}))_{s,t} = (F_n^* \Delta_n(\tilde{f_\theta}^{-1}) D_n(f_\theta))_{s,t}
\]

\[
= \begin{cases}
\sigma^{-2} \sum_{t=0}^{p-t} \phi_{\ell+t,\tilde{\omega}}(\ell+s) \mod n & 1 \leq t \leq p \\
\sigma^{-2} \sum_{t=0}^{p-(n-t)} \phi_{\ell+(n-t),\tilde{\omega}}(\ell-s) \mod n & n-p+1 \leq t \leq n \\
0 & \text{otherwise}
\end{cases}
\]

(2.16)

11
Theorem 2.3 shows that for AR($p$) models, the predictive DFT only involves the $p$ observations on each side of the observational boundary $X_1, \ldots, X_p$ and $X_{n-p+1}, \ldots, X_n$, where the coefficients in the prediction are a linear combination of the AR parameters (excluding the denominator $\phi_p(\omega)$). The well known result, that $F_n^* \Delta_n(f_{\theta}^{-1}) D_n(f_{\theta})$ is non-zero only at the $(p \times p)$ submatrices located in the four corners of $F_n^* \Delta_n(f_{\theta}^{-1}) D_n(f_{\theta})$ follows from equation (2.16).

By using (2.14) we obtain an analytic expression for the Gaussian likelihood of the AR($p$) model in terms of the autoregressive coefficients. In particular, the Gaussian likelihood (written in the frequency domain) corresponding to the AR($p$) model $X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t$ is

$$L_n(\phi) = \frac{\sigma^{-2}}{n} \sum_{k=1}^{n} \left| J_n(\omega_{k,n}) \right|^2 |\phi(\omega_{k,n})|^2 + \frac{\sigma^{-2}}{n} \sum_{\ell=1}^{p} X_{\ell} \sum_{s=0}^{p-\ell} \phi_{\ell+s} \left( X_{(s+n) \mod n} - \sum_{j=1}^{p} \phi_j X_{(j-s+n) \mod n} \right) \right.$$ 

$$+ \frac{\sigma^{-2}}{n} \sum_{\ell=1}^{p} X_{n+1-\ell} \sum_{s=0}^{p-\ell} \phi_{\ell+s} \left( X_{(s+1+n) \mod n} - \sum_{j=1}^{p} \phi_j X_{(s+1+j-n) \mod n} \right),$$

(2.17)

where $\phi = (\phi_1, \ldots, \phi_p)'$ and $\phi(\omega) = 1 - \sum_{j=1}^{p} \phi_j \exp(-ij\omega)$. A proof of the above identity can be found in Supplementary A.1. Equation (2.17) offers a simple representation of the Gaussian likelihood in terms of a Whittle likelihood plus an additional term in terms of the AR($p$) coefficients.

3 Frequency domain approximations of the Gaussian likelihood

In Theorem 2.2 we rewrote the Gaussian likelihood within the frequency domain. This allowed us to obtain an expression for the difference between the Gaussian and Whittle likelihoods for AR($p$) models (see (2.17)). This is possible because the predictive DFT $\hat{J}_n(\cdot, f_{\theta})$ has a simple analytic form.

It would be of interest to generalize this result to general time series models. However, for infinite order autoregressive models the predictions across the boundary involve the prediction coefficients which are unwieldy functions of the underlying autocovariance $f_{\theta}(\omega)$. Thus, in general, the predictive DFT given in (2.9) does not have a simple, analytic form. In this section, we show that we can obtain an approximation of the predictive DFT in terms of the AR($\infty$) coefficients corresponding to $f_{\theta}$. In turn, this allows us to obtain an approximation for $\Gamma_n(f_{\theta})^{-1} - C_n(f_{\theta}^{-1})$, which is analogous to equation (2.16) for AR($p$) models. Such a result proves to be very useful from both a theoretical and practical perspective. Theoretically, we use it to show that the difference between the Whittle and Gaussian likelihood is of order $O(n^{-1})$. However, arguably, the practical implications may be of greater interest. In Section 4 we show that these approx-
imations can be used to motivate an alternative quasi-likelihood defined within the frequency domain.

We require the following set of assumptions on the spectral density \( f_\theta \).

**Assumption 3.1**

(i) The spectral density \( f \) is bounded away from zero.

(ii) For some \( K \geq 1 \), the autocovariance function is such that \( \sum_{r \in \mathbb{Z}} |r^K c(r)| < \infty \).

Under the above assumptions, we can write \( f(\omega) = \sigma^2 |\psi(\omega; f)|^2 = \sigma^2 |\phi(\omega; f)|^{-2} \) where

\[
\psi_f(\omega) = 1 + \sum_{j=1}^{\infty} \psi_j(f) \exp(-i j \omega) \quad \text{and} \quad \phi_f(\omega) = 1 - \sum_{j=1}^{\infty} \phi_j(f) \exp(-i j \omega).
\]  

(3.1)

Further, under Assumption 3.1, we have \( \sum_{r=1}^{\infty} |r^K \psi_r(f)| < \infty \) and \( \sum_{r=1}^{\infty} |r^K \phi_r(f)| < \infty \) (see Kreiss et al. (2011)). Thus if \( \|f\|_K < \infty \), and \( f \) is bounded away from zero, then \( \|\psi_f\|_K \) and \( \|\phi_f\|_K < \infty \) are both finite.

In the remainder of this section we use the notation \( \rho_{n,K}(f) = \sum_{r=1}^{\infty} |r^K \phi_r(f)|, A_K(f,g) = 2\sigma^2 \|\psi_f\|_0 \|\phi_g\|_0 \|\phi_f\|_K \) and \( C_{f,K} = \frac{3-\epsilon}{1-\epsilon} \|\phi_f\|_K^2 \|\psi_f\|_K^2 \) (for some \( 0 < \epsilon < 1 \), its precise role is explained in Supplementary B.2).

### 3.1 Approximations and bounds

In order to obtain a result analogous to Theorem 2.3, we replace \( \phi_{s,n}(\tau; f_\theta) \) in \( D_n(f_\theta) \) with \( \phi_s(\tau; f_\theta) \) which are the coefficients of the best linear predictor of \( X_\tau \) (for \( \tau \leq 0 \)) given \( \{X_t\}_{t=1}^{\infty} \) i.e. \( \hat{X}_\tau = \sum_{t=1}^{\infty} \phi_t(\tau; f_\theta) X_t \). This gives the matrix \( D_{\infty,n}(f_\theta) \), where

\[
(D_{\infty,n}(f_\theta))_{k,t} = n^{-1/2} \sum_{\tau=0}^{\infty} (\phi_{t}(\tau; f_\theta) e^{i\tau \omega_k,n} + \phi_{n+1-t}(\tau; f_\theta) e^{-i(\tau-1) \omega_k,n}) \quad 1 \leq k, t \leq n.
\]

It can be shown that

\[
(D_{\infty,n}(f_\theta))_{k,t} = n^{-1/2} \frac{\phi_{k}^\infty(\omega_k,n; f_\theta)}{\phi(\omega_k,n; f_\theta)} + n^{-1/2} e^{i\omega_k,n} \frac{\phi_{n+1-t}^\infty(\omega_k,n; f_\theta)}{\phi(\omega_k,n; f_\theta)} \quad 1 \leq k, t \leq n,
\]  

(3.2)

where \( \phi_{k}^\infty(\omega; f_\theta) = \sum_{s=0}^{\infty} \phi_{t+s}(f_\theta) e^{-is \omega} \). The proof of the above identity can be found in Supplementary A.2. Using the above we can show that \( (D_{\infty,n}(f_\theta) X_n)_k = \hat{J}_{\infty,n}(\omega_k,n; f_\theta) \) where

\[
\hat{J}_{\infty,n}(\omega; f_\theta) = \frac{n^{-1/2}}{\phi(\omega; f_\theta)} \sum_{t=1}^{n} X_t \sum_{s=0}^{\infty} \phi_{t+s}(f_\theta) e^{-is \omega} + \frac{n^{-1/2}}{\phi(\omega; f_\theta)} \sum_{t=1}^{n} X_{n+1-t} \sum_{s=0}^{\infty} \phi_{t+s}(f_\theta) e^{i(s+1) \omega}.
\]  

(3.3)

We show below that \( \hat{J}_{\infty,n}(\omega_k,n; f_\theta) \) is an approximation of \( \hat{J}_n(\omega_k,n; f_\theta) \).
Theorem 3.1 (An AR(∞) approximation for general processes) Suppose \( f \) satisfies Assumption 3.1, \( f_\theta \) is bounded away from zero and \( \|f_\theta\|_{0} < \infty \) (with \( f_\theta(\omega) = \sigma_\theta^2 |\phi_\theta(\omega)|^{-2} \)). Let \( D_n(f) \), \( D_{\infty,n}(f) \) and \( \tilde{J}_{\infty,n}(\omega_k,n;f) \) be defined as in (2.6) and (3.2) and (3.3) respectively. Then we have

\[
X_n' F_n^* \Delta_n(f_\theta^{-1}) (D_n(f) - D_{\infty,n}(f)) X_n = \sum_{k=1}^{n} \frac{J_n(\omega_k,n)}{f_\theta(\omega_k,n)} \left( \tilde{J}_n(\omega_k,n;f) - \tilde{J}_{\infty,n}(\omega_k,n;f) \right)
\]

and

\[
\left\| F_n^* \Delta_n(f_\theta^{-1}) (D_n(f) - D_{\infty,n}(f)) \right\|_{1} \leq \frac{C_{f,0}\rho_{n,K}(f)}{n^{K-1}} A_K(f, f_\theta).
\]

Further, if \( \{X_t\} \) is a time series where \( \sup_t \|X_t\|_{2q} = \|X\|_{2q} < \infty \) (for some \( q > 1 \)). Then

\[
n^{-1} \left\| X_n' F_n^* \Delta_n(f_\theta^{-1}) (D_n(f) - D_{\infty,n}(f)) X_n \right\|_{q} \leq \frac{C_{f,0}\rho_{n,K}(f)}{n^{K}} A_K(f, f_\theta) \|X\|_{2q}.
\]

PROOF See Section A.2 \hfill \Box

We mention that we state the above theorem in the general case that the spectral density \( f \) is used to construct the predictors \( D_n(f) \). It does not necessarily have to be the same as \( f_\theta \). This is to allow generalisations on the Whittle and Gaussian likelihoods, which we discuss in Section 4.

Applying the above theorem to the Gaussian likelihood gives an approximation which is analogous to (2.17)

\[
\mathcal{L}_n(\theta) = K_n(\theta) + \frac{1}{n} \sum_{k=1}^{n} \frac{J_{\infty,n}(\omega_k,n;f_\theta)J_n(\omega_k,n)}{f_\theta(\omega_k,n)} + O_p(n^{-K})
\]

\[
= K_n(\theta) + \frac{1}{n} \sum_{t,s=1}^{n} X_t X_s \frac{1}{n} \sum_{k=1}^{n} e^{-is\omega_k,n} \varphi_{t,n}(\omega_k,n;f_\theta) + O_p(n^{-K}),
\]

(3.7)

where \( \varphi_{t,n}(\omega_k,n;f_\theta) = \sigma^{-2} \left[ \phi(\omega_k,n;f_\theta) \phi_t^\infty(\omega_k,n;f_\theta) + e^{i\omega_k,n} \phi(\omega_k,n;f_\theta) \phi_t^{\infty}_{n+1-t}(\omega_k,n;f_\theta) \right] \). The above approximation shows that if the autocovariances, corresponding to \( f_\theta \), decay sufficiently fast in the sense that \( \sum_{r \in \mathbb{Z}} |r^K c_{f_\theta}(r)| < \infty \) for some \( K \geq 1 \), then replacing the finite predictions with the predictors using the infinite past (or future) gives a very close approximation of the Gaussian likelihood.

Remark 3.1 The entrywise difference between the two matrices is approximately

\[
(\Gamma_n(f_\theta)^{-1} - C_n(f_\theta^{-1}))_{s,t} \approx (F_n^* \Delta_n(f_\theta^{-1}) D_{\infty,n}(f_\theta))_{s,t} = \frac{1}{n} \sum_{k=1}^{n} e^{-is\omega_k,n} \varphi_{t,n}(\omega_k,n;f_\theta),
\]

14
thus giving an analytic approximation to (2.13).

In the following theorem, we use this expansion to obtain a bound between the Gaussian and Whittle likelihood.

**Theorem 3.2 (The $L_1$-norm of $\Gamma_n(f_\theta)^{-1} - C_n(f_\theta^{-1})$)** Suppose $f_\theta$ satisfies Assumption 3.1. Let $D_n(f_\theta)$ and $D_{\infty,n}(f_\theta)$ be defined as in (2.6) and (3.2) respectively. Then we have

$$\|F_n^* \Delta_n(f_\theta^{-1}) D_{\infty,n}(f_\theta)\|_1 \leq A_1(f_\theta, f_\theta)$$

(3.8)

and

$$\|\Gamma_n(f_\theta)^{-1} - C_n(f_\theta^{-1})\|_1 \leq \left(A_1(f_\theta, f_\theta) + \frac{C_{f_\theta,0}(f_\theta)}{nK^{-1}} A_K(f_\theta, f_\theta)\right).$$

(3.9)

Further, suppose $\{X_t\}$ is a time series where $\sup_t \|X_t\|_2 = \|X\|_2 < \infty$ (for some $q > 1$). Then we have

$$\|L_n(\theta) - K_n(\theta)\|_q \leq n^{-1} \left(A_1(f_\theta, f_\theta) + \frac{C_{f_\theta,0}(f_\theta)}{nK^{-1}} A_K(f_\theta, f_\theta)\right) \|X\|_2^q.$$  

(3.10)

**PROOF** See Appendix A.2.

The above result shows that under the stated conditions

$$n^{-1} \|\Gamma_n(f_\theta)^{-1} - C_n(f_\theta^{-1})\|_1 = O(n^{-1}),$$

and the difference between the Whittle and Gaussian likelihoods is of order $O(n^{-1})$. Further, the approximation of the Gaussian likelihood in equation (3.7) can be used to give a mixed Whittle/Gaussian likelihood estimator, which reduces the difference between the two likelihoods. Specifically, by truncating the AR($\infty$) parameters to the first $p$ coefficients $\{\phi_j(f_\theta)\}_{j=1}^p$, we can define an approximation which is analogous to (2.14). This is a compromise between the “simplicity” of the Whittle likelihood and the accuracy of the Gaussian likelihood.

In the following section we propose a variant of this idea, which is simple to evaluate and leads to estimators which tend to have a smaller bias than the Whittle likelihood.

### 4 New frequency domain quasi-likelihoods

In this section we apply the approximations from the previous section to define two new spectral divergence criteria.

To motivate the criterion, we recall from Theorem 2.2 that the Gaussian likelihood can be written as a contrast between $\tilde{J}_n(\omega; f_\theta)\tilde{J}_n(\omega)$ and $f_\theta(\omega)$. The resulting estimator is based on simultaneously predicting and fitting the spectral density. In the case, that the model is correctly
specified in the sense there exists a \( \theta \in \Theta \) where \( f = f_{\theta} \) (and \( f \) is the true spectral density). Then
\[
\mathbb{E}_{f_{\theta}}[\tilde{J}_n(\omega; f_{\theta})J_n(\omega)] = f_{\theta}(\omega)
\]

and the Gaussian criterion has a clear interpretation. However, if the model is misspecified (which for real data is likely), \( \mathbb{E}_f[\tilde{J}_n(\omega; f_{\theta})J_n(\omega)] \) has no clear interpretation. Instead, to understand what the Gaussian likelihood is estimating, we use that
\[
\mathbb{E}_{f}[\hat{J}_n(\omega; f_{\theta})J_n(\omega)] = \mathcal{O}(n^{-1}),
\]

which leads to the approximation
\[
\mathbb{E}_f[\tilde{J}_n(\omega; f_{\theta})J_n(\omega)] = f(\omega) + \mathcal{O}(n^{-1}).
\]

From this we observe that the Gaussian likelihood is
\[
n^{-1}\mathbb{E}_f[X_n'\Gamma_n(f_{\theta})^{-1}X_n] + n^{-1}\det(\Gamma_n(f_{\theta})) = I(f, f_{\theta}) + \mathcal{O}(n^{-1}),
\]

where
\[
I_n(f; f_{\theta}) = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{f(\omega_{k,n})}{f_{\theta}(\omega_{k,n})} + \log f_{\theta}(\omega_{k,n}) \right).
\]

Since \( I_n(f; f_{\theta}) \) is the spectral divergence between the true spectral \( f \) density and parametric spectral density \( f_{\theta} \), asymptotically the mispecified Gaussian likelihood estimator has a meaningful interpretation. However, there is still a finite sample bias in the Gaussian likelihood of order \( O(n^{-1}) \). This can have an knock on effect, by increasing the finite sample bias in the resulting Gaussian likelihood estimator. To remedy this, in the following section, we obtain a frequency domain criterion which approximates the spectral divergence \( I_n(f; f_{\theta}) \) to a greater degree of accuracy. This has the potential to lead to a class of estimators which give a more accurate fit of the underlying spectral density.

We mention that strictly, the spectral divergence is
\[
n^{-1}\sum_{k=1}^{n} \left( \frac{f(\omega_{k,n})}{f_{\theta}(\omega_{k,n})} - \log f_{\theta}(\omega_{k,n}) - 1 \right),
\]

which is zero when \( f_{\theta} = f \) and positive for other values of \( f_{\theta} \). But since \( -\log f(\omega) - 1 \) does not depend on \( \theta \) we ignore these terms.

### 4.1 The boundary corrected Whittle likelihood

In order to address some of the issues raised above, we recall from Theorem 2.1 that
\[
\mathbb{E}_f[\tilde{J}_n(\omega; f)J_n(\omega)] = f(\omega).
\]

In other words, by predicting over the boundary using the (unobserved) spectral density which generates the data, the “complete periodogram” \( \tilde{J}_n(\omega; f)J_n(\omega) \) is an inconsistent but unbiased of the true spectral density \( f \). This motivates the infeasible criterion
\[
W_n(\theta) = \frac{1}{n} \sum_{k=1}^{n} \frac{\tilde{J}_n(\omega_{k,n}; f)J_n(\omega_{k,n})}{f_{\theta}(\omega_{k,n})} + \frac{1}{n} \sum_{k=1}^{n} \log f_{\theta}(\omega_{k,n}).
\]
Thus, if \( \{X_t\} \) is a second order stationary time series with spectral density \( f \), then we have \( \mathbb{E}_f[W_n(\theta)] = I_n(f; f_\theta) \).

Of course \( f \) and thus \( \tilde{J}_n(\omega_{k,n}; f) \) are unknown. However, we recall that \( \tilde{J}_n(\omega_{k,n}; f) \) is comprised of the best linear predictors based on the unobserved time series. The coefficients of the best linear predictors can be replaced with the \( h \)-step ahead predictors evaluated with the best fitting autoregressive parameters of order \( p \) (the so called plug-in estimators; see Bhansali (1996) and Kley et al. (2019)). This is equivalent to replacing \( f \) in \( \tilde{J}_n(\omega_{k,n}; f) \) with the spectral density function corresponding to the best fitting AR(\( p \)) process \( \tilde{J}_n(\omega_{k,n}; f_p) \), where an analytic form is given in (2.14). Since we have replaced \( f \) with \( f_p \), the “periodogram” \( \tilde{J}_n(\omega_{k,n}; f_p)J_n(\omega_{k,n}) \) does have a bias, but it is considerably smaller than the bias of the usual periodogram. In particular, it follows from the proof of Lemma 4.1 below, that

\[
\mathbb{E}_f[\tilde{J}_n(\omega_{k,n}; f_p)J_n(\omega_{k,n})] = f(\omega_{k,n}) + O\left(\frac{1}{np^{K-1}}\right).
\]

This result leads in the boundary corrected Whittle likelihood

\[
W_{p,n}(\theta) = \frac{1}{n} \sum_{k=1}^{n} \frac{\tilde{J}_n(\omega_{k,n}; f_p)J_n(\omega_{k,n})}{f(\omega_{k,n})} + \frac{1}{n} \sum_{k=1}^{n} \log f(\omega_{k,n}). \tag{4.3}
\]

In the following lemma we obtain a bound between the “ideal” boundary corrected Whittle likelihood \( W_n(\theta) \) and \( W_{p,n}(\theta) \).

**Lemma 4.1** Suppose \( f \) satisfies Assumption 3.1, \( f_\theta \) is bounded away from zero and \( \| f_\theta \|_0 < \infty \). Let \( \{a_j(p)\} \) denote the coefficients of the best fitting AR(\( p \)) model corresponding to the spectral density \( f \) and define \( f_p(\omega) = |1 - \sum_{j=1}^{p} a_j(p)e^{-ij\omega}|^{-2} \). Suppose \( p < n \), then we have

\[
\|F_n^\ast \Delta_n(\theta^{-1}) (D_n(f) - D_n(f_p))\|_1 \leq \rho_{p,K}(f) A_K(f, f_\theta) \left(\frac{(C_{f,1} + 1)}{p^{K-1}} + \frac{2(C_{f,1} + 1)^2}{p^K} \|\psi_f\|_0 \|\phi_f\|_1 + \frac{C_{f,0}}{n^{K-1}}\right). \tag{4.4}
\]

If for some \( q > 1 \), \( \sup_t \|X_t\|_{2q} = \|X\|_{2q} < \infty \), then

\[
\|W_n(\theta) - W_{p,n}(\theta)\|_q \leq \|X\|_{2q} \rho_{p,K}(f) A_K(f, f_\theta) \left(\frac{(C_{f,1} + 1)}{np^{K-1}} + \frac{2(C_{f,1} + 1)^2}{n^K} \|\psi_f\|_0 \|\phi_f\|_1 + \frac{C_{f,0}}{n^K}\right). \tag{4.5}
\]

**PROOF** See Appendix A.2

It follows from the lemma above that if \( p < n \), \( \mathbb{E}_f[W_{p,n}(\theta)] = I_n(f; f_\theta) + O((np^{K-1})^{-1}) \) and

\[
W_{p,n}(\theta) = W_n(\theta) + O_p\left(\frac{1}{np^{K-1}}\right).
\]
Thus $W_{p,n}(\theta)$ gives a higher degree of approximation to the “ideal” $W_n(\theta)$ than both the Whittle and the Gaussian likelihood.

Since $f$ is unknown, $f_p$ is also unknown. But $f_p$ is easily estimated from the data. Using the Yule-Walker estimator we fit an AR($p$) process to the observed time series, where we select the order $p$ using the AIC. We denote this estimator as $\hat{\phi}_p$ and the corresponding spectral density as $\hat{f}_p$. Using this we define $\tilde{J}_n(\omega_{k,n}; \hat{f}_p)$ where

$$
\tilde{J}_n(\omega; \hat{f}_p) = \frac{n^{-1/2}}{\hat{\phi}_p(\omega)} \sum_{\ell=1}^{p} X_{\ell} \sum_{s=0}^{p-\ell} \hat{\phi}_{\ell+s,p} \exp(-is\omega) + \frac{n^{-1/2}}{\hat{\phi}_p(\omega)} \sum_{\ell=1}^{p} X_{n+1-\ell} \sum_{s=0}^{p-\ell} \hat{\phi}_{\ell+s,p} \exp((s+1)\omega),
$$

and $\hat{\phi}_p(\omega) = 1 - \sum_{u=1}^{p} \hat{\phi}_{u,p} \exp(-iu\omega)$. This estimator allows us to replace $\tilde{J}_n(\omega_{k,n}; f_p)$ in $W_{p,n}(\theta)$ with $\tilde{J}_n(\omega_{k,n}; \hat{f}_p)$ to give the “observed” boundary corrected Whittle likelihood

$$
\tilde{W}_{p,n}(\theta) = \frac{1}{n} \sum_{k=1}^{n} \frac{\tilde{J}_n(\omega_{k,n}; \hat{f}_p) \tilde{J}_n(\omega_{k,n})}{f_\theta(\omega_{k,n})} + \frac{1}{n} \sum_{k=1}^{n} \log f_\theta(\omega_{k,n}). \quad (4.6)
$$

We use as an estimator of $\theta$, $\hat{\theta}_n = \arg \min \tilde{W}_{p,n}(\theta)$. It is worth bearing in mind that

$$
\Im \frac{\tilde{J}_n(\omega_{k,n}; \hat{f}_p) \tilde{J}_n(\omega_{k,n})}{f_\theta(\omega_{k,n})} = -\Im \frac{\tilde{J}_n(\omega_{n-k,n}; \hat{f}_p) \tilde{J}_n(\omega_{n-k,n})}{f_\theta(\omega_{n-k,n})}
$$

thus $\tilde{W}_{p,n}(\theta)$ is real for all $\theta$. However, due to rounding errors it is prudent to use $\Re \tilde{W}_{p,n}(\theta)$ in the minimisation algorithm. Sometimes $\Re \tilde{J}_n(\omega_{k,n}; \hat{f}_p) \tilde{J}_n(\omega_{k,n})$ can be negative, when this arises we threshold it to be positive (the method we use is given in Section 5).

For a finite order $p$ it is well known that $\|\hat{\phi}_p - \phi_p\|_2 = O_p(n^{-1/2})$. Thus we may expect that $|\tilde{W}_{p,n}(\theta) - W_{p,n}(\theta)| = O_p(n^{-3/2})$. However, the proof of this result is complex and we leave this for future research.

### 4.2 The tapered hybrid Whittle estimator

The simulations in Section 5 show that the boundary corrected Whittle likelihood estimator (defined in (4.6)) yields an estimator with a smaller bias than the regular Whittle likelihood. However, the bias of the tapered Whittle likelihood (and often the Gaussian likelihood) is in some cases lower. The tapered Whittle likelihood (first proposed in [Dahlhaus (1988)] gives a better resolution at the peaks in the spectral density. It also “softens” the observed domain of observation. With this in mind, we propose a hybrid Whittle likelihood which incorporates the notion of tapering.

Let $h_n = \{h_{t,n}\}_{t=1}^{n}$ denote a data taper, where the weights $\{h_{t,n}\}$ are non-negative and
\[ \sum_{t=1}^{n} h_{t,n} = n. \] We define the tapered DFT as

\[ J_{n,h_{n}}(\omega_{k,n}) = n^{-1/2} \sum_{t=1}^{n} h_{t,n} X_t \exp(it\omega_{k,n}). \]

Suppose \( f \) is the best fitting spectral density function. Using that \( \sum_{t=1}^{n} h_{t,n} = n \) and \( \text{cov}_f(X_t, \hat{X}_{\tau,n}) = c_f(t - \tau) \) we have

\[ \mathbb{E}_f[J_n(\omega; f)J_{n,h_{n}}(\omega)] = f(\omega), \tag{4.7} \]

which is analogous to the non-tapered result \( \mathbb{E}_f[J_n(\omega; f)\hat{J}_n(\omega)] = f(\omega) \). Based on the above result we define the infeasible hybrid Whittle likelihood which combines the regular DFT of the tapered time series and the complete DFT (which is not tapered)

\[ H_n(\theta) = \frac{1}{n} \sum_{k=1}^{n} \frac{\tilde{J}_n(\omega_{k,n}; f)J_{n,h_{n}}(\omega_{k,n})}{f_\theta(\omega_{k,n})} + \frac{1}{n} \sum_{k=1}^{n} \log f_\theta(\omega_{k,n}). \tag{4.8} \]

Using (4.7), it can be shown that \( \mathbb{E}_f[H_n(\theta)] = I_n(f; f_\theta) \). Thus \( W_n(\theta) \) and \( H_n(\theta) \) are both unbiased estimators of \( I_n(f; f_\theta) \). Clearly, it is not possible to estimate \( \theta \) using the (unobserved) criterion \( H_n(\theta) \). Instead we replace \( \tilde{J}_n(\omega_{k,n}; f) \) with its estimator \( \tilde{J}_n(\omega_{k,n}; \hat{f}_p) \) and define

\[ \hat{H}_{p,n}(\theta) = \frac{1}{n} \sum_{k=1}^{n} \frac{\tilde{J}_n(\omega_{k,n}; \hat{f}_p)J_{n,h_{n}}(\omega_{k,n})}{f_\theta(\omega_{k,n})} + \frac{1}{n} \sum_{k=1}^{n} \log f_\theta(\omega_{k,n}). \tag{4.9} \]

We then use as an estimator of \( \theta \),

\[ \tilde{\theta}_n = \arg \min \hat{H}_{p,n}(\theta). \]

An illustration which visualises and compares the boundary corrected Whittle likelihood and hybrid Whittle likelihood is given in Figure 4.

Figure 4: Left: The regular DFT and the estimated complete DFT which yields the boundary corrected Whittle likelihood. Right: The tapered DFT and the estimated complete DFT which form the hybrid Whittle likelihood.
We show in the simulations in Section 5 that both the boundary corrected Whittle and hybrid Whittle likelihoods tend to reduce the bias in the classical Whittle likelihood.

5 Empirical results

To substantiate our theoretical results, we conduct some simulations. To compare different methods, we evaluate six different quasi-likelihoods: the Gaussian likelihood (equation (1.1)), the Whittle likelihood (equation (1.3)), the boundary corrected Whittle likelihood (equation (4.6)), the hybrid Whittle likelihood (equation (4.9)), the tapered Whittle likelihood (p.810 of Dahlhaus (1988)) and the debiased Whittle likelihood (equation (7) in Sykulski et al. (2019)).

The tapered and hybrid Whittle likelihood require the use of data tapers. We use a Cosine-bell taper, where the proportion of tapering at each end of the time series is 0.1 (this is the default in R).

When evaluating the boundary corrected Whittle likelihood and hybrid Whittle likelihood, we choose the order \( p \) and \( \hat{f}_p \) using the AIC and the best fitting AR\((p)\) spectral density estimator. This is done with the Yule-Walker estimator. Unlike the Whittle, tapered Whittle and debiased Whittle likelihood, \( \mathbb{R} \hat{J}_n(\omega_{k,n}, \hat{f}_p) \hat{J}_n(\omega_{k,n}) \) and \( \mathbb{R} \hat{J}_{n,h}(\omega_{k,n}, \hat{f}_p) \hat{J}_n(\omega_{k,n}) \) can be negative. To avoid negative values, we apply the thresholding function \( f(t) = \max(t, 10^{-3}) \) to \( \mathbb{R} \hat{J}_n(\omega_{k,n}, \hat{f}_p) \hat{J}_n(\omega_{k,n}) \) and \( \mathbb{R} \hat{J}_{n,h}(\omega_{k,n}, \hat{f}_p) \hat{J}_n(\omega_{k,n}) \) over all the frequencies. All simulations are conducted over 500 replications.

5.1 Estimation with correctly specified models

We first study the AR(1) and MA(1) parameter estimates when the models are correctly specified. We generate two types of time series models \( X_n \) and \( Y_n \), which satisfy the following recursions

\[
\begin{align*}
\text{AR}(1) : & \quad X_t = \theta X_{t-1} + e_t; \quad \phi_X(\omega) = 1 - \theta e^{-i\omega} \\
\text{MA}(1) : & \quad Y_t = e_t + \theta e_{t-1}; \quad \phi_Y(\omega) = (1 + \theta e^{-i\omega})^{-1},
\end{align*}
\]

where \(|\theta| < 1\), \( \{e_t\} \) are independent, identically distributed random variables. We consider the cases that \( \{e_t\} \) are (i) standard normal and (ii) a centered chi-squared distribution with 2 degrees of freedom. To compare different behaviour, we generate the AR(1) and MA(1) models with parameters \( \theta = 0.1, 0.3, 0.5, 0.7 \) and 0.9. For the time series generated by an AR(1) process we fit an AR(1) model, similarly for the time series generated by a MA(1) process we fit an MA(1) model. The length of all time series is \( n = 50 \).

For each simulation, we evaluate the six different parameter estimators. The empirical bias and standard deviation are calculated. Figures 5 and 6 give the bias (left panels) and the standard deviation (right panels) of each estimated parameter \( \theta \). We focus on positive \( \theta \), similar results are obtained for negative \( \theta \). The results are also summarized in Table 1.
The sample size is small, thus we observe a stark difference between the bias of the Whittle likelihood estimator (blue line) and the other five other methods, which in most cases have a lower bias. As is expected, the tapered Whittle estimator performs uniformly well for all the models, whereas the Gaussian likelihood estimator performs very well for all the autoregressive models but not quite as well for the moving average models. The debiased Whittle likelihood performs very well for both the AR(1) and MA(1) models when the innovations are Gaussian. However, the performance of the debiased Whittle is not as good when the innovations are non-Gaussian. We conjecture that the kurtosis in the non-Gaussian time series may lead to an increase in the bias and standard error of the debiased Whittle estimator. For the Gaussian time series, the boundary corrected and hybrid Whittle likelihoods perform well but in general, do not have the smallest bias. On the other hand, when the time series is non-Gaussian, these two new methods are a lot more competitive, with usually the smallest or second smallest bias. This is reassuring as it shows that the proposed estimators appear to be competitive for a range of models both Gaussian and non-Gaussian.

Figure 5: Bias (left panels) and the standard deviation (right panels) of the parameter estimates for the AR(1) models. Length of the time series $n = 50$ and the total number of repetitions are 1000. Top: Gaussian innovation; Bottom: $\chi^2(2) - 2$ innovation.
Figure 6: Same as Figure 5 but for the MA(1) models.
<table>
<thead>
<tr>
<th>Likelihoods</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
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<tr>
<td>AR(1), ({\epsilon_t} \sim \mathcal{N}(0, 1))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Gaussian</td>
<td>-0.010(0.14)</td>
<td>-0.014(0.14)</td>
<td>-0.026(0.12)</td>
<td>-0.025(0.11)</td>
<td>-0.030(0.07)</td>
<td>-0.008(0.15)</td>
<td>0.012(0.16)</td>
<td>0.010(0.14)</td>
<td>0.012(0.12)</td>
<td>0.013(0.08)</td>
</tr>
<tr>
<td>Whittle</td>
<td>-0.012(0.14)</td>
<td>-0.018(0.14)</td>
<td>-0.035(0.12)</td>
<td>-0.043(0.11)</td>
<td>-0.054(0.08)</td>
<td>-0.009(0.16)</td>
<td>0.004(0.15)</td>
<td>-0.005(0.14)</td>
<td>-0.026(0.13)</td>
<td>-0.073(0.11)</td>
</tr>
<tr>
<td>Boundary</td>
<td>-0.012(0.14)</td>
<td>-0.015(0.14)</td>
<td>-0.027(0.12)</td>
<td>-0.032(0.11)</td>
<td>-0.040(0.08)</td>
<td>-0.009(0.16)</td>
<td>0.010(0.16)</td>
<td>0.006(0.14)</td>
<td>-0.006(0.12)</td>
<td>-0.042(0.09)</td>
</tr>
<tr>
<td>Hybrid</td>
<td>-0.009(0.14)</td>
<td>-0.017(0.14)</td>
<td>-0.029(0.13)</td>
<td>-0.029(0.11)</td>
<td>-0.038(0.08)</td>
<td>-0.005(0.16)</td>
<td>0.008(0.16)</td>
<td>0.008(0.14)</td>
<td>0.001(0.12)</td>
<td>-0.019(0.07)</td>
</tr>
<tr>
<td>Tapered</td>
<td>-0.009(0.14)</td>
<td>-0.020(0.14)</td>
<td>-0.032(0.13)</td>
<td>-0.032(0.11)</td>
<td>-0.042(0.08)</td>
<td>-0.004(0.16)</td>
<td>0.004(0.16)</td>
<td>0.006(0.14)</td>
<td>-0.003(0.12)</td>
<td>-0.018(0.08)</td>
</tr>
<tr>
<td>Debiased</td>
<td>-0.011(0.14)</td>
<td>-0.013(0.14)</td>
<td>-0.027(0.12)</td>
<td>-0.033(0.11)</td>
<td>-0.034(0.09)</td>
<td>-0.007(0.16)</td>
<td>0.012(0.16)</td>
<td>0.010(0.14)</td>
<td>0.012(0.14)</td>
<td>-0.015(0.11)</td>
</tr>
<tr>
<td>MA(1), ({\epsilon_t} \sim \mathcal{N}(0, 1))</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gaussian</td>
<td>0.001(0.15)</td>
<td>-0.004(0.14)</td>
<td>0.015(0.13)</td>
<td>0.095(0.19)</td>
<td>0.083(0.07)</td>
<td>0.010(0.17)</td>
<td>0.022(0.17)</td>
<td>0.063(0.20)</td>
<td>0.135(0.19)</td>
<td>0.063(0.12)</td>
</tr>
<tr>
<td>Whittle</td>
<td>-0.006(0.14)</td>
<td>-0.017(0.13)</td>
<td>-0.009(0.12)</td>
<td>-0.025(0.11)</td>
<td>-0.035(0.08)</td>
<td>0.002(0.16)</td>
<td>0.004(0.15)</td>
<td>0.009(0.14)</td>
<td>0.006(0.11)</td>
<td>-0.023(0.07)</td>
</tr>
<tr>
<td>Boundary</td>
<td>-0.006(0.14)</td>
<td>-0.017(0.13)</td>
<td>-0.011(0.12)</td>
<td>-0.027(0.11)</td>
<td>-0.039(0.08)</td>
<td>0.000(0.16)</td>
<td>0.003(0.15)</td>
<td>0.005(0.14)</td>
<td>-0.003(0.12)</td>
<td>-0.022(0.08)</td>
</tr>
<tr>
<td>Hybrid</td>
<td>-0.006(0.14)</td>
<td>-0.019(0.13)</td>
<td>-0.011(0.12)</td>
<td>-0.027(0.11)</td>
<td>-0.039(0.08)</td>
<td>0.000(0.16)</td>
<td>0.003(0.15)</td>
<td>0.005(0.14)</td>
<td>-0.003(0.12)</td>
<td>-0.022(0.08)</td>
</tr>
<tr>
<td>Tapered</td>
<td>-0.006(0.14)</td>
<td>-0.017(0.13)</td>
<td>-0.011(0.12)</td>
<td>-0.027(0.11)</td>
<td>-0.039(0.08)</td>
<td>0.000(0.16)</td>
<td>0.003(0.15)</td>
<td>0.005(0.14)</td>
<td>-0.003(0.12)</td>
<td>-0.022(0.08)</td>
</tr>
<tr>
<td>Debiased</td>
<td>-0.001(0.15)</td>
<td>-0.004(0.14)</td>
<td>0.015(0.13)</td>
<td>0.095(0.19)</td>
<td>0.083(0.07)</td>
<td>0.010(0.17)</td>
<td>0.022(0.17)</td>
<td>0.063(0.20)</td>
<td>0.135(0.19)</td>
<td>0.063(0.12)</td>
</tr>
</tbody>
</table>

Table 1: Bias and the standard deviation (in the parentheses) of six different quasi-likelihoods for an AR(1) (left) and MA(1) (right) model for the standard normal (top) and the centered chi-squared distribution with 2 degrees of freedom (bottom). Length of the time series \(n = 50\) and 1000 replications. We use red to denote the smallest bias and magenta to denote the second smallest bias.
5.2 Estimation under misspecification

Next, we turn our attention to the case that the model is misspecified (which in reality is the most realistic situation). We generate the time series \( Z_n (n = 50) \), from an ARMA(3,2) process with spectral density \( f_Z(\omega) = \sigma^2 |\psi_Z(e^{i\omega})|^2 / |\phi_Z(e^{i\omega})|^2 \), where AR and MA characteristic polynomials are

\[
\phi_Z(z) = (1 - 0.7z)(1 - 0.9e^{i}z)(1 - 0.9e^{-i}z) \quad \text{and} \quad \psi_Z(z) = (1 + 0.5z + 0.5z^2).
\]

We chose this spectral density since it had quite a few interesting characteristics: a pronounced peak, large amount of power at the low frequencies and a sudden drop in power at the higher frequencies. To do the estimation, we fit an ARMA(2,1) model with spectral density

\[
f_\theta(\omega) = \sigma^2 |1 + \psi_1 e^{-i\omega}|^2 |1 - \phi_1 e^{-i\omega} - \phi_2 e^{-2i\omega}|^{-2},
\]

where \( \theta = (\phi_1, \phi_2, \psi_1) \).

Figure 7 shows the (logarithm of) theoretical ARMA(3,2) spectral density (solid line, \( f_Z \)) and the corresponding spectral density of the best fitting ARMA(2,1) process (dashed line, \( f_{\theta, \text{best}} \)) obtained by minimizing the spectral divergence \( \theta^{\text{Best}} = \arg \min_{\theta \in \Theta} I_n(f; f_\theta) \), where \( I_n(f, f_\theta) \) is defined in (4.1) and \( \Theta \) is the parameter space.

For each simulation, we calculate six different parameter estimators and the spectral divergence. The result of the estimators using the six different quasi-likelihoods is given in Table 2 (for Gaussian innovations) and Table 3 (for centered chi-squared distribution with 2 degrees of
freedom). Our calculations suggest that in the case of misspecification the new quasi-likelihoods should result in a smaller bias than the Gaussian likelihood. The simulations we have conducted give some credence to this claim (at least for these models). For this model, the hybrid Whittle likelihood yields a smaller bias than the Gaussian likelihood. And the average spectral divergence criterion, $I_n(f, \hat{f})$, for the hybrid Whittle, also appears to be closer to the true divergence than the Gaussian likelihood. Intriguingly, it is not clear why the hybrid Whittle likelihood should significantly outperform the boundary corrected Whittle likelihood.

<table>
<thead>
<tr>
<th></th>
<th>Best</th>
<th>Gaussian</th>
<th>Whittle</th>
<th>Boundary</th>
<th>Hybrid</th>
<th>Tapered</th>
<th>Debiased</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
<td>1.267</td>
<td>1.269(0.08)</td>
<td>1.170(0.10)</td>
<td>1.243(0.08)</td>
<td>1.262(0.08)</td>
<td>1.226(0.08)</td>
<td>1.265(0.12)</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>-0.751</td>
<td>-0.758(0.08)</td>
<td>-0.666(0.11)</td>
<td>-0.732(0.09)</td>
<td>-0.748(0.08)</td>
<td>-0.720(0.09)</td>
<td>-0.733(0.13)</td>
</tr>
<tr>
<td>$\psi_1$</td>
<td>0.632</td>
<td>0.649(0.11)</td>
<td>0.436(0.18)</td>
<td>0.586(0.14)</td>
<td>0.636(0.08)</td>
<td>0.657(0.11)</td>
<td>0.852(0.11)</td>
</tr>
<tr>
<td>$I_n(f; f_\theta)$</td>
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<td>1.965(0.28)</td>
<td>2.277(0.46)</td>
<td>1.981(0.27)</td>
<td>1.938(0.22)</td>
<td>1.956(0.24)</td>
<td>2.830(0.51)</td>
</tr>
</tbody>
</table>

Table 2: Best fitting and the average of estimated coefficients for six different methods for the Gaussian ARMA(2,1) misspecified case. Standard deviations are in the parentheses. We use red to denote the closest estimate to the best fitting coefficient.

<table>
<thead>
<tr>
<th></th>
<th>Best</th>
<th>Gaussian</th>
<th>Whittle</th>
<th>Boundary</th>
<th>Hybrid</th>
<th>Tapered</th>
<th>Debiased</th>
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<tbody>
<tr>
<td>$\phi_1$</td>
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<td>1.292(0.08)</td>
<td>1.171(0.11)</td>
<td>1.245(0.08)</td>
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<td>1.229(0.08)</td>
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<td>$\psi_1$</td>
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<td>0.647(0.11)</td>
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<td>0.632(0.08)</td>
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<tr>
<td>$I_n(f; f_\theta)$</td>
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<td>1.965(0.28)</td>
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<td>1.938(0.22)</td>
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<td>3.406(0.55)</td>
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</table>

Table 3: Same as in Table 2 but for $\chi^2(2) - 2$ innovations.

**Concluding remarks**

In this paper we have derived an exact expression for the differences $\Gamma_n(f_\theta)^{-1} - C_n(f_\theta^{-1})$ and \(X'_n[\Gamma_n(f_\theta)^{-1} - C_n(f_\theta^{-1})]X_n\). These expressions are simple, with an intuitive interpretation, in terms of predicting outside the boundary of observation. They also provide a new perspective to the Whittle likelihood as an approximation of the biorthogonal transform. We have used these expansions and approximations to define two new spectral divergence criteria (in the frequency domain). Our simulations show that both new estimators tend to outperform the Whittle likelihood. But intriguingly the hybrid Whittle likelihood tends to perform better than the boundary corrected Whittle likelihood. This will certainly be an avenue of future research.

It would be interesting to investigate how these methods generalize to spatial data defined on a regular grid. By considering the best linear predictors (see Meyer et al. (2017)) given the observed spatial data, it may be possible to generalize Theorems 2.1 and 2.2 to spatial processes. It is also of interest to consider the setting of missing observations. It is unlikely that one
can easily find the biorthogonal transform associated with the Fourier transform in this case. However, it is interesting to note that by predicting all the missing observations (both inside and outside the domain of observation) we can define a “complete DFT” (analogous to Theorem 2.1). The product of the usual Fourier transform of the observed spatial data together with the “complete DFT” will give an unbiased estimator of the spectral density function. Recently, the method of imputation of missing values for both time series and spatial data has recently been developed within the spectral domain (see Lee and Zhu (2009) and Guinness (2019)). It would be of interest to investigate how these methods are connected to the results of this paper.

In summary, the notion of biorthogonality and its application to the inversion of certain variance matrices may be of value in future research.

Acknowledgements

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References


A The main proofs

A.1 Proof of Theorems 2.1 and 2.3

PROOF of Theorems 2.1 To prove the result, we recall that it entails obtaining a transform $U_nX_n$ where $\text{cov}_f(U_nX_n, F_nX_n) = \Delta_n(f)$. Pre and post multiplying this covariance with $F_n^*$ and $F_n$ gives

$$F_n^*\text{cov}_f(U_nX_n, F_nX_n) F_n = \text{cov}_\theta(F_n^*U_nX_n, X_n) = F_n^*\Delta_n(f)F_n = C_n(f).$$

Thus our objective is to find the transform $Y_n = F_n^*U_nX_n$ such that $\text{cov}_f(Y_n, X_n) = C_n(f)$. Then, the vector $F_nY_n = U_nX_n$ will be biorthogonal to $F_nX_n$, as required. We observe that the entries of the circulant matrix $C_n(f)$ are

$$(C_n(f))_{u,v} = n^{-1}\sum_{k=1}^{n} f(\omega_{k,n}) \exp(-i(u-v)\omega_{k,n}) = \sum_{\ell \in \mathbb{Z}} c_f(u-v+\ell n),$$

where the second equality is due to the Poisson summation. The random vector $Y_n = \{Y_{u,n}\}_{u=1}^{n}$ is such that $\text{cov}_f(Y_{u,n}, X_v) = \sum_{\ell \in \mathbb{Z}} c_f(u-v+\ell n)$ and $Y_u \in \text{sp}(X_n)$. Since $\text{cov}_f(X_{u+\ell n}, X_v) = c_f(u-v+\ell n)$, at least “formally” $\text{cov}_f(\sum_{\ell \in \mathbb{Z}} X_{u+\ell n}, X_v) = \sum_{\ell \in \mathbb{Z}} c_f(u-v+\ell n)$. However, $\sum_{\ell \in \mathbb{Z}} X_{u+\ell n}$ is neither a well defined random variable nor does it belong to $\text{sp}(X_n)$. We replace each element in the sum $\sum_{\ell \in \mathbb{Z}} X_{u+\ell n}$ with an element that belongs to $\text{sp}(X_n)$ and gives the same covariance. To do this we use the following well known result. Let $Z$ and $X$ denote a random variable and vector respectively. Let $P_X(Z)$ denote the projection of $Z$ onto $\text{sp}(X)$, i.e., the best linear predictor of $Z$ given $X$, then $\text{cov}_f(Z, X) = \text{cov}_f(P_X(Z), X)$. Let $\hat{X}_{\tau,n}$ denote best linear predictor of $X_{\tau}$ given $X_n = (X_1, \ldots, X_n)$ (as defined in (2.4)). $\hat{X}_{\tau,n}$ retains the pertinent properties of $X_{\tau}$ in the sense that $\text{cov}_f(\hat{X}_{\tau,n}, X_t) = c_f(\tau-t)$ for all $\tau \in \mathbb{Z}$ and $1 \leq t \leq n$. Define

$$Y_{u,n} = \sum_{\ell \in \mathbb{Z}} \hat{X}_{u+\ell n} = \sum_{s=1}^{n} \left( \sum_{\ell \in \mathbb{Z}} \phi_{s,n}(u+\ell n; f) \right) X_s \in \text{sp}(X_n),$$

where we note that $Y_{u,n}$ a well defined random variable, since by using Lemma A.2 it can be shown that $\sup_n \sum_{s=1}^{n} \sum_{\ell = -\infty}^{\infty} |\phi_{s,n}(u+\ell n; f)| < \infty$. Thus by definition of $Y_{u,n}$ the following holds

$$\text{cov}_f(Y_{u,n}, X_v) = \sum_{\ell \in \mathbb{Z}} c_f(u-v+\ell n) = (C_n(f))_{u,v}, \quad (A.1)$$

and $Y_n = F_n^*U_nX_n$, gives the desired transformation of the time series. Thus based on this construction, $F_nY_n = U_nX_n$ and $F_nX_n$ are biorthogonal transforms, with entries $(F_nX_n)_k =$
Using these two observations we can decompose \( U_{m,n} \) and \( U_t \). It immediately follows from the above decomposition that \( \phi \) to second order stationarity the coefficients defined in (2.6). Thus proving (2.5).

The entries of the matrix \( U_n \) are \( (U_n)_{k,t} = n^{-1/2} \sum_{\tau \in \mathbb{Z}} \phi_t,n(\tau;f) \exp(i\tau \omega_{k,n}) \). To show that \( U_n \) “embeds” the regular DFT, we observe that for \( 1 \leq \tau \leq n \), \( \phi_t,n(\tau;f) = \delta_{r,t} \), furthermore, due to second order stationarity the coefficients \( \phi_t,n(\tau;f) \) are reflective i.e. the predictors of \( X_m \) (for \( m > n \)) and \( X_{n+1-m} \) share the same set of prediction coefficients (just reflected) such that

\[
\phi_t,n(m;f) = \phi_{n+1-t,n}(n+1-m;f) \quad \text{for } m > n. \tag{A.3}
\]

Using these two observations we can decompose \( (U_n)_{k,t} \) as

\[
(U_n)_{k,t} = n^{-1/2} \left( \exp(it \omega_{k,n}) + \sum_{\tau \leq 0} \phi_{t,n}(\tau;f) \exp(i\tau \omega_{k,n}) + \sum_{\tau \geq n+1} \phi_{t,n}(\tau;f) \exp(i\tau \omega_{k,n}) \right)
= n^{-1/2} e^{i\omega_{k,n}} + n^{-1/2} \sum_{\tau \leq 0} \left( \phi_{t,n}(\tau;f) e^{i\tau \omega_{k,n}} + \phi_{n+1-t,n}(\tau;f) e^{-i(\tau-1)\omega_{k,n}} \right) \quad 1 \leq k, t \leq n.
\]

It immediately follows from the above decomposition that \( U_n = F_n + D_n(f) \) where \( D_n(f) \) is defined in (2.6). Thus proving (2.5).

To prove (2.7), we first observe that (2.5) implies

\[
\text{cov}_f \left( (F_n + D_n(f))X_n \right)_{k_1} = f(\omega_{k_1,n}) \delta_{k_1,k_2}.
\]

It is clear that \( (F_nX_n)_k = J_n(\omega_{k,n}) \) and from the representation of \( F_nX_n \) given in (A.2) we have

\[
(F_nX_n)_k = n^{-1/2} \sum_{\tau=1}^{n} X_{\tau} \exp(i\tau \omega_{k,n}) + n^{-1/2} \sum_{\tau \notin \{1,n\}} \hat{X}_{\tau,n} \exp(i\tau \omega_{k,n})
= J_n(\omega_{k,n}) + \hat{J}_n(\omega_{k,n};f).
\]

This immediately proves (2.7). \( \square \)

Note that equation (2.7) can be verified directly by using the properties of linear predictors and covariances discussed in the above proof.

In order to prove Theorem 2.3 we study the predictive DFT for autoregressive processes. We start by obtaining an explicit expression for \( \hat{J}_n(\omega;f_\theta) \) where \( f_\theta(\omega) = \sigma^2|1 - \sum_{u=1}^{p} \phi_u e^{-i\omega u}|^{-2} \) (the spectral density corresponding to an AR\((p)\) process). It is straightforward to show that
predictive DFT predictor based on the AR(1) model is
\[
\hat{J}_n(\omega; f_\theta) = n^{-1/2} \sum_{\tau=-\infty}^{0} \phi^{-\tau+1} X_1 \exp(i\tau \omega) + n^{-1/2} \sum_{\tau=n+1}^{\infty} \phi^{\tau+1-n} X_n \exp(i\tau \omega)
\]
\[
= \frac{n^{-1/2}}{\phi_1(\omega)} X_1 + \frac{n^{-1/2}}{\phi_1(\omega)} X_n e^{i\omega},
\]
where \(\phi_1(\omega) = 1 - \phi \exp(-i\omega)\). In order to prove Theorem 2.3, which generalizes the above expression to AR(\(p\)) processes, we partition \(\hat{J}_n(\omega; f_\theta)\) into the predictions involving the past and future terms
\[
\hat{J}_n(\omega; f_\theta) = \hat{J}_{n,L}(\omega; f_\theta) + \hat{J}_{n,R}(\omega; f_\theta)
\]
where
\[
\hat{J}_{n,L}(\omega; f_\theta) = n^{-1/2} \sum_{\tau=-\infty}^{0} \hat{X}_{\tau,n} e^{i\tau \omega} \quad \text{and} \quad \hat{J}_{n,R}(\omega; f_\theta) = n^{-1/2} \sum_{\tau=n+1}^{\infty} \hat{X}_{\tau,n} e^{i\tau \omega}.
\]
We now obtain expressions for \(\hat{J}_{n,L}(\omega; f_\theta)\) and \(\hat{J}_{n,R}(\omega; f_\theta)\) separately, in the case the predictors are based on the AR(\(p\)) parameters. To do so, we define the \(p\)-dimension vector \(\phi' = (\phi_1, \ldots, \phi_p)\) and the matrix \(A_p(\phi)\) as
\[
A_p(\phi) = \begin{pmatrix}
\phi_1 & \phi_2 & \ldots & \phi_{p-1} & \phi_p \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & 0 & 0 \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}.
\]
Therefore, for \(\tau \leq 0\), since \(\hat{X}_{\tau,n} = [A_p(\phi)^{|\tau|+1} X_p]_{(1)}\), where \(X_p = (X_1, \ldots, X_p)\), we can write
\[
\hat{J}_{n,L}(\omega; f_\theta) = n^{-1/2} \sum_{\tau=-\infty}^{0} [A_p(\phi)^{|\tau|+1} X_p]_{(1)} e^{i\tau \omega}.
\]

**Lemma A.1** Let \(\hat{J}_{n,L}(\omega)\) be defined as in (A.5), where the parameters \(\phi\) are such that the roots of \(\phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j\) lie outside the unit circle. Then an analytic expression for \(\hat{J}_{n,L}(\omega; f_\theta)\) is
\[
\hat{J}_{n,L}(\omega; f_\theta) = \frac{n^{-1/2}}{\phi_p(\omega)} \sum_{\ell=1}^{p} X_\ell \sum_{s=0}^{p-\ell} \phi_{\ell+s} \exp(-i s \omega).
\]

where \(\phi_p(\omega) = 1 - \sum_{s=1}^{p} \phi_s e^{-i s \omega}\).
PROOF. By using (B.1) we have

$$[A_p(\phi)|\tau|+1 X_p](1) = \sum_{\ell=1}^p X_\ell \sum_{s=0}^{p-\ell} \phi_{\ell+s} \theta_{|\tau|-s}. $$

Therefore, using (A.5) and the change of variables $\tau \leftarrow -\tau$

$$\hat{J}_{n,L}(\omega; f_\theta) = n^{-1/2} \sum_{\ell=1}^p X_\ell \sum_{s=0}^{p-\ell} \phi_{\ell+s} \sum_{\tau=0}^\infty \theta_{\tau-s} \exp(-i\tau \omega)$$

$$= n^{-1/2} \sum_{\ell=1}^p X_\ell \sum_{s=0}^{p-\ell} \phi_{\ell+s} \exp(-is\omega) \sum_{\tau=0}^\infty \theta_{\tau-s} \exp(-i(\tau-s)\omega) \quad \text{use that $\theta_{\tau} = 0$ for $\tau < 0$}$$

$$= n^{-1/2} \sum_{\ell=1}^p X_\ell \sum_{s=0}^{p-\ell} \phi_{\ell+s} \exp(-is\omega) \sum_{\tau=0}^\infty \theta_{\tau-s} \exp(-i(\tau-s)\omega).$$

Let $\sum_{s=0}^\infty \theta_s e^{-is\omega} = \theta(\omega) = \phi_p(\omega)^{-1}$, and substitute this into the above to give

$$\hat{J}_{n,L}(\omega; f_\theta) = \frac{n^{-1/2}}{\phi_p(\omega)} \sum_{\ell=1}^p X_\ell \sum_{s=0}^{p-\ell} \phi_{\ell+s} \exp(-is\omega), \quad (A.7)$$

Thus we obtain the desired result. \hfill \Box

**PROOF of Theorem 2.3** To prove (2.14), we note that the same proof as that above can be used to prove that the right hand side predictive DFT $\hat{J}_{n,R}(\omega; f_\theta)$ has the representation

$$\hat{J}_{n,R}(\omega; f_\theta) = \frac{n^{-1/2}}{\phi_p(\omega)} \sum_{\ell=1}^p X_{n+1-\ell} \sum_{s=0}^{p-\ell} \phi_{\ell+s} \exp(i(s+1)\omega).$$

Since $\hat{J}_n(\omega; f_\theta) = \hat{J}_{n,L}(\omega; f_\theta) + \hat{J}_{n,R}(\omega; f_\theta)$, Lemma B.1 and the above give an explicit expression for $\hat{J}_n(\omega; f_\theta)$, thus proving equation (2.14).

To prove (2.15) we use that

$$(\hat{J}_n(\omega_{1,n}; f_\theta), \ldots, \hat{J}_n(\omega_{n,n}; f_\theta))' = D_n(f_\theta)X_n.$$  

Now by using (2.14) together with the above we immediately obtain (2.15).

Finally, we prove (2.16). We use the result $n^{-1} \sum_{k=1}^n \phi_p(\omega_{k,n}) \exp(i\omega_{k,n}) = \tilde{\phi}_{s \mod n}$ where
\( \phi_p(\omega) = \sum_{r=0}^{n-1} \phi_r e^{-ir\omega} \) and \( \bar{\phi}_r = 0 \) for \( p + 1 \leq r \leq n \). For \( 1 \leq t \leq p \) we use have

\[
(F_n^* \Delta_n (f_{\tilde{\theta}}^{-1}) D_n (f_{\tilde{\theta}}))_{s,t} = \frac{1}{n} \sum_{k=1}^{n} \frac{\phi_{t,p}(\omega_{k,n})}{f_{\tilde{\theta}}(\omega_{k,n})} \exp(-is\omega_{k,n})
\]

\[
= \frac{\sigma^{-2}}{n} \sum_{k=1}^{n} \phi_p(\omega_{k,n}) \sum_{\ell=0}^{p-t} \phi_{\ell+t} \exp(-i\ell\omega_{k,n}) \exp(-is\omega_{k,n})
\]

\[
= \sigma^{-2} \sum_{\ell=0}^{p-t} \phi_{\ell+t+1} \frac{1}{n} \sum_{k=1}^{n} \phi_p(\omega_{k,n}) \exp(-i(\ell+1)s\omega_{k,n})
\]

Similarly, for \( 1 \leq t \leq p \),

\[
(F_n^* \Delta_n (f_{\tilde{\theta}}^{-1}) D_n (f_{\tilde{\theta}}))_{s,n-t+1} = \frac{1}{n} \sum_{k=1}^{n} \frac{\phi_{t,p}(\omega_{k,n})}{f_{\tilde{\theta}}(\omega_{k,n})} \exp(i(1-s)\omega_{k,n})
\]

\[
= \frac{\sigma^{-2}}{n} \sum_{k=1}^{n} \phi_p(\omega_{k,n}) \sum_{\ell=0}^{p-t} \phi_{\ell+t} \exp(i\ell\omega_{k,n}) \exp(i(1-s)\omega_{k,n})
\]

\[
= \sigma^{-2} \sum_{\ell=0}^{p-t} \phi_{\ell+t+1} \frac{1}{n} \sum_{k=1}^{n} \phi_p(\omega_{k,n}) \exp(i(\ell+1-s)\omega_{k,n})
\]

\[
= \sigma^{-2} \sum_{\ell=0}^{p-t} \phi_{\ell+t} \bar{\phi}_{\ell+1-s \mod n}.
\]

\( \square \)

**PROOF of Equation (2.17)** We use that \( \frac{1}{\phi_p(\omega)} f_p(\omega)^{-1} = \sigma^{-2} \bar{\phi}_p(\omega) \). This gives

\[ L_n(\phi) - K_n(\phi) = I + II \]

where

\[
I = \frac{1}{n^{3/2}} \sum_{k=1}^{n} \frac{J_n(\omega_{k,n})}{f_{\tilde{\theta}}(\omega_{k,n})} \left\{ \frac{1}{\phi_p(\omega_{k,n})} \sum_{\ell=1}^{p} \sum_{s=0}^{-\ell} X_{\ell} \phi_{s+\ell} e^{-i\ell\omega_{k,n}} \right\}
\]

\[
= \frac{\sigma^{-2}}{n^{3/2}} \sum_{k=1}^{n} J_n(\omega_{k,n}) \phi_p(\omega_{k,n}) \sum_{\ell=1}^{p} \sum_{s=0}^{-\ell} \phi_{s+\ell} e^{-i\ell\omega_{k,n}} \phi_{s+\ell} e^{-i\ell\omega_{k,n}}
\]

\[
= \frac{\sigma^{-2}}{n} \sum_{\ell=1}^{p} \sum_{s=0}^{-\ell} \phi_{s+\ell} \frac{1}{n^{1/2}} \sum_{k=1}^{n} \frac{J_n(\omega_{k,n}) \phi_p(\omega_{k,n})}{\phi_p(\omega_{k,n})} e^{-i\ell\omega_{k,n}}
\]

33
and

\[
II = \frac{\sigma^{-2}}{n^{3/2}} \sum_{k=1}^{n} \frac{J_n(\omega_{k,n})}{f_p(\omega_{k,n})} \frac{1}{\phi_p(\omega_{k,n})} \sum_{\ell=1}^{p} X_{n+1-\ell} \sum_{s=0}^{p-\ell} \phi_{\ell+s} \exp(i(s + 1)\omega_{k,n}).
\] (A.8)

We first consider I. Using that \(\phi_p(\omega_{k,n}) = 1 - \sum_{j=1}^{p} \phi_j e^{ij\omega_{k,n}}\) and \(n^{-1/2} \sum_{k=1}^{n} J_n(\omega_{k,n}) \exp(i s \omega_{k,n}) = X_s \mod n\), gives

\[
I = -\frac{\sigma^{-2}}{n} \sum_{\ell=1}^{p} X_{\ell} \sum_{s=0}^{p-\ell} \sum_{j=0}^{p} \phi_j \phi_{s+j} \frac{1}{n^{1/2}} \sum_{k=1}^{n} J_n(\omega_{k,n}) e^{-i(s-r)\omega_{k,n}}
\]

(set \(\phi_0 = -1\))

\[
= -\frac{\sigma^{-2}}{n} \sum_{\ell=1}^{p} X_{\ell} \sum_{s=0}^{p-\ell} \sum_{j=0}^{p} \phi_j \phi_{s+j} X_{-(s-j) \mod n}.
\]

The proof of II is similar. Altogether this proves the result. \(\square\)

### A.2 Proof of Theorems 3.1, 3.2 and Lemma 4.1

Many of the results below hinge on a small generalisation of Baxter’s inequality which we summarize below.

**Lemma A.2 (Extended Baxter’s inequality)** Suppose \(f(\cdot)\) is a spectral density function which satisfies Assumption 3.1. Let \(\psi(\cdot)\) and \(\phi(\cdot)\) be defined as in (3.1) (for the simplicity, we omit the notation \(f\) inside the \(\psi(\cdot)\) and \(\phi(\cdot)\)). Let \(\phi_{\infty}^{p+1}(\omega) = \sum_{s=p+1}^{\infty} \phi_s e^{-i s \omega}\). Further, let \(\{\phi_{s,n}(\tau)\}\) denote the coefficients in the best linear predictor of \(X_{\tau}\) given \(X_n = \{X_t\}_{t=1}^{n}\) and \(\{\phi_s(\tau)\}\) the corresponding the coefficients in the best linear predictor of \(X_{\tau}\) given \(X_{\infty} = \{X_t\}_{t=1}^{\infty}\), where \(\tau \leq 0\).

Suppose \(p\) is large enough such that \(\|\phi_{\infty}^{p+1}\|_K \|\psi\|_K \leq \varepsilon < 1\). Then for all \(n > p\) we have

\[
\sum_{s=1}^{n} (2^K + s^K) |\phi_{s,n}(\tau) - \phi_s(\tau)| \leq C_{f,K} \sum_{s=n+1}^{\infty} (2^K + s^K) |\phi_s(\tau)|;
\] (A.9)

where \(C_{f,K} = \frac{3 - \varepsilon}{1 - \varepsilon} \|\phi\|_K^2 \|\psi\|_K^2\) and \(\phi_s(\tau) = \sum_{j=0}^{\infty} \phi_{s+j} \psi_{|\tau| - j}\) (we set \(\psi_0 = 1\) and \(\psi_j = 0\) for \(j < 0\)).

**PROOF.** For completeness we give the proof in Supplementary B.2 \(\square\)

**PROOF of Equation (3.2)** Since

\[
(D_{\infty,n}(f_{\theta}))_{k,t} = n^{-1/2} \sum_{\tau \leq n} (\phi_t(\tau) e^{i\tau \omega_{k,n}} + \phi_{n+1-t}(\tau) e^{-i(\tau-1)\omega_{k,n}}) \quad 1 \leq k, t \leq n
\] (A.10)

we replace \(\phi_t(\tau)\) in the above with the coefficients of the MA and AR infinity expansions; \(\phi_t(\tau) = \)
\[
\sum_{s=0}^{\infty} \phi_{t+s} \psi_{|\tau| - s}. \text{ Substituting this into the first term in (A.10) gives }
\]
\[
n^{-1/2} \sum_{\tau \leq 0} \phi_t(\tau) e^{i \tau \omega_k n} = n^{-1/2} \sum_{\tau \leq 0} \sum_{s=0}^{\infty} \phi_{t+s} \psi_{\tau-s} e^{i \tau \omega_k n}
\]
\[
= n^{-1/2} \sum_{s=0}^{\infty} \phi_{t+s} e^{-i s \omega_k n} \sum_{\tau \leq 0} \psi_{\tau-s} e^{-i (\tau-s) \omega_k n}
\]
\[
= n^{-1/2} \psi(\omega_k n) \sum_{s=0}^{\infty} \phi_{t+s} e^{-i s \omega_k n} = n^{-1/2} \phi(\omega_k n)^{-1} \phi_t^\infty(\omega_k n),
\]
which gives the first term in (3.2). The second term follows similarly. Thus giving the identity in equation (3.2).

Next we prove Theorem 3.1. To do this we note that the entries of \(F_n^* \Delta_n(f^{-1}_\theta) D_n(f)\) are
\[
(F_n^* \Delta_n(f^{-1}_\theta) D_n(f))_{s,t} = \sum_{\tau \leq 0} [\phi_t(\tau; f) G_{1,n}(s, \tau; f_\theta) + \phi_{n+1-t}(\tau; f) G_{2,n}(s, \tau; f_\theta)],
\]
where \(G_{1,n}\) and \(G_{2,n}\) are defined as in (2.13). Thus
\[
(F_n^* \Delta_n(f^{-1}_\theta) [D_n(f) - D_n(\infty,f)])_{s,t} = \sum_{\tau \leq 0} \{[\phi_{t,n}(\tau; f) - \phi_t(\tau; f)] G_{1,n}(s, \tau; f_\theta) + [\phi_{n+1-t,n}(\tau; f) - \phi_{n+1-t}(\tau; f)] G_{2,n}(s, \tau; f_\theta)\}.
\]

To prove Theorem 3.1 we bound the above terms.

**PROOF of Theorem 3.1** To simplify notation we only emphasis the coefficients associated with \(f_\theta\) and not the coefficients associated with \(f\). I.e. we set \(\phi_{s,n}(\tau; f) = \phi_{s,n}(\tau), \phi_{s}(\tau; f) = \phi_s(\tau), \phi_f = \phi\) and \(\psi_f = \psi\).

The proof of (3.4) simply follows from the definitions of \(D_n(f)\) and \(D_n(\infty,f)\).

Next we prove (3.5). By using (2.13) and (A.11) (or simply (A.12)) we have
\[
\|F_n^* \Delta_n(f^{-1}_\theta) D_n(f) - F_n^* \Delta_n(f^{-1}_\theta) D_n(\infty,f)\|_1 \leq T_{1,n} + T_{2,n},
\]
where
\[
T_{1,n} = \sum_{s,t=1}^{n} \sum_{\tau=-\infty}^{0} |\phi_{s,n}(\tau) - \phi_{s}(\tau)||G_{1,n}(t, \tau; f_\theta)|
\]
and
\[
T_{2,n} = \sum_{s,t=1}^{n} \sum_{\tau=-\infty}^{0} |\phi_{n+1-s,n}(\tau) - \phi_{n+1-s}(\tau)||G_{2,n}(t, \tau; f_\theta)|.
\]

We focus on \(T_{1,n}\), noting that the method for bounding \(T_{2,n}\) is similar. Exchanging the summands
we have 

\[ T_{1,n} \leq \sum_{\tau = -\infty}^{0} \sum_{t=1}^{n} |G_{1,n}(t, \tau; f_{\theta})| \sum_{s=1}^{n} |\phi_{s,n}(\tau) - \phi_s(\tau)|. \]

To bound \( \sum_{s=1}^{n} |\phi_{s,n}(\tau) - \phi_s(\tau)| \) we require the generalized Baxter’s inequality stated in Lemma A.2. Substituting the bound in Lemma A.2 into the above (and for a sufficiently large \( n \)) we have

\[ T_{1,n} \leq C_{f,0} \sum_{\tau = -\infty}^{0} \sum_{t=1}^{n} |G_{1,n}(t, \tau; f_{\theta})| \sum_{s=n+1}^{\infty} |\phi_s(\tau)|. \]

Using that \( G_{1,n}(t, \tau) = \sum_{a \in \mathbb{Z}} K_{f_{\theta}^{-1}}(\tau - t + an) \) we have the bound

\[ T_{1,n} \leq C_{f,0} \sum_{\tau = -\infty}^{0} \sum_{t=1}^{n} \sum_{a \in \mathbb{Z}} |K_{f_{\theta}^{-1}}(t - \tau + an)| \sum_{s=n+1}^{\infty} |\phi_s(\tau)| \]

\[ = C_{f,0} \sum_{r \in \mathbb{Z}} |K_{f_{\theta}^{-1}}(r)| \sum_{\tau = -\infty}^{0} \sum_{s=n+1}^{\infty} |\phi_s(\tau)|. \]

Therefore,

\[ T_{1,n} \leq C_{f,0} \sum_{r \in \mathbb{Z}} |K_{f_{\theta}^{-1}}(r)| \sum_{\tau = -\infty}^{0} \sum_{s=n+1}^{\infty} |\phi_s(\tau)| \]

\[ \leq C_{f,0} \sum_{r \in \mathbb{Z}} |K_{f_{\theta}^{-1}}(r)| \sum_{\tau = -\infty}^{0} \sum_{s=n+1}^{\infty} \sum_{j=0}^{\infty} |\phi_{s+j}| |\psi_{\tau-j}| \quad \text{(use } \phi_s(\tau) = \sum_{j=0}^{\infty} \phi_{s+j} \psi_{\tau-j} \text{)} \]

\[ = C_{f,0} \sum_{r \in \mathbb{Z}} |K_{f_{\theta}^{-1}}(r)| \sum_{\tau = 0}^{\infty} |\psi_{\tau-j}| \sum_{s=n+1}^{\infty} \sum_{j=0}^{\infty} |\phi_{s+j}| \quad \text{(change limits of } \sum \text{)} \]

\[ \leq C_{f,0} \sum_{r \in \mathbb{Z}} |K_{f_{\theta}^{-1}}(r)| \sum_{\ell} |\psi_\ell| \sum_{u=n+1}^{\infty} |u\phi_u| \quad \text{(change of variables } u = s + j \text{).} \]

Next we use Assumption 3.1(i) to give

\[ T_{1,n} \leq C_{f,0} \sum_{r \in \mathbb{Z}} |K_{f_{\theta}^{-1}}(r)| \sum_{\ell} |\psi_\ell| \sum_{s=n+1}^{\infty} s^{K} |\phi_s| \]

\[ \leq \frac{C_{f,0}}{n^{K-1}} \sum_{r \in \mathbb{Z}} |K_{f_{\theta}^{-1}}(r)| \sum_{\ell} |\psi_\ell| \sum_{s=n+1}^{\infty} s^{K} |\phi_s| \]

\[ \leq \frac{C_{f,0}}{n^{K-1}} \rho_{n,K}(f) \|\psi\|_0 \|\phi\|_K \sum_{r \in \mathbb{Z}} |K_{f_{\theta}^{-1}}(r)|. \]
We note that \( K_{f\theta^{-1}}(r) = \int_0^{2\pi} f_{\theta^{-1}}(\omega)e^{ir\omega}d\omega = \sigma_{f\theta}^{-2} \int_0^{2\pi} |\phi_{f\theta}(\omega)|^2e^{ir\omega}d\omega = \sigma_{f\theta}^{-2} \sum_j \phi_j(f\theta)\phi_{j+1}(f\theta). \)

Therefore

\[
\sum_{r=-\infty}^{\infty} |K_{f\theta}(r)| \leq \sigma_{f\theta}^{-2} \|\phi_{f\theta}\|^2_0. \tag{A.13}
\]

Substituting this into the above yields the bound

\[
T_{1,n} \leq \frac{C_{f,0}}{\sigma_{f\theta}^2 nK-1} \rho_n(f) \|\psi\|_0 \|\phi_{f\theta}\|^2_0 \|\phi\|_K.
\]

The same bound holds for \( T_{2,n} \). Together the bounds for \( T_{1,n} \) and \( T_{2,n} \) give

\[
\|F_n^s \Delta_n(f\theta^{-1})D_n(f\theta) - F_n^s \Delta_n(n\theta^{-1})D_{\infty,n}(f\theta)\|_1 \leq \frac{2C_{f,0}}{\sigma_{f\theta}^2 nK-1} \rho_{n,K}(f) \|\psi\|_0 \|\phi_{f\theta}\|^2_0 \|\phi\|_K.
\]

Replacing \( \|\psi\|_0 = \|\psi\|_0 \) and \( \|\phi_f\|_K = \|\phi\|_K \), this proves (3.5).

To prove (3.6) we recall

\[
\|X_sX_t\|_q = (\mathbb{E}|X_sX_t|^q)^{1/q} \leq (\mathbb{E}X_s^{2q})^{1/2q} (\mathbb{E}X_t^{2q})^{1/2q} \leq \|X\|_{2q}^2.
\]

Therefore,

\[
n^{-1} \|X_n^j F_n \Delta_n(f\theta^{-1}) (D_n(f) - D_{\infty,n}(f)) X_n\|_q \leq n^{-1} \sum_{s,t=1}^n \| (F_n^s \Delta_n(f\theta^{-1}) (D_n(f) - D_{\infty,n}(f)))_{s,t}\| X_sX_t\|_q \leq n^{-1} \| F_n^s \Delta_n(f\theta^{-1}) (D_n(f) - D_{\infty,n}(f))\|_1 \|X\|_{2q}^2 \leq \frac{2C_{f,0}}{\sigma_{f\theta}^2 nK} \rho_{n,K}(f) \|\psi\|_0 \|\phi_{f\theta}\|^2_0 \|\phi\|_K \|X\|_{2q}^2,
\]

where the last line follows from the inequality in (3.5). This proves (3.6). \( \square \)

**PROOF of Theorem 3.2** For notational simplicity, we omit the parameter dependence on \( f\theta \).

We first prove (3.8). We observe that

\[
\|F_n^s \Delta_n(f\theta^{-1})D_{\infty,n}(f\theta)\|_1 \leq \sum_{s,t=1}^n \sum_{\tau=-\infty}^0 (|\phi_s(\tau)||G_{1,n}(t,\tau)| + |\phi_{n+1-s}(\tau)||G_{2,n}(t,\tau)|) = S_{1,n} + S_{2,n}.
\]

As in the proof of Theorem 3.1, we bound each term separately. Using a similar set of bounds...
to those used in the proof of Theorem 3.1 we have

$$ S_{1,n} \leq \sum_{r \in \mathbb{Z}} |K_{f_\theta^{-1}}(r)| \sum_{\ell} |\psi_{\ell}| \sum_{s=1}^{n} \sum_{j=0}^{\infty} |s \phi_s| $$

$$ \leq \sum_{r \in \mathbb{Z}} |K_{f_\theta^{-1}}(r)| \sum_{\ell} |\psi_{\ell}| \sum_{s=1}^{\infty} |s \phi_s| \leq \frac{1}{\sigma_1^2} \|\psi_{f_\theta}\|_0 \|\phi_{f_\theta}\|_0^2 \|\phi_{f_\theta}\|_1, $$

where the bound $\sum_{r \in \mathbb{Z}} |K_{f_\theta^{-1}}(r)| \leq \sigma_1^{-2} \|\phi_{f_\theta}\|_0^2$ follows from (A.13). Using a similar method we obtain the bound $S_{2,n} \leq \sigma_1^{-2} \|\psi_{f_\theta}\|_0 \|\phi_{f_\theta}\|_0^2 \|\phi_{f_\theta}\|_1$. Altogether the bounds for $S_{1,n}$ and $S_{2,n}$ give

$$ \|F_n^* \Delta_n(f_\theta^{-1})D_{\infty,n}(f_\theta)\|_1 \leq \frac{2}{\sigma_1^2} \|\psi_{f_\theta}\|_0 \|\phi_{f_\theta}\|_0^2 \|\phi_{f_\theta}\|_1, $$

this proves (3.8).

The proof of (3.9) uses the triangle inequality

$$ \|\Gamma_n(f_\theta)^{-1} - C_n(f_\theta^{-1})\|_1 = \|F_n^* \Delta_n(f_\theta)D_n(f_\theta)\|_1 $$

$$ \leq \|F_n^* \Delta_n(f_\theta) (D_n(f_\theta) - D_{\infty,n}(f_\theta))\|_1 + \|F_n^* \Delta_n(f_\theta)D_{\infty,n}(f_\theta)\|_1. $$

Substituting the bound Theorem 3.1 (equation (3.5)) and (3.8) into the above gives (3.9).

The proof of (3.10) uses the bound in (3.9) together with similar arguments to those in the proof of Theorem 3.1 we omit the details. □

We now prove Lemma 4.1. The proof is similar to the proof of Theorem 3.1 but with some subtle differences. Rather than bounding the best finite predictors with the best infinite predictors, we bound the best infinite predictors with the plug-in estimators based on the best fitting AR($p$) parameters. For example, the bounds use the regular Baxter’s inequality rather than the generalized Baxter’s inequality.

PROOF of Lemma 4.1 We first prove (4.4). By using the triangular inequality we have

$$ \|F_n^* \Delta_n(f_\theta^{-1}) (D_n(f) - D_n(f_p))\|_1 $$

$$ \leq \|F_n^* \Delta_n(f_\theta^{-1}) (D_n(f) - D_{\infty,n}(f))\|_1 + \|F_n^* \Delta_n(f_\theta^{-1}) (D_{\infty,n}(f) - D_n(f_p))\|_1 $$

$$ \leq \frac{C_{f,0,\rho_n,K}(f)}{nK^{-1}} A_K(f, f_\theta) + \|F_n^* \Delta_n(f_\theta^{-1}) (D_{\infty,n}(f) - D_n(f_p))\|_1, \quad (A.14) $$

where the first term of the right hand side of the above follows from (3.5). Now we bound the second term on the right hand side of the above. We observe that since the AR($p$) process only uses the first and last $p$ observations for the predictions that $D_n(f_p) = D_{\infty,n}(f_p)$, thus we can
write the second term as
\[
F_n^* \Delta_n (f \theta^{-1}) (D_{\infty,n}(f) - D_n(f_p)) = F_n^* \Delta_n (f \theta^{-1}) (D_{\infty,n}(f) - D_{\infty,n}(f_p)) \cdot (A.15)
\]

Recall that \( \{a_j(p)\}_{j=1}^p \) are the best fitting AR\( (p) \) parameters based on the autocovariance function associated with the spectral density \( f \). Let \( a_\phi(\omega) = 1 - \sum_{s=1}^{p} a_s(p) e^{-i s \omega}, \ a_\phi(\omega) = 1 - \sum_{s=1}^{p-j} a_{s+j}(p) e^{-i j \omega} \) and \( a_p(\omega)^{-1} = \sum_{j=0}^{\infty} \psi_j p e^{-i j \omega} \). By using the expression for \( D_{\infty,n}(f) \) given in (3.2) we have
\[
[F_n^* \Delta_n (f \theta^{-1}) (D_{\infty,n}(f) - D_{\infty,n}(f_p))]_{t,j} = U_{1,n}^{j,t} + U_{2,n}^{j,t}
\]
where
\[
U_{1,n}^{j,t} = \frac{1}{n} \sum_{k=1}^{n} e^{-i t \omega_k,n} \left( \phi_j^\infty(\omega_k,n) - \frac{a_{j,p}(\omega_k,n)}{a_p(\omega_k,n)} \right)
\]
\[
U_{2,n}^{j,t} = \frac{1}{n} \sum_{k=1}^{n} e^{-i (t-1) \omega_k,n} \left( \frac{a_{j+1,p}(\omega_k,n)}{a_p(\omega_k,n)} \phi_j^\infty(\omega_k,n) - \frac{a_{j+1,p}(\omega_k,n)}{a_p(\omega_k,n)} \right).
\]

We focus on \( U_{1,n}^{j,t} \), and partition it into two terms \( U_{1,n,1}^{j,t} = U_{1,n,1}^{j,t} + U_{1,n,2}^{j,t} \), where
\[
U_{1,n,1}^{j,t} = \frac{1}{n} \sum_{k=1}^{n} e^{-i t \omega_k,n} \phi(\omega_k,n) \phi_j^\infty(\omega_k,n) - a_{j,p}(\omega_k,n)
\]
and
\[
U_{1,n,2}^{j,t} = \frac{1}{n} \sum_{k=1}^{n} e^{-i t \omega_k,n} a_{j,p}(\omega_k,n) \left( \phi(\omega_k,n)^{-1} - a_p(\omega_k,n)^{-1} \right)
\]
\[
= \frac{1}{n} \sum_{k=1}^{n} e^{-i t \omega_k,n} a_{j,p}(\omega_k,n) \left( \psi(\omega_k,n) - \psi_p(\omega_k,n) \right).
\]

We first consider \( U_{1,n,1}^{j,t} \). We observe \( \phi(\omega_k,n)^{-1} = \psi(\omega_k,n) = \sum_{\ell=0}^{\infty} \psi_\ell e^{-i \ell \omega_k,n} \). Substituting this into \( U_{1,n,1}^{j,t} \) gives
\[
U_{1,n,1}^{j,t} = \sum_{s=0}^{\infty} (\phi_{j+s} - a_{j+s}(p)) \frac{1}{n} \sum_{k=1}^{n} e^{-i (t+s) \omega_k,n} \phi(\omega_k,n) \phi_j^\infty(\omega_k,n)
\]
\[
= \sum_{s=0}^{\infty} (\phi_{j+s} - a_{j+s}(p)) \sum_{\ell=0}^{\infty} \psi_\ell \frac{1}{n} \sum_{k=1}^{n} \phi(\omega_k,n) \phi_j^\infty(\omega_k,n) e^{-i (t+\ell+s) \omega_k,n}
\]
\[
= \sum_{s=0}^{\infty} (\phi_{j+s} - a_{j+s}(p)) \sum_{\ell=0}^{\infty} \psi_\ell \sum_{r \in \mathbb{Z}} K_{f \theta}^{-1}(t+\ell+s+rn),
\]
where $K_{f_\theta}^{-1}(r) = \int_0^{2\pi} f_\theta(\omega)^{-1} \exp(ir\omega) d\omega$. Therefore, the absolute sum of the above gives

$$
\sum_{j,t=1}^{n} |U_{1,n,1}^{j,t}| \leq \sum_{j,t=1}^{n} \sum_{s=0}^{\infty} |\phi_{j+s} - a_{j+s}(p)| \sum_{\ell=0}^{\infty} |\psi_\ell| \sum_{r \in \mathbb{Z}} |K_{f_\theta}^{-1}(t + \ell + s + rn)|
= \sum_{j=1}^{n} \sum_{s=0}^{\infty} |\phi_{j+s} - a_{j+s}(p)| \sum_{\ell=0}^{\infty} |\psi_\ell| \sum_{t=1}^{n} \sum_{r \in \mathbb{Z}} |K_{f_\theta}^{-1}(t + \ell + s + rn)|
\leq \left( \sum_{j=1}^{n} \sum_{s=0}^{\infty} |\phi_{j+s} - a_{j+s}(p)| \right) \|\psi_\ell\|_0 \sum_{\tau \in \mathbb{Z}} |K_{f_\theta}^{-1}(\tau)|
\leq \left( \sum_{s=1}^{\infty} s|\phi_s - a_s(p)| \right) \|\psi_\ell\|_0 \sum_{\tau \in \mathbb{Z}} |K_{f_\theta}^{-1}(\tau)|.
$$

By using (A.13) we have $\sum_{\tau \in \mathbb{Z}} |K_{f_\theta}^{-1}(\tau)| \leq \sigma_{f_\theta}^{-2} \|f_\theta\|_0^2$. Further, by using the regular Baxter inequality we have

$$
\sum_{s=1}^{\infty} s|\phi_s - a_s(p)| \leq (1 + C_{f,1}) \sum_{s=p+1}^{\infty} s|\phi_s| \leq (1 + C_{f,1}) p^{-K+1} \rho_{p,K} \|f_\ell\|_K.
$$

Substituting these two bounds into $\sum_{j,t=1}^{n} |U_{1,n,1}^{j,t}|$ yields

$$
\sum_{j,t=1}^{n} |U_{1,n,1}^{j,t}| \leq \frac{(1 + C_{f,1})}{\sigma_{f_\theta}^{-2} p^{K-1} \rho_{p,K}} \|f_\ell\|_K \|\psi_\ell\|_0 \|f_\theta\|_0^2.
$$

Next we consider the second term $U_{1,n,2}^{j,t}$. Using that $\psi(\omega_{k,n}) = \sum_{s=0}^{\infty} \psi_s e^{-is\omega}$ and $\psi_p(\omega_{k,n}) = \sum_{s=0}^{\infty} \psi_{s,p} e^{-is\omega}$ we have

$$
U_{1,n,2}^{j,t} = \frac{1}{n} \sum_{k=1}^{\infty} e^{-it\omega_{k,n}} a_{j+p}^\infty(\omega_{k,n}) \left( \psi(\omega_{k,n}) - \psi_p(\omega_{k,n}) \right)
= \sum_{s=0}^{\infty} (\psi_s - \psi_{s,p}) \frac{1}{n} \sum_{k=1}^{\infty} e^{-it(s+\ell)\omega_{k,n}} a_{j+p}^\infty(\omega_{k,n})
= \sum_{s=0}^{\infty} (\psi_s - \psi_{s,p}) \sum_{\ell=0}^{\infty} a_{j+p}(p) \frac{1}{n} \sum_{k=1}^{\infty} e^{-it(s+\ell)\omega_{k,n}} K_{f_\theta}^{-1}(t + s + \ell + rn).
$$
Substituting the bound in (A.16) and (A.17) into 

\[ \sum_{j,t=1}^{\infty} |U_{1,n,2}^{j,t}| \]

\[ \leq \sum_{j,t=1}^{\infty} |\psi_s - \psi_{s,p}| \sum_{s=0}^{\infty} |a_{j+\ell}(p)| \sum_{r \in \mathbb{Z}} |K_{f_s^{-1}(t + s + \ell + rn)}| \]

\[ \leq \sum_{s=0}^{\infty} |\psi_s - \psi_{s,p}| \sum_{j=1}^{\infty} \sum_{\ell=0}^{\infty} |a_{j+\ell}(p)| \sum_{r \in \mathbb{Z}} |K_{f_s^{-1}(r)}| \quad \text{(apply the bound (A.13))} \]

\[ \leq \sigma_{f_s}^{-2} \|\phi_{f_s}\|_0^2 \left( \sum_{s=0}^{\infty} |\psi_s - \psi_{s,p}| \right) \sum_{u=0}^{\infty} |ua_u(p)| \]

\[ \leq \sigma_{f_s}^{-2} \|\phi_{f_s}\|_0^2 \|a_p\|_1 \sum_{s=0}^{\infty} |\psi_s - \psi_{s,p}|. \]

Next we bound \( \|a_p\|_1 \) and \( \sum_{s=0}^{\infty} |\psi_s - \psi_{s,p}| \). Let \( \phi_p(\omega) = 1 - \sum_{j=1}^{p} \phi_j \exp(ij\omega) \) (the truncated AR(\( \infty \)) process). Then by applying Baxter’s inequality, it is straightforward to show that

\[ \|a_p\|_1 \leq \|\phi_p\|_1 + \|a_p - \phi_p\|_1 \leq (C_{f,1} + 1)\|\phi_f\|_1. \]  

(A.16)

To bound \( \sum_{s=0}^{\infty} |\psi_s - \psi_{s,p}| \) we use the inequality in [Kreiss et al. (2011), page 2126]

\[ \sum_{s=0}^{\infty} |\psi_s - \psi_{s,p}| \leq \frac{\|\psi_f\|_0^2 \sum_{j=1}^{\infty} |\phi_j - a_j(p)|}{1 - \|\psi_f\|_0 \cdot \|a_p - \phi\|_0}. \]

Applying Baxter’s inequality to the numerator of the above gives

\[ \sum_{s=0}^{\infty} |\psi_s - \psi_{s,p}| \leq \frac{\|\psi_f\|_0^2 (C_{f,0} + 1)\rho_{p,K}(f)\|\phi_f\|_K}{p^K(1 - \|\psi_f\|_0 \cdot \|a_p - \phi\|_0)} \]  

(A.17)

Substituting the bound in (A.16) and (A.17) into \( \sum_{j,t=1}^{n} |U_{1,n,2}^{j,t}| \) gives

\[ \sum_{j,t=1}^{n} |U_{1,n,2}^{j,t}| \leq \frac{(C_{f,1} + 1)^2}{\sigma_{f_s}^2 p^K} \cdot \frac{\|\psi_f\|_0^2 \|\phi_f\|_1 \|\phi_f\|_K \|\phi_{f_s}\|_0^2 \rho_{p,K}(f)}{1 - \|\psi_f\|_0 \|a_p - \phi\|_0}. \]

Altogether, for sufficiently large \( p \), where \( \|\psi_f\|_0 \cdot \|a_p - \phi\|_0 \leq 1/2 \) we have

\[ \sum_{t,j=1}^{n} |U_{1,n}^{j,t}| \leq \frac{1 + C_{f,1}}{\sigma_{f_s}^2 p^{K-1}} \rho_{p,K}(f) \|\phi_f\|_K \|\psi_f\|_0 \|\phi_{f_s}\|_0^2 + \frac{2(C_{f,1} + 1)^2}{\sigma_{f_s}^2 p^K} \|\psi_f\|_0^2 \|\phi_f\|_1 \|\phi_f\|_K \|\phi_{f_s}\|_0^2 \rho_{p,K}(f) \]

\[ \leq \frac{(C_{f,1} + 1)}{\sigma_{f_s}^2 p^{K-1}} \rho_{p,K}(f) \|\phi_f\|_K \|\phi_{f_s}\|_0^2 \|\psi_f\|_0 (1 + 2p^{-1}(1 + C_{f,1}) \|\psi_f\|_0 \|\phi_f\|_1) \]
The same bound holds for \( \sum_{t,j=1}^n |U_{t,j}^j| \), thus using \( (A.14) \) and \( \rho_{n,K}(f) \leq \rho_{p,K}(f) \) gives

\[
\|F_n^* \Delta_n(f^{-1}) (D_{\infty,n}(f) - D_n(f_p))\|_1 \leq \rho_{p,K}(f) A_K(f,f_p) \left( \frac{(C_{f,1} + 1)}{p^{K-1}} + \frac{2(C_{f,1} + 1)^2}{p^K} \|\psi_f\|_0 \|\phi_f\|_1 \right).
\]

Substituting the above into \( (A.14) \) gives \( (4.4) \).

The proof of \( (4.5) \) is similar to the proof of Theorem 3.1, we omit the details. \( \square \)

## B Additional Lemmas

### B.1 Some additional proofs

**Lemma B.1** Let \( \widetilde{J}_{n,L}(\omega) \) be defined as in \( (A.5) \), where the parameters \( \phi \) are such that the roots of \( \phi(z) = 1 - \sum_{j=1}^p \phi_j z^j \) lie outside the unit circle. Let \( A_p(\phi) \) be defined as in \( (A.4) \). Then

\[
[A_p(\phi)]^{\tau+1} X_p(1) = \sum_{\ell=1}^p X_{\ell} \sum_{s=0}^{p-\ell} \phi_{\ell+s} \psi_{|\tau|-s}, \tag{B.1}
\]

where \( \{\psi_j\} \) are the coefficients in the expansion \( (1 - \sum_{j=1}^p \phi_j e^{-ij\omega})^{-1} = \sum_{j=0}^\infty \psi_j e^{-is\omega}. \)

**Proof.** To simplify notation let \( A = A_p(\phi) \). The proof is based on the observation that the \( j \)th row of \( A^m \) \( (m \geq 1) \) is the \((j-1)\)th row of \( A^{m-1} \) (due to the structure of \( A \)). Let \( (a_{1,m}, \ldots, a_{p,m}) \) denote the first row of \( A^m \). Using this notation we have

\[
\begin{pmatrix}
a_{1,m} & a_{2,m} & \cdots & a_{p,m} \\
a_{1,m-1} & a_{2,m-1} & \cdots & a_{p,m-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1,m-p+1} & a_{2,m-p+1} & \cdots & a_{p,m-p+1}
\end{pmatrix} = \begin{pmatrix}
a_{1,m-1} & a_{2,m-1} & \cdots & a_{p,m-1} \\
a_{1,m-2} & a_{2,m-2} & \cdots & a_{p,m-2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1,m-p} & a_{2,m-p} & \cdots & a_{p,m-p}
\end{pmatrix} \begin{pmatrix}
\phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}
\]

From the above we see that \( a_{\ell,m} \) satisfies the recursion

\[
a_{\ell,m} = \phi_{\ell} a_{1,m-1} + a_{\ell+1,m-1} \quad 1 \leq \ell \leq p - 1
\]

\[
a_{p,m} = \phi_p a_{1,m-1}. \tag{B.2}
\]

Our aim is to obtain an expression for \( a_{\ell,m} \) in terms of \( \{\phi_j\}_{j=1}^p \) and \( \{\psi_j\}_{j=0}^\infty \) which we now define. Since the roots of \( \phi(\cdot) \) lies outside the unit circle the function \( (1 - \sum_{j=1}^p \phi_j z^j)^{-1} \) is well defined for \( |z| \leq 1 \) and has the power series expansion \( (1 - \sum_{i=1}^p \phi_i z^i)^{-1} = \sum_{i=0}^\infty \psi_i z^i \) for \( |z| \leq 1 \). By mathematical induction, it can be proved that \( [A^m]_{1,1} = a_{1,m} = \psi_m \). We now obtain a formula for the remaining coefficients \( \{a_{\ell,m}; 2 \leq \ell \leq p\} \) in terms of \( \{\phi_i\} \) and \( \{\psi_i\} \). Solving the system of equations in \( (B.2) \), starting with \( a_{1,m-1} = \psi_{m-1} \) and recursively solving for \( a_{p,m}, \ldots, a_{2,m} \) we
have

\[
\begin{align*}
    a_{p,m} &= \phi_p \psi_{m-1} \\
    a_{p-1,m} &= \phi_{p-1} \psi_{m-1} + \phi_p \psi_{m-2} \\
    a_{p-2,m} &= \phi_{p-2} \psi_{m-1} + \phi_{p-1} \psi_{m-2} + \phi_p \psi_{m-3} \\
    \Rightarrow a_{p-r,m} &= \sum_{s=0}^{r} \phi_{p-r+s} \psi_{m-1-s} \quad 0 \leq r \leq p-1.
\end{align*}
\]

In the last line of the above we change variables with \( \ell = p - r \) to give for \( m \geq 1 \)

\[
a_{\ell,m} = \sum_{s=0}^{p-\ell} \phi_{\ell+s} \psi_{m-1-s} \quad 1 \leq \ell \leq p,
\]

where \( \theta_0 = 1 \) and for \( t < 0, \psi_t = 0 \). Therefore

\[
[A^{\tau+1} X_p]_{(1)} = \sum_{\ell=1}^{p} X_\ell \sum_{s=0}^{p-\ell} \phi_{\ell+s} \psi_{|\tau|-s}.
\]

Thus we obtain the desired result. \( \square \)

In the proof of Lemma [B.1] we obtained an expression in terms for the best linear predictor based on the parameters of an AR(\( p \)) process. For completeness, we obtain an expression for the left hand side predictive DFT for a general second order stationary time series which is based on infinite future. For \( \tau \leq 0 \), let \( \hat{X}_\tau \) be the best linear predictor of \( X_\tau \) given the infinite future \( \{X_t\}_{t=1}^{\infty} \) i.e.

\[
\hat{X}_\tau = \sum_{s=1}^{\infty} \phi_s(\tau) X_s. \quad (B.3)
\]

The left hand side predictive DFT given the infinite future is defined as

\[
J_L(\omega; f_\theta) = \sum_{\tau=-\infty}^{0} \hat{X}_\tau \exp(i\tau \omega). \quad (B.4)
\]

**Corollary B.1** Suppose that \( f_\theta \) satisfies Assumption [3.1] with \( K \geq 1 \). Let \( \hat{J}_L(\omega) \) be defined as in \((B.4)\). Then, the best linear predictor of \( X_\tau \) given \( \{X_t\}_{t=1}^{\infty} \) where \( \tau \leq 0 \) (defined in \((B.3)\)) can be evaluated using the recursion \( \hat{X}_\tau = \sum_{s=1}^{\infty} \phi_s \hat{X}_{\tau+s} \), where we set \( \hat{X}_t = X_t \) for \( t \geq 1 \). Further, \( \phi_\ell(\tau) = \sum_{s=0}^{\infty} \phi_{\ell+s} \psi_{|\tau|-s} \) and

\[
\hat{J}_L(\omega; f_\theta) = \phi(\omega)^{-1} \sum_{\ell=1}^{\infty} X_\ell \sum_{s=0}^{\infty} \phi_{\ell+s} \exp(-is\omega). \quad (B.5)
\]
PROOF. We recall that for general processes \( X_t \) with \( f_\theta \) bounded away from 0 has the AR(\( \infty \)) representation 
\[
X_t = \sum_{j=1}^{\infty} \phi_j X_{j+t} + \varepsilon_t \quad \text{where} \quad \{\varepsilon_t\} \text{ are uncorrelated random variables.}
\]
This immediately implies that the best linear prediction of \( X_\tau \) given \( \{X_t\}_{t=1}^{\infty} \) can be evaluated using 
the recursion 
\[
\hat{X}_\tau = \sum_{s=1}^{\infty} \phi_s \hat{X}_{\tau+s}.
\]
By using (B.1) where we let \( p \to \infty \) (This is is possible by Assumption 3.1 and the Baxter’s inequality) we have that 
\[
\phi(\tau) = \sum_{s=0}^{\infty} \phi_{\tau+s} \psi_{|\tau|-s}.
\]
This gives the first part of the result.

To obtain an expression for \( \hat{J}_L(\cdot; f_\theta) \) we use (A.6) where we let \( p \to \infty \) to obtain the desired result. \(\square\)

### B.2 An extension of Baxter’s inequality

Let \( \{X_t\} \) be a second order stationary time series with absolutely summable autocovariance and 
spectral density \( f \). We can represent \( f \) as 
\[
f(\omega) = \psi(\omega)\overline{\psi(\omega)} = 1/\left(\phi(\omega)\overline{\phi(\omega)}\right)
\]
where 
\[
\phi(\omega) = 1 - \sum_{s=1}^{\infty} \phi_s e^{-is\omega} \quad \text{and} \quad \psi(\omega) = 1 + \sum_{s=1}^{\infty} \psi_s e^{-is\omega}.
\]
Note that \( \{\phi_s\} \) and \( \{\psi_s\} \) are the corresponding AR(\( \infty \)) and MA(\( \infty \)) coefficients respectively and 
\( \psi(\omega) = \phi(\omega)^{-1} \). To simplify notation we have ignored the variance of the innovation.

Let \( \{\phi_{s,p}(\tau)\}_{p=1}^{\infty} \) denote the the coefficients of the best linear predictor of \( X_{t+\tau} \) (for \( \tau \geq 0 \)) 
given \( \{X_s\}_{t-p}^{\infty} \)
\[
E \left[ \left( X_{t+\tau} - \sum_{s=1}^{p} \phi_{s,p}(\tau) X_{t-s} \right) X_{t-k} \right] = 0 \quad \text{for} \quad k = 1, \ldots, p. \quad (B.6)
\]
and \( \{\phi_s(\tau)\} \) denote the coefficients of the best linear predictor of \( X_{t+\tau} \) given the infinite past 
\( \{X_s\}_{s=-\infty}^{t-1} \)
\[
E \left[ \left( X_{t+\tau} - \sum_{s=1}^{\infty} \phi_s(\tau) X_{t-s} \right) X_{t-k} \right] = 0 \quad \text{for} \quad k = 1, 2, \ldots \quad (B.7)
\]

Before we begin, we define an appropriate norm on the subspace of \( L_2[0, 2\pi] \).

**Definition B.1 (Norm on the subspace of \( L_2[0, 2\pi] \))** Suppose the sequence of positive weights 
\( \{v(k)\}_{k \in \mathbb{Z}} \) satisfies 2 conditions: (1) \( v(n) \) is even, i.e., \( v(-n) = v(n) \) for all \( n \geq 0 \); (2) 
\( v(n+m) \leq v(n)v(m) \) for all \( n, m \in \mathbb{Z} \).

Given \( \{v(k)\} \) satisfies 2 conditions above, define a subspace \( A_v \) of \( L_2[0, 2\pi] \) by 
\[
A_v = \{ f \in L_2[0, 2\pi] : \sum_{k \in \mathbb{Z}} v(k)|f_k| < \infty \}.
\]
where, \( f(\omega) = \sum_{k \in \mathbb{Z}} f_k \exp(ik\omega) \). We define a norm \( \|f\| \) on \( A_v \) by \( \|f\| = \sum_{k \in \mathbb{Z}} v(k)|f_k| \), then it is easy to check this is a valid norm.

**Remark B.1 (Properties of \( \| \cdot \| \))** Suppose the sequence \( \{v(k)\}_{k \in \mathbb{Z}} \) satisfies 2 conditions in Definition B.1 and define the norm \( \| \cdot \| \) with respect to \( \{v(k)\} \). Then, beside the triangle inequality, this norm also satisfies \( \|1\| = v(0) \leq 1 \), \( \|f\| = \|f\| \), and \( \|fg\| \leq \|f\|\|g\| \) (which does not hold for all norms but is an important component of the (extended) Baxter’s proof), i.e., \( (A_v, \| \cdot \| ) \) is a Banach algebra with involution operator. The proof for the multiplicative inequality follows from the fact that \( (fg)_k = \sum_r f_r g_{k-r} \), where \( f_k \) and \( g_k \) are \( k \)th Fourier coefficient of \( f \) and \( g \). Thus

\[
\|fg\| \leq \sum_{k \in \mathbb{Z}} v(k) \left| \sum_{r \in \mathbb{Z}} f_r g_{k-r} \right| \leq \sum_{k \in \mathbb{Z}} v(r) v(k-r) \left| \sum_{r \in \mathbb{Z}} f_r g_{k-r} \right| \leq \sum_{k, r \in \mathbb{Z}} v(r) v(k-r) |f_r| |g_{k-r}| = \|f\|\|g\|.
\]

Examples of weights include \( v(r) = (2^q + |r|^q) \) or \( v(r) = (1 + |r|)^q \) for some \( q \geq 0 \). In these two examples, when \( q = K \), under Assumption B.1, \( \psi(\omega) \), \( \phi(\omega) \in A_v \) where \( \psi(\omega) = 1 + \sum_{j=1}^{\infty} \psi_j \exp(-i\omega) \) and \( \phi(\omega) = 1 - \sum_{j=1}^{\infty} \phi_j \exp(-i\omega) \) (see Kreiss et al. (2011)).

The proof below follows closely the proof of Baxter (1962) and Baxter (1963).

**PROOF of Lemma A.2** We use the same proof as Baxter, which is based on rewriting the normal equations in (B.6) within the frequency domain to yield

\[
\frac{1}{2\pi} \int_0^{2\pi} \left( e^{i\tau \omega} - \sum_{s=1}^{p} \phi_{s,p}(\tau) e^{-i\omega_s} \right) f(\omega) e^{-i\omega_d} d\omega = 0, \text{ for } k = 1, \ldots, p
\]

Similarly, using the infinite past to do prediction yields the normal equations

\[
\frac{1}{2\pi} \int_0^{2\pi} \left( e^{i\tau \omega} - \sum_{s=1}^{\infty} \phi_s(\tau) e^{-i\omega_s} \right) f(\omega) e^{-i\omega_d} d\omega = 0, \text{ for } k \geq 1.
\]

Thus taking differences of the above two equations for \( k = 1, \ldots, p \) gives

\[
\frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{s=1}^{p} |\phi_{s,p}(\tau) - \phi_s(\tau)| e^{-i\omega} \right) f(\omega) e^{-i\omega_d} d\omega
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{s=p+1}^{\infty} \phi_s(\tau) e^{-i\omega_s} \right) f(\omega) e^{-i\omega_d} d\omega \text{ for } k = 1, \ldots, p. \tag{B.8}
\]

These \( p \)-equations give rise to Baxter’s Weiner-Hopf equations and allow one to find a bound for \( \sum_{s=1}^{p} |\phi_{s,p}(\tau) - \phi_s(\tau)| \) in terms of \( \sum_{s=p+1}^{\infty} |\phi_s(\tau)| \). Interpreting the above, we have two different functions \( (\sum_{s=1}^{p} |\phi_{s,p}(\tau) - \phi_s(\tau)| e^{-i\omega_s} \) \( f(\omega) \) and \( (\sum_{s=p+1}^{\infty} \phi_s(\tau) e^{-i\omega}) \) \( f(\omega) \) whose first \( p \) Fourier coefficients are the same.
Define the polynomials

\[ h_p(\omega) = \sum_{s=1}^{p} [\phi_{s,p}(\tau) - \phi_s(\tau)] e^{-i\omega} \quad \text{and} \quad g_p(\omega) = \sum_{k=1}^{p} g_{k,p} e^{ik\omega} \]  

(B.9)

where

\[ g_{k,p} = (2\pi)^{-1} \int_0^{2\pi} \left( \sum_{s=p+1}^{\infty} \phi_s(\tau) e^{-i\omega} \right) f(\omega) e^{-ik\omega} d\omega. \]  

(B.10)

We will show that for a sufficiently large \( p \), \( \| h_p \| \leq C_f \| g_p \| \), where the constant \( C_f \) is a function of the spectral density (that we will derive).

The Fourier expansion of \( h_p f \) is

\[ h_p(\omega) f(\omega) = \sum_{k=-\infty}^{\infty} \tilde{g}_{k,p} e^{ik\omega}, \]

where \( \tilde{g}_{k,p} = (2\pi)^{-1} \int_0^{2\pi} h_p(\omega) f(\omega) e^{-ik\omega} d\omega. \) Then, by (B.8) for \( 1 \leq k \leq p \), \( \tilde{g}_{k,p} = g_{k,p} \) (where \( g_{k,p} \) is defined in (B.10)). Thus

\[ h_p(\omega) f(\omega) = G_{-\infty}^0(\omega) + g_p(\omega) + G_{p+1}^\infty(\omega) \]  

(B.11)

where

\[ G_{-\infty}^0(\omega) = \sum_{k=-\infty}^{0} \tilde{g}_{k,p} e^{ik\omega} \quad \text{and} \quad G_{p+1}^\infty(\omega) = \sum_{s=p+1}^{\infty} \tilde{g}_{k,p} e^{ik\omega}. \]

Dividing by \( f^{-1} = \phi \phi \) and taking the \( \| \cdot \| \)-norm we have

\[ \| h_p \| \leq \| f^{-1} G_{-\infty}^0 \| + \| f^{-1} g_p \| + \| f^{-1} G_{p+1}^\infty \| \leq \| f^{-1} G_{-\infty}^0 \| + \| f^{-1} || g_p \| + \| f^{-1} G_{p+1}^\infty \| \leq \| \tilde{\phi} \| \| \phi G_{-\infty}^0 \| + \| f^{-1} || g_p \| + \| \phi \| \| \phi G_{p+1}^\infty \|. \]  

(B.12)

First we obtain bounds for \( \| \phi G_{-\infty}^0 \| \) and \( \| \phi G_{p+1}^\infty \| \) in terms of \( \| g_p \| \). We will show that for a sufficiently large \( p \)

\[ \| \phi G_{-\infty}^0 \| \leq \| \phi \| \| g_p \| + \varepsilon \| \phi G_{p+1}^\infty \| \]  

and

\[ \| \phi G_{p+1}^\infty \| \leq \| \phi \| \| g_p \| + \varepsilon \| \phi G_{-\infty}^0 \|. \]

The bound for these terms hinges on the Fourier coefficients of a function being unique, which allows us to compare coefficients across functions. Some comments are in order that will help in
the bounding of the above. We recall that \( f(\omega)^{-1} = \phi(\omega)\bar{\phi}(\omega) \), where

\[
\phi(\omega) = 1 - \sum_{s=1}^{\infty} \phi_s \exp(-i\omega) \quad \bar{\phi}(\omega) = 1 - \sum_{s=1}^{\infty} \phi_s \exp(i\omega).
\]

Thus \( \phi(\omega)G_{-\infty}^{0}(\omega) \) and \( \bar{\phi}(\omega)G_{p+1}^{\infty}(\omega) \) have Fourier expansions with only less than 1st and greater than \( p \)th frequencies respectively. This observation gives the important insight into the proof.

Suppose \( b(\omega) = \sum_{j=-\infty}^{\infty} b_j \exp(ij\omega) \), we will make the use of the notation \( \{b(\omega)\}_+ = \sum_{j=1}^{\infty} b_j \exp(ij\omega) \) and \( \{b(\omega)\}_- = \sum_{j=-\infty}^{0} b_j \exp(ij\omega) \), thus \( b(\omega) = \{b(\omega)\}_- + \{b(\omega)\}_+ \).

We now return to (B.11) using that \( \|G\| \) we multiply (B.11) by \( \psi(\omega)^{-1} = \phi(\omega) \) to give

\[
h_p(\omega)\psi(\omega) = \phi(\omega)G_{-\infty}^{0}(\omega) + \phi(\omega)g_p(\omega) + \phi(\omega)G_{p+1}^{\infty}(\omega). \quad \text{(B.13)}
\]

Rearranging the above gives

\[
-\phi(\omega)G_{-\infty}^{0}(\omega) = -h_p(\omega)\psi(\omega) + \phi(\omega)g_p(\omega) + \phi(\omega)G_{p+1}^{\infty}(\omega).
\]

We recall that \( h_p(\omega)\psi(\omega) \) only contain positive frequencies, whereas \( \phi(\omega)G_{-\infty}^{0}(\omega) \) only contains nonpositive frequencies. Based on these observations we have

\[
-\phi(\omega)G_{-\infty}^{0}(\omega) = \{-\phi(\omega)G_{-\infty}^{0}(\omega)\}_- = \{\phi(\omega)g_p(\omega)\}_- + \{\phi(\omega)G_{p+1}^{\infty}(\omega)\}_-. \quad \text{(B.14)}
\]

We further observe that \( G_{p+1}^{\infty} \) only contains non-zero coefficients for positive frequencies of \( p + 1 \) and greater, thus only the coefficients of \( \phi(\omega) \) with frequencies less or equal to \( -(p + 1) \) will give nonpositive frequencies when multiplied with \( G_{p+1}^{\infty} \). Therefore

\[
-\phi(\omega)G_{-\infty}^{-1}(\omega) = \{\phi(\omega)g_p(\omega)\}_- + \{\phi_{p+1}^{\infty}(\omega)G_{p+1}^{\infty}(\omega)\}_-,
\]

where \( \phi_{p+1}^{\infty}(\omega) = \sum_{a=p+1}^{\infty} \phi_a e^{-i\omega} \). Evaluating the norm of the above (using both the triangle and the multiplicative inequality) we have

\[
\|\phi G_{-\infty}^{0}\| \leq \|\phi\| \|g_p\| + \|\phi_{p+1}^{\infty} G_{p+1}^{\infty}\| \\
\leq \|\phi\| \|g_p\| + \|\phi_{p+1}^{\infty}\| \|\psi\| \|\phi_{p+1}^{\infty}\| \quad \text{since} \quad \psi(\omega)\bar{\phi}(\psi) = 1.
\]

This gives a bound for \( \|\phi G_{-\infty}^{0}\| \) in terms of \( \|g_p\| \) and \( \|\phi_{p+1}^{\infty}\| \). Next we obtain a similar bound for \( \|\phi_{p+1}^{\infty}\| \) in terms of \( \|g_p\| \) and \( \|\phi G_{-\infty}^{0}\| \).

Again using (B.11), \( f(\omega) = \psi(\omega)\bar{\psi}(\omega) \), but this time multiplying (B.11) by \( \psi(\omega)^{-1} = \phi(\omega) \),
we have
\[ h_p(\omega)\psi(\omega) = \overline{\phi(\omega)}G^0_{-\infty}(\omega) + \overline{\phi(\omega)}g_p(\omega) + \overline{\phi(\omega)}G^\infty_{p+1}(\omega). \]

Rearranging the above gives
\[ \Rightarrow \overline{\phi(\omega)}G^\infty_{p+1}(\omega) = h_p(\omega)\psi(\omega) - \overline{\phi(\omega)}G^0_{-\infty}(\omega) - \overline{\phi(\omega)}g_p(\omega). \]

We observe that \( \overline{\phi(\omega)}G^\infty_{p+1}(\omega) \) contains frequencies greater than \( p \) whereas \( h_p(\omega)\psi(\omega) \) only contains frequencies less or equal to the order \( p \) (since \( h_p \) is a polynomial up to order \( p \)). Therefore multiply \( e^{-ip\omega} \) on both side and take \( \{ \}_+ \) gives
\[ e^{-ip\omega}\overline{\phi(\omega)}G^\infty_{p+1}(\omega) = -\left\{ e^{-ip\omega}\overline{\phi(\omega)}G^0_{-\infty}(\omega) \right\}_+ - \left\{ e^{-ip\omega}\overline{\phi(\omega)}g_p(\omega) \right\}_+. \tag{B.15} \]

By the similar technique from the previous, it is easy to show \( \left\{ e^{-ip\omega}\overline{\phi(\omega)}G^0_{-\infty}(\omega) \right\}_+ = \left\{ e^{-ip\omega}\overline{\phi^\infty_{p+1}(\omega)}G^0_{-\infty}(\omega) \right\}_+ \). Multiplying \( e^{ip\omega} \) and evaluating the \( \| \cdot \| \)-norm of the above yields the inequality
\[ \|\overline{\phi^\infty_{p+1}}\| \leq \|\overline{\phi}\|_g + \|\overline{\phi^\infty_{p+1}}G^0_{-\infty}\| \]
\[ \leq \|\overline{\phi}\|_g + \|\overline{\phi^\infty_{p+1}}\| \|\psi\| \|\phi G^{-\infty}\| . \]

We note that \( \|\phi^\infty_{p+1}\| = \|\overline{\phi^\infty_{p+1}}\| \). For \( \phi \in A_v \) (see Definition B.1 and Remark B.1), \( \|\phi^\infty_{p+1}\| = \sum_{s=p+1}^\infty \|e^{is\omega}\phi_s\| \to 0 \) as \( p \to \infty \), for a large enough \( p \), \( \|\psi(\omega)\| \cdot \|\phi^\infty_{p+1}\| < 1 \). Suppose that \( p \) is such that \( \|\phi^\infty_{p+1}(\omega)\| \|\psi(\omega)\| \leq \varepsilon < 1 \), then we have the desired bounds
\[ \|\phi G^{-\infty}\| \leq \|\phi\| \|g_p\| + \varepsilon \|\overline{\phi^\infty_{p+1}}\| \]
\[ \|\overline{\phi^\infty_{p+1}}\| \leq \|\overline{\phi}\| \|g_p\| + \varepsilon \|\phi G^{-\infty}\| . \]

The above implies that \( \|\phi G^{-\infty}\| + \|\overline{\phi G^\infty_{p+1}}\| \leq 2(1 - \varepsilon)^{-1} \|\phi\| \|g_p\| \). Substituting the above in (B.12), and using that \( \|\phi\| \geq 1 \) (since \( \phi = 1 - \sum_{s=1}^\infty \phi_s e^{-is\omega}, \|\phi\| \geq 1 \| = v(0) \geq 1 \)) we have
\[ \|h_p\| \leq \frac{2 \|\phi\| \|g_p\|}{1 - \varepsilon} + \|f^{-1}\| \|g_p\| \]
\[ \leq (1 - \varepsilon)^{-1} (2 \|\phi\| + (1 - \varepsilon) \|\phi\|^2) \|g_p\| \leq \frac{3 - \varepsilon}{1 - \varepsilon} \|\phi\|^2 \|g_p\| . \]

Thus based on the above we have
\[ \|h_p\| \leq \frac{3 - \varepsilon}{1 - \varepsilon} \|\phi\|^2 \|g_p\| . \tag{B.16} \]

Finally, we obtain a bound for \( \|g_p\| \) in terms of \( \sum_{s=p+1}^\infty \|\phi_s(\tau)\| \). We define an extended version of the function \( g_p(\omega) \). Let \( \tilde{g}_p(\omega) = \sum_{k \in \mathbb{Z}} g_{k,p} e^{i k \omega} \) where \( g_{k,p} \) is as in (B.9). By definition, \( \tilde{g}_p(\omega) = \)
\( \left( \sum_{s=p+1}^{\infty} \phi_s(\tau)e^{-is\omega} \right) f(\omega) \) and the Fourier coefficients of \( g_p(\omega) \) are contained within \( \tilde{g}_p(\omega) \), which implies

\[
\|g_p\| \leq \|	ilde{g}_p\| = \left\| \phi_{p+1}^\infty(\tau) f \right\| \leq \left\| \phi_{p+1}^\infty(\tau) \right\| \|f\| \leq \left\| \phi_{p+1}^\infty \right\| \|\psi\|^2.
\]

(B.17)

where \( \phi_{p+1}^\infty(\tau) = \sum_{s=p+1}^{\infty} \phi_s(\tau)e^{-is\omega} \). Finally, substituting (B.17) into (B.16), implies that if \( p \) is large enough such that

\[
\left\| \phi_{p+1}^\infty \right\| \|\psi\| \leq \varepsilon < 1,
\]

then

\[
\|h_p\| \leq \frac{3-\varepsilon}{1-\varepsilon} \left\| \phi \right\|^2 \|\psi\|^2 \left\| \phi_{p+1}^\infty(\tau) \right\|.
\]

Thus, if the weights in the norm are \( v(m) = (2^q + m^q) \) we have

\[
\sum_{s=1}^{p} (2^q + s^q) |\phi_{s,p}(\tau) - \phi_s(\tau)| \leq \frac{3-\varepsilon}{1-\varepsilon} \left\| \phi \right\|^2 \|\psi\|^2 \sum_{s=p+1}^{\infty} (2^q + s^q) |\phi_s(\tau)|.
\]

(B.18)

Using Corollary B.1 we have for \( \tau \geq 0 \phi_s(\tau) = \sum_{j=0}^{\infty} \phi_{s+j} \psi_{\tau-j} \) (noting that \( \psi_{\tau-j} = 0 \) for \( \tau - j < 0 \), and the desired result.

\textbf{Corollary B.2} Suppose Assumption \textbf{3.1} holds with \( K = 2 \). Let \( C_f \) be defined as in Lemma A.2. Let \( a_{j,p} \) be the best fitting AR(p) coefficients to the spectral density \( f(\cdot) = |\phi(\omega)|^2 \) and \( a_p(\omega) = 1 - \sum_{j=1}^{p} a_{j,p} \exp(-ij\omega) \). Then we have

\[
\left| |\phi(\omega)|^2 - |a_p(\omega)|^2 \right| \leq \frac{(C_f + 1)^2}{p^K} \|\phi\|_K \|\phi\|_0
\]

PROOF. We use the inequality

\[
\|A\|^2 - |B|^2 \leq |A - B| \cdot ||A| + |B|| \leq |A - B| (2|A| + |A - B|).
\]

Let \( A = \phi(\omega) \) and \( B = a_p(\omega) \). Then by using the above and Lemma A.2, we have

\[
\sup \left| |\phi(\omega)|^2 - |a_p(\omega)|^2 \right| \leq \left[ C_f + 1 \right] \sum_{s=p+1}^{\infty} |\phi_s| \left( 2\|\phi\|_0 + \left[ C_f + 1 \right] \sum_{s=p+1}^{\infty} |\phi_s| \right)
\]

\[
\leq \frac{[C_f + 1][C_f + 3]}{p^K} \|\phi\|_K \|\phi\|_0.
\]

Thus giving the required result. \( \Box \)
C The bias of the different criteria

In this section we derive the approximate bias of the Gaussian, Whittle and the boundary corrected criteria under quite general assumptions on the underlying time series \{X_t\}. We mention that the bias we evaluate will be in the sense of Bartlett (1952) and will be based on the second order expansion of the loss function.

We assume that \{X_t\} is a stationary time series with spectral density \(f\), where \(f\) is bounded away from zero (and bounded above). We fit the model with spectral density \(f_\theta\) to the observed time series. We do not necessarily assume that there exists a \(\theta_0 \in \Theta\) where \(f = f_{\theta_0}\). To avoid the use tensors we focus on the case that \(\theta\) is a univariate parameter. Since we allow for the misspecified case, for a given \(n\), it seems natural that the “ideal” best fitting parameter is

\[
\theta_n = \arg\min_{\theta} I_n(f, f_\theta).
\]

where \(I_n(f, f_\theta)\) is defined in (4.1). Note that in the case the spectral density is not misspecified, then \(\theta_n = \theta_0\) for all \(n\) where \(f = f_{\theta_0}\).

**Assumption C.1**

(i) For all \(\theta \in \Theta\), \(f_\theta\) is bounded away from zero and is also bounded from above.

(ii) The parameter \(\theta\) is not a function of the one-step ahead predictions error \(\sigma^2 = \exp(\int \log f_\theta(\omega) d\omega)\).

(iii) Let \(\phi_j(\theta)\) and \(\psi_j(\theta)\) denote the AR(\(\infty\)) and MA(\(\infty\)) expansion corresponding to the spectral density \(f_\theta\) respectively. Then for all \(\theta \in \Theta\) and for \(1 \leq s \leq 3\) we have

\[
\sum_{j=1}^{\infty} |j^K \nabla_{\theta}^s \phi_j(\theta)| < \infty \quad \sum_{j=1}^{\infty} |j^K \nabla_{\theta}^s \psi_j(\theta)| < \infty,
\]

where \(K > 3/2\).

(iv) \(\{X_t\}\) is a stationary time series with spectral density \(f\). Let \(\kappa_s(\cdot)\) denote the \(s\) order cumulant associated with \(\{X_t\}\). For \(1 \leq s \leq 6\) and \(1 \leq j \leq s\) we assume \(\sum_{t_1,\ldots,t_{s-1}} |(1 + t_j)\kappa_s(t_1,\ldots,t_{s-1})| < \infty\), where \(\kappa_s(t_1,\ldots,t_{s-1})\) denotes the joint cumulant of \(\text{cum}(X_0, X_{t_1}, \ldots, X_{t_{s-1}})\).

In order to derive the limiting bias, we require the following definitions

\[
I(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{d^2 f_\theta(\omega)^{-1}}{d\theta^2} \right) f(\omega) d\omega \quad \text{and} \quad J(g) = \frac{1}{2\pi} \int_0^{2\pi} g(\omega) f(\omega) d\omega.
\]

For real functions \(g, h \in L^2[0, 2\pi]\) we define

\[
V(g, h) = \frac{2}{2\pi} \int_0^{2\pi} g(\omega) h(\omega) f(\omega)^2 d\omega + \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} g(\omega_1) h(\omega_2) f_4(\omega_1, -\omega_1, \omega_2)d\omega_1 d\omega_2,
\]
where \( f_4 \) denotes the fourth order cumulant density of the time series \( \{X_t\} \). Further, we define

\[
B_{G,n}(\theta) = \frac{2}{n\sigma^2} \Re \sum_{t,j=1}^n c(t-j) \frac{1}{n} \sum_{k=1}^n e^{-it\omega_k} \frac{d}{d\theta} \phi(\omega_k; f_\theta) \phi_j^{(\infty)}(\omega_k; f_\theta)
\]

\[
B_{K,n}(\theta) = \frac{1}{n} \sum_{k=1}^n f_n(\omega_k) \frac{d^2f_\theta(\omega_k)^{-1}}{d\theta}
\]

where \( \phi(\omega; \theta) \) and \( \phi_j^{(\infty)}(\omega; \theta) \) are defined in Section 3. The results for the hybrid Whittle likelihood mirror those for the boundary corrected likelihood. Observe that these criteria do not contain the term \( B_{K,n}(\theta_n) + B_{G,n}(\theta_n) \) (which is not equal to zero when the spectral density is misspecified). This may explain why in

\[
\begin{align*}
\mathbb{E}_\theta (\hat{\theta}_n^G - \theta_n) &= I(\theta)^{-1} (B_{K,n}(\theta_n) + B_{G,n}(\theta_n)) + n^{-1}G(\theta_n) + O(n^{-2} + n^{-K+1/2}) \\
\mathbb{E}_\theta (\hat{\theta}_n^K - \theta_n) &= I(\theta)^{-1}B_{K,n}(\theta_n) + n^{-1}G(\theta_n) + O(n^{-2}) \\
\text{and } \mathbb{E}_\theta (\hat{\theta}_n^W - \theta_n) &= n^{-1}G(\theta_n) + O(n^{-2} + n^{-1}p^{-K+1})
\end{align*}
\]

where

\[
G(\theta) = I(\theta)^{-2} \left( \frac{d^2f^{-1}_\theta}{d\theta^2} , \frac{d^3f^{-1}_\theta}{d\theta^3} \right) + 2^{-1}I(\theta)^{-3} \left( \frac{df^{-1}_\theta}{d\theta} , \frac{d^2f^{-1}_\theta}{d\theta^2} \right) \cdot J \left( \frac{d^2f^{-1}_\theta}{d\theta^2} , \frac{d^3f^{-1}_\theta}{d\theta^3} \right)
\]

PROOF. See Supplementary C. \( \square \)

Remark C.1  
(i) In the case that the model is correctly specified and linear, then \( f_4(\omega_1, -\omega_1, \omega_2) = \frac{\partial^4 f(\omega_1)f(\omega_2)}{\partial \omega_1^2 \partial \omega_2^2} \). Straightforward calculations show that in this case, the fourth order cumulant term in \( \mathbb{V} \left( \frac{df^{-1}_\theta}{d\theta}, \frac{d^2f^{-1}_\theta}{d\theta^2} \right) \) and \( \mathbb{V} \left( \frac{df^{-1}_\theta}{d\omega_1}, \frac{d^2f^{-1}_\theta}{d\omega_1^2} \right) \) is zero. This results in the fourth order cumulant term in \( G(\theta) \) being zero too.

(ii) In the case of the Gaussian likelihood, if the spectral density \( f_\theta \) is correctly specified, then \( B_{K,n}(\theta_n) + B_{G,n}(\theta_n) = 0 \).

(iii) The results for the hybrid Whittle likelihood mirror those for the boundary corrected likelihood. Observe that these criteria do not contain the term \( B_{K,n}(\theta_n) + B_{G,n}(\theta_n) \) (which is not equal to zero when the spectral density is misspecified). This may explain why in
the simulations the hybrid Whittle likelihood tends to have a slightly smaller bias than the Gaussian likelihood when the spectral density is misspecified.

C.1 The bias for the AR(1) model

In general, it is difficult to obtain a simple expression for $B_{G,n}(\theta)$ and $I(\theta)$ defined in Theorem C.1, but in the special case for correctly specified AR(1) process, we are able to simplify $B_n(\theta)$ and $I(\theta)$ by

$$B_{G,n}(\theta) = \frac{4\theta}{1 - \theta^2}, \quad I(\theta) = \frac{2}{1 - \theta^2}. \quad (C.3)$$

Therefore, by Theorem C.1, we have

$$\mathbb{E}_\theta \left( \tilde{\theta}_n^G - \hat{\theta}_n^K \right) = \frac{2\theta}{n} + O(n^{-2} + n^{-K+1/2}).$$

PROOF of Equation (C.3)

Note that the characteristic function and spectral density is given by $\phi(\omega) = 1 - \theta e^{-i\omega}$, $f_\theta(\omega) = \sigma^2|\phi(\omega)|^{-2}$ respectively. It is easy to show

$$\phi_\infty^j(\omega) = 0, j \geq 2.$$

Therefore,

$$B_{G,n}(\theta) = -\frac{2}{\sigma^2} \sum_{t,j=1}^n c(t-j) \frac{1}{n} \sum_{k=1}^n e^{-it\omega_k,n} \frac{d}{d\theta} \phi(\omega_k,n;\theta) \phi_j^\infty(\omega_k,n;\theta)$$

$$= -\frac{2}{\sigma^2} \sum_{t=1}^n c(t-1) \frac{1}{n} \sum_{k=1}^n e^{-it\omega_k,n} \frac{d}{d\theta} (1 - \theta e^{i\omega_k,n}) \theta$$

$$= -\frac{2}{\sigma^2} \sum_{t=1}^n c(t-1) \frac{1}{n} \sum_{k=1}^n (e^{-it\omega_k,n} - 2\theta e^{-i(t-1)\omega_k,n}).$$

The second summation (over $k$) is 0 unless $t = 1, n$ and $c(t-1) = 0$ unless $t = 0, 1, 2$. Therefore,

$$B_{G,n}(\theta) = -\frac{2}{\sigma^2} c(0) \frac{1}{n} \sum_{k=1}^n (e^{-i\omega_k,n} - 2\theta e^{-i0\omega_k,n}) = -\frac{2}{\sigma^2} c(0) \frac{1}{n} (-2\theta n) = \frac{4\theta}{1 - \theta^2}.$$ 

To calculate $I(\theta)$, first note that

$$\frac{d^2}{d\theta^2} f_\theta(\omega)^{-1} = \frac{d^2}{d\theta^2} \sigma^{-2}(1 + \theta^2 - 2\theta \cos \omega) = 2\sigma^{-2}.$$
Therefore,

\[ I(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{d^2 f_\theta(\omega)}{d\theta^2} \right) f(\omega) d\omega = \frac{1}{\pi \sigma^2} \int_0^{2\pi} f(\omega) d\omega = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{1 + \theta^2 - 2\theta \cos \omega} d\omega. \]

To calculate the integral above, we parametrize \( z = e^{i\omega} \) and use the Residue theorem that

\[ I(\theta) = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{1 + \theta^2 - 2\theta \cos \omega} d\omega = \frac{2}{1 - \theta^2}. \]

\[ \square \]

### C.2 Bias proofs

We first state three lemmas, that are required to prove Theorem C.1.

**Lemma C.1** Suppose Assumption C.1 holds. Suppose the matrix \( \|A_n\|_1 < \rho_n \) and the entries of \( B_n \) are bounded. Then we have

\[ \left| \text{cov} \left( \sum_{t_1, t_2 = 1}^{n} (A_n)_{t_1, t_2} X_{t_1} X_{t_2}, \sum_{\tau_1, \tau_2 = 1}^{n} (B_n)_{\tau_1, \tau_2} X_{\tau_1} X_{\tau_2} \right) \right| = O(\rho_n). \]

**PROOF.** Expanding the covariance gives

\[
\sum_{t_1, t_2 = 1}^{n} \sum_{\tau_1, \tau_2 = 1}^{n} (A_n)_{t_1, t_2} (B_n)_{\tau_1, \tau_2} \text{cov}_\theta(X_{t_1} X_{t_2}, X_{\tau_1} X_{\tau_2})
= \sum_{t_1, t_2 = 1}^{n} \sum_{\tau_1, \tau_2 = 1}^{n} (A_n)_{t_1, t_2} (B_n)_{\tau_1, \tau_2} \left[ c(t_1 - \tau_1)c(t_2 - \tau_2) + c(t_1 - \tau_2)c(t_2 - \tau_1) + \kappa_4(t_2 - t_1, \tau_1 - t_1, \tau_2 - t_1) \right].
\]

Now by using \( \sum_{r} |c(r)| < \infty \), \( \sum_{t_1, t_2, t_3} |\kappa_4(t_1, t_2, t_3)| < \infty \) and \( \max_{\tau_1, \tau_2} |(B_n)_{\tau_1, \tau_2}| < \infty \) we obtain the result. \[ \square \]

**Lemma C.2** Suppose Assumption C.1 holds and \( g_1, g_2 \) are bounded functions. Define \( I_n(\omega) = |J_n(\omega)|^2 \) be the periodogram. Then

\[
\left| \text{cov} \left[ \left( n^{-1} \sum_{k_1 = 1}^{n} g_1(\omega_{k_1}) I_n(\omega_{k_1, n}) \right)^2, n^{-1} \sum_{k_2 = 1}^{n} g_2(\omega_{k_2}) I_n(\omega_{k_2, n}) \right] \right| = O(n^{-2})
\]

53
Lemma C.3 Suppose Assumption C.1 holds. Let \( \hat{\phi}_n \) as in Lemma A.2. In the lemma below we give a Baxter-type inequality on derivatives of the linear predictors.

\[
\text{PROOF. Expanding the covariance gives}
\]
\[
\text{cov} \left( \left( n^{-1} \sum_{k_1=1}^{n} g_1(\omega_{k_1}) I_n(\omega_{k_1,n}) \right)^2, n^{-1} \sum_{k_2=1}^{n} g_2(\omega_{k_2}) I_n(\omega_{k_2,n}) \right) = \frac{1}{n^3} \sum_{k_1,k_2,k_3=1}^{n} g_1(\omega_{k_1}) g_1(\omega_{k_2}) g_2(\omega_{k_3}) \text{cov} \left( I_n(\omega_{k_1,n}) I_n(\omega_{k_2,n}), I_n(\omega_{k_3,n}) \right).
\]

Now by applying indecomposable partitions to the above covariance we obtain the result. \( \square \)

Next our aim is to obtain a Baxter-type inequality for the derivatives of the linear predictors. The bound requires the following definitions

\[
U_{0,n}(\tau) = \sum_{j=n+1}^{\infty} |\phi_j(\tau)|,
\]

\[
U_{1,n}(\tau) = n^{1/2} f_0^* \left( [C_{f_0,0} f_1^* + C_1] \sum_{j=n+1}^{\infty} |\phi_j(\tau)| + C_0 n^{1/2} \sum_{j=n+1}^{\infty} \left| \frac{d\phi_j(\tau)}{d\theta} \right| \right)
\]

\[
U_{2,n}(\tau) = n^{1/2} f_0^* \left( f_2^* U_{0,n}(\tau) + 2 f_1^* U_{n}(\tau) + \sum_{j=n+1}^{\infty} \left[ \left| \frac{d^2 \phi_j(\tau)}{d\theta^2} \right| C_0 + \left| \frac{d\phi_j(\tau)}{d\theta} \right| C_1 + |\phi_j(\tau)| C_2 \right] \right)
\]

and

\[
U_{3,n}(\tau) = n^{1/2} f_0^* \left( \sum_{i_1,i_2=0}^{3} \left( \frac{3}{i_1} \right) f_1^* U_{i_1,n}(\tau) + \sum_{j=n+1}^{\infty} \sum_{i_1,i_2=0}^{3} \left( \frac{3}{i_1} \right) \left| \frac{d^i \phi_j(\tau)}{d\theta^i} \right| C_{i_2} \right) \quad (C.4)
\]

where \( f_0^* = (\inf_{\omega} f_0(\omega))^{-1} \), \( f_i^* = \max \left| \frac{d^i f_0(\omega)}{d\theta^i} \right| \), \( C_i = \sum_{\tau} \left| \frac{d^i c(\tau)}{d\theta^i} \right| \) for \( 0 \leq i \leq 3 \) and \( C_{f,0} \) is defined as in Lemma A.2. In the lemma below we give a Baxter-type inequality on derivatives of the predictors.

**Lemma C.3** Suppose Assumption C.1 holds. Let \( U_{i,n}(\tau) \) for \( i = 0, 1, 2, 3 \), be defined as in \( (C.4) \). Then for \( 1 \leq i \leq 3 \), we have

\[
\sum_{s=1}^{n} \left| \frac{d^i \phi_{s,n}(\tau)}{d\theta^i} - \frac{d^i \phi_s(\tau)}{d\theta^i} \right| \leq U_{i,n}(\tau) \quad (C.5)
\]

**PROOF** For simplicity we suppose \( \tau \leq 0 \). To prove (C.5), we define the vectors

\[
\tilde{\phi}_n(\tau)' = (\phi_{1,n}(\tau), \ldots, \phi_{n,n}(\tau)) \quad \text{(best linear finite future predictor)}
\]

\[
\phi_n(\tau)' = (\phi_1(\tau), \ldots, \phi_n(\tau)) \quad \text{(truncated best linear infinite future predictor)}
\]

\[
\zeta_{n,\tau}' = (c(\tau - 1), c(\tau - 2), \ldots, c(\tau - n)).
\]

54
We first prove (C.5) for the case $i = 1$. By definition of the best linear predictors given above we have the two expansions

$$
\Gamma_n(f_\theta) \tilde{\phi}_n(\tau) = \zeta_{n,\tau} \quad \text{and} \quad \Gamma_n(f_\theta) \phi_n(\tau) + \sum_{j=n+1}^{\infty} \phi_j(\tau) \zeta_{n,-j+\tau} = \zeta_{n,\tau}.
$$

We now follow the same expansions given in the proof of Theorem 3.2, Meyer et al. (2017) (who proves a Baxter-type inequality for spatial processes). Taking differences of the above gives

$$
\Gamma_n(f_\theta) \left[ \tilde{\phi}_n(\tau) - \phi_n(\tau) \right] = \sum_{j=n+1}^{\infty} \phi_j(\tau) \zeta_{n,-j+\tau}.
$$

As our aim is to bound the derivative of the difference of $\left[ \tilde{\phi}_n(\tau) - \phi_n(\tau) \right]$. Thus we differentiate the above with respect to $\theta$:

$$
\frac{d\Gamma_n(f_\theta)}{d\theta} \left[ \tilde{\phi}_n(\tau) - \phi_n(\tau) \right] + \Gamma_n(f_\theta) \frac{d}{d\theta} \left[ \tilde{\phi}_n(\tau) - \phi_n(\tau) \right] = \sum_{j=n+1}^{\infty} \left[ \frac{d\phi_j(\tau)}{d\theta} \zeta_{n,-j+\tau} + \phi_j(\tau) \frac{d\zeta_{n,-j+\tau}}{d\theta} \right].
$$

(C.6)

Isolating $\frac{d}{d\theta} \left[ \tilde{\phi}_n(\tau) - \phi_n(\tau) \right]$ gives

$$
\frac{d}{d\theta} \left[ \tilde{\phi}_n(\tau) - \phi_n(\tau) \right] = -\Gamma_n(f_\theta)^{-1} \frac{d\Gamma_n(f_\theta)}{d\theta} \left[ \tilde{\phi}_n(\tau) - \phi_n(\tau) \right] + \Gamma_n(f_\theta)^{-1} \sum_{j=n+1}^{\infty} \left[ \frac{d\phi_j(\tau)}{d\theta} \zeta_{n,-j+\tau} + \phi_j(\tau) \frac{d\zeta_{n,-j+\tau}}{d\theta} \right].
$$

Evaluating the $\ell_1$ norm of the above (and using that $\|A_\infty\|_2 \leq n^{1/2}\|A_\infty\|_1 \leq n^{1/2}\|A\|_{\text{spec}} \|x\|_1$) gives the bound

$$
\sum_{s=1}^{n} \left| \frac{d\phi_{s,n}(\tau)}{d\theta} - \frac{d\phi_{s}(\tau)}{d\theta} \right| 
\leq n^{1/2}\|\Gamma_n(f_\theta)^{-1}\|_{\text{spec}} \left( \left\| \frac{d\Gamma_n(f_\theta)}{d\theta} \right\|_{\text{spec}} \sum_{s=1}^{n} |\phi_{s,n}(\tau) - \phi_{s}(\tau)| + \sum_{j=n+1}^{\infty} \left| \frac{d\phi_j(\tau)}{d\theta} \right| \|\zeta_{n,-j+\tau}\|_1 + 
\sum_{j=n+1}^{\infty} \left| \frac{d\zeta_{n,-j+\tau}}{d\theta} \right|_1 \right).
$$

By using the same arguments used to show that $\|\Gamma_n(f_\theta)^{-1}\|_{\text{spec}} \leq (\inf_{\omega} f(\omega))^{-1}$ we can show that
\[ \left\| \frac{d^2 \Gamma_n(f_\theta)}{d\theta^2} \right\|_1 \leq n^{1/2} f_0^* \left( \left\| f_1^* \right\|_1 \sum_{j=n+1}^{\infty} |\phi_j(\tau)| + C_0 \sum_{j=n+1}^{\infty} \left| \frac{d\phi_j(\tau)}{d\theta} \right| \right) + \sum_{j=n+1}^{\infty} \left[ \left| \frac{d^2 \phi_j(\tau)}{d\theta^2} \right| C_0 + 2 \left| \frac{d\phi_j(\tau)}{d\theta} \right| C_1 + |\phi_j(\tau)| C_2 \right] \].

Setting \( U_\tau = n^{1/2} f_0^* (|C_f f_1^* + C_1| \sum_{j=n+1}^{\infty} |\phi_j(\tau)| + C_0 \sum_{j=n+1}^{\infty} \left| \frac{d\phi_j(\tau)}{d\theta} \right|) \), the above gives

\[ \left\| \frac{d^2 \Gamma_n(f_\theta)}{d\theta^2} \left[ \tilde{\phi}_n(\tau) - \phi_n(\tau) \right] \right\|_1 \leq n^{1/2} f_0^* \left( f_2^* \sum_{s=n+1}^{\infty} |\phi_s(\tau)| + 2f_1^* \left\| \frac{d\tilde{\phi}_n(\tau) - \phi_n(\tau)}{d\theta} \right\|_1 \right) + \sum_{j=n+1}^{\infty} \left[ \left| \frac{d^2 \phi_j(\tau)}{d\theta^2} \right| C_0 + 2 \left| \frac{d\phi_j(\tau)}{d\theta} \right| C_1 + |\phi_j(\tau)| C_2 \right]. \]
and Gaussian likelihoods i.e. a result similar to Theorem 3.1 but applied to
\[
\left[ F_n^*(g) \frac{d^i}{d\theta^i} (D_n(f_\theta) - D_{\infty,n}(f_\theta)) \right],
\]
where \(g\) is a bounded function.

**Lemma C.4** Suppose Assumptions 3.1 and C.1 hold. Suppose \(g\) is bounded. Then
\[
\left\| \left[ F_n^*(g) \frac{d^i}{d\theta^i} (D_n(f_\theta) - D_{\infty,n}(f_\theta)) \right] \right\|_1 = O(n^{-K+3/2}), \quad (C.10)
\]
for \(i = 1, 2 \text{ and } 3\).

To prove the result we first note that \(\phi_j(\tau; \theta) = \sum_{s=0}^{\infty} \phi_{s+j}(\theta)\theta^{s-j}(\theta)\). Using this expansion we have
\[
\sum_{j=n+1}^{\infty} \left| \frac{d\phi_j(\tau)}{d\theta} \right| \leq \sum_{s=0}^{\infty} \sum_{j=n+1}^{\infty} \left| \frac{d\phi_{s+j}}{d\theta} \theta^{s-j} + \phi_{s+j} \frac{d\theta^{s-j}}{d\theta} \right|.
\]
Substituting the above bound into \(U_{1,n}\) gives
\[
\sum_{s=1}^{n} \left| \frac{d\phi_{s,n}(\tau)}{d\theta} - \frac{d\phi_{s}(\tau)}{d\theta} \right| \leq n^{1/2} f_0^* \left( [C_1f_1^* + C_1] \sum_{j=n+1}^{\infty} |\phi_j(\tau)| + C_0 \sum_{s=0}^{\infty} \sum_{j=n+1}^{\infty} \left| \frac{d\phi_{s+j}}{d\theta} \theta^{s-j} + \phi_{s+j} \frac{d\theta^{s-j}}{d\theta} \right| \right). \quad (C.11)
\]
Using the expansion for \(F_n^*\Delta_n(g)(D_n(f_\theta) - D_{\infty,n}(f_\theta))\) given in (A.12) we have
\[
F_n^*\Delta_n(g)(D_n(f_\theta) - D_{\infty,n}(f_\theta)) = \sum_{\tau \leq 0} [\phi_t(\tau; f_\theta)G_{1,n}(s, \tau; g) + \phi_{n+1-t}(\tau; f_\theta)G_{2,n}(s, \tau; g)].
\]
To simplify notation we remove the dependence of \(G_{1,n}\) and \(G_{2,n}\) on \(g\). Differentiating the above wrt \(\theta\) gives
\[
\left[ F_n^*\Delta_n(f_\theta^{-1}) \frac{d}{d\theta} (D_n(f_\theta) - D_{\infty,n}(f_\theta)) \right]_{s,t}\]
\[
= \sum_{\tau \leq 0} \left[ G_{1,n}(s, \tau) \frac{d}{d\theta} [\phi_{t,n}(\tau) - \phi_{t}(\tau)] + G_{2,n}(s, \tau) \frac{d}{d\theta} [\phi_{n+1-t}(\tau) - \phi_{n+1-t}(\tau)] \right].
\]
Therefore, substituting (C.11) into the above and using the same techniques used to prove Theorem 3.1 proves (C.10) for the case \(i = 1\). The proof for the cases \(i = 2\) and \(i = 3\) is similar (and follows from Lemma C.3). \(\square\)
Using Lemmas C.1, C.2 and C.4 we prove Theorem C.1.

We derive the bias of the following three criterions (we ignore the log term in the estimation, thanks to Assumption C.1(ii))

\[
\begin{align*}
\mathcal{L}_n(\theta) &= n^{-1}X_n'\Gamma_n(f_\theta)^{-1}X_n = \frac{1}{n} \sum_{k=1}^{n} \frac{J_n(\omega_{k,n})\tilde{J}_n(\omega_{k,n};f_\theta)}{f_\theta(\omega_{k,n})} \\
K_n(\theta) &= \frac{1}{n} \sum_{k=1}^{n} \frac{|J_n(\omega_{k,n})|^2}{f_\theta(\omega_{k,n})} \\
W_{p,n}(\theta) &= \frac{1}{n} \sum_{k=1}^{n} \frac{J_n(\omega_{k,n})\tilde{J}_n(\omega_{k,n};f_p)}{f_\theta(\omega_{k,n})}
\end{align*}
\]

**PROOF of Theorem C.1** Let $L_n(\cdot)$ denote the general minimisation criterion (it can be $\mathcal{L}_n(\theta)$, $K_n(\theta)$ or $W_{p,n}(\theta)$) and $\hat{\theta} = \arg\min L_n(\theta)$. The first order expansion of the estimator gives

\[
(\hat{\theta} - \theta) \approx U(\theta) - \frac{dL_n(\theta)}{d\theta},
\]

where $U(\theta) = -\mathbb{E}\left(\frac{d^2L_n}{d\theta^2}\right)$. To obtain the “bias” we make a second order expansion

\[
\frac{dL_n(\theta)}{d\theta} + (\hat{\theta} - \theta)\frac{d^2L_n(\theta)}{d\theta^2} + \frac{1}{2}(\hat{\theta} - \theta)^2\frac{d^3L_n(\theta)}{d\theta^3} \approx 0.
\]

Taking expectation of the above and using (C.12) (this is the Bartlett correction described in Bartlett (1952) and Cox and Snell (1968)) gives

\[
\mathbb{E}\left(\frac{dL_n(\theta)}{d\theta}\right) - \mathbb{E}(\hat{\theta} - \theta)U(\theta) + U(\theta)^{-1}\text{cov}\left(\frac{dL_n(\theta)}{d\theta}, \frac{d^2L_n(\theta)}{d\theta^2}\right) \\
+ 2^{-1}U(\theta)^{-2}\mathbb{E}\left(\frac{dL_n(\theta)}{d\theta}\right)^2 \mathbb{E}\left(\frac{d^3L_n(\theta)}{d\theta^3}\right) + 2^{-1}U(\theta)^{-2}\text{cov}\left(\left(\frac{dL_n(\theta)}{d\theta}\right)^2, \frac{d^3L_n(\theta)}{d\theta^3}\right) \approx 0.
\]
Therefore, solving for $\mathbb{E}(\hat{\theta} - \theta)$ gives

$$\mathbb{E}(\hat{\theta} - \theta) = U(\theta)^{-1}\mathbb{E} \left( \frac{dL_n(\theta)}{d\theta} \right) + U(\theta)^{-2} \text{cov} \left( \frac{dL_n(\theta)}{d\theta}, \frac{d^2 L_n(\theta)}{d\theta^2} \right)$$

$$+ 2^{-1} U(\theta)^{-3} \mathbb{E} \left( \frac{dL_n(\theta)}{d\theta} \right)^2 \mathbb{E} \left( \frac{d^3 L_n(\theta)}{d\theta^3} \right) + 2^{-1} U(\theta)^{-3} \text{cov} \left( \left( \frac{dL_n(\theta)}{d\theta} \right)^2, \frac{d^3 L_n(\theta)}{d\theta^3} \right)$$

$$= U(\theta)^{-1} \mathbb{E} \left( \frac{dL_n(\theta)}{d\theta} \right) + U(\theta)^{-2} \text{cov} \left( \frac{dL_n(\theta)}{d\theta}, \frac{d^2 L_n(\theta)}{d\theta^2} \right)$$

$$+ 2^{-1} U(\theta)^{-3} \left[ \text{var} \left( \frac{dL_n(\theta)}{d\theta} \right) + \left\{ \mathbb{E} \left[ \frac{dL_n(\theta)}{d\theta} \right] \right\}^2 \right] \mathbb{E} \left( \frac{d^3 L_n(\theta)}{d\theta^3} \right)$$

$$+ 2^{-1} U(\theta)^{-3} \text{cov} \left( \left( \frac{dL_n(\theta)}{d\theta} \right)^2, \frac{d^3 L_n(\theta)}{d\theta^3} \right).$$

Thus

$$\mathbb{E}(\hat{\theta} - \theta) = U(\theta)^{-1} \mathbb{E} \left( \frac{dL_n(\theta)}{d\theta} \right) + 2^{-1} U(\theta)^{-3} \left\{ \mathbb{E} \left( \frac{dL_n(\theta)}{d\theta} \right) \right\}^2 \mathbb{E} \left( \frac{d^3 L_n(\theta)}{d\theta^3} \right)$$

$$+ U(\theta)^{-2} \text{cov} \left( \frac{dL_n(\theta)}{d\theta}, \frac{d^2 L_n(\theta)}{d\theta^2} \right) + 2^{-1} U(\theta)^{-3} \text{var} \left( \frac{dL_n(\theta)}{d\theta} \right) \mathbb{E} \left( \frac{d^3 L_n(\theta)}{d\theta^3} \right)$$

$$+ 2^{-1} U(\theta)^{-3} \text{cov} \left( \left( \frac{dL_n(\theta)}{d\theta} \right)^2, \frac{d^3 L_n(\theta)}{d\theta^3} \right).$$

Note that the terms which contain $\mathbb{E} \left( \frac{dL_n(\theta)}{d\theta} \right)$ will differ for all the three different quasi-likelihoods. The remaining terms, are asymptotically the same for the three quasi-likelihoods. Thus we first obtain expressions for $\mathbb{E} \left( \frac{dL_n(\theta)}{d\theta} \right)$ for the three quasi-likelihoods

$$\mathbb{E} \left( \frac{dK_n(\theta)}{d\theta} \right) = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left[ \left| J_n(\omega_{k,n}) \right|^2 \right] \frac{d}{d\theta} f_\theta(\omega_{k,n})^{-1}$$

$$\mathbb{E} \left( \frac{dL_n(\theta)}{d\theta} \right) = \mathbb{E} \left( \frac{dK_n(\theta)}{d\theta} \right) + \frac{1}{n} \mathbb{E} \left[ X_n F_n^* \left\{ \Delta_n \left( \frac{df_\theta}{d\theta} \right) D_{\infty,n}(f_\theta) + \Delta_n \left( f_\theta^{-1} \right) \frac{dD_{\infty,n}(f_\theta)}{d\theta} \right\} \right] X_n$$

$$+ \mathbb{E} \left[ R_{1,n}(\theta) \right]$$

$$\mathbb{E} \left( \frac{dW_{p,n}(\theta)}{d\theta} \right) = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left( J_n(\omega_{k,n}) \tilde{J}_n(\omega_{k,n}; f_p) \right) \frac{d}{d\theta} f_\theta(\omega_{k,n})^{-1}$$

$$= \frac{1}{n} \sum_{k=1}^{n} f(\omega_{k,n}) \frac{d}{d\theta} f_\theta(\omega_{k,n})^{-1} + O((np^{K-1})^{-1})$$
where
\[
R_{i,n}(\theta) = \frac{1}{n} X_n' F_n^* \sum_{\ell=0}^i \binom{i}{\ell} d^\ell \Delta_n(f_\theta) \frac{d^{\ell-i}}{d\theta^{\ell-i}} (D_n(f_\theta) - D_{\infty,n}(f_\theta)) X_n.
\]

Let \( c(r) = \text{cov}(X_0, X_r) \) and \( f_n(\omega) = \int F_n(\omega - \lambda)f(\lambda)d\lambda \) and \( F_n \) is the Fejér kernel of order \( n \). Then by using (3.2) we can write the above as
\[
\mathbb{E} \left( \frac{dK_n(\theta)}{d\theta} \right) = \frac{1}{n} \sum_{k=1}^n f_n(\omega_{k,n}) \frac{d}{d\theta} f_\theta(\omega_{k,n})^{-1} = B_{K,n}(\theta)
\]
\[
\mathbb{E} \left( \frac{dL_n(\theta)}{d\theta} \right) = B_{K,n}(\theta) + 2\sigma^{-2} \mathbb{R} \sum_{t,j=1}^n c(t - j) \frac{1}{n} \sum_{k=1}^n e^{-it\omega_{k,n}} \frac{d}{d\theta} \phi(\omega_{k,n}; f_\theta) \phi_j(\omega_{k,n}; f_\theta) + \mathbb{E}[R_{1,n}(\theta)]
\]
\[
= B_{K,n}(\theta) + B_{G,n}(\theta) + O(n^{-K+1/2})
\]
\[
\mathbb{E} \left( \frac{dW_{p,n}(\theta)}{d\theta} \right) = \frac{1}{n} \sum_{k=1}^n f(\omega_{k,n}) \frac{d}{d\theta} f_\theta(\omega_{k,n})^{-1} + O((np^{K-1})^{-1}).
\]

where the the bound for \( \mathbb{E}[R_{1,n}(\theta)] \) follows from Lemma C.4 and the bound for \( \mathbb{E} \left( \frac{dW_{p,n}(\theta)}{d\theta} \right) \) follows from Lemma 4.1.

Evaluating the above at the best fitting parameter \( \theta_n \) and by Assumption C.1(ii) gives
\[
\mathbb{E} \left( \frac{dK_n(\theta)}{d\theta} \right) \bigg|_{\theta=\theta_n} = B_{K,n}(\theta_n)
\]
\[
\mathbb{E} \left( \frac{dL_n(\theta)}{d\theta} \right) \bigg|_{\theta=\theta_n} = B_{K,n}(\theta_n) + B_{G,n}(\theta_n) + O(n^{-K+1/2})
\]
\[
\mathbb{E} \left( \frac{dW_{p,n}(\theta)}{d\theta} \right) \bigg|_{\theta=\theta_n} = O((np^{K-1})^{-1}). \quad (C.14)
\]

It can be shown that \( B_{K,n}(\theta_n) = O(n^{-1}) \) and \( B_{G,n}(\theta_n) = O(n^{-1}) \). Though these terms could be negative or positive so there is no clear cut answer if \( B_{K,n}(\theta_n) \) is larger or \( B_{K,n}(\theta_n) + B_{G,n}(\theta_n) \) is larger (our simulations results suggest that often \( B_{K,n}(\theta_n) \) tends to be larger).

We now consider the remaining terms in the expansion. It is easily shown that
\[
\frac{d^r L_n(\theta)}{d\theta^r} = \frac{d^r W_n(\theta)}{d\theta^r} + \frac{1}{n} X_n' F_n^* \sum_{\ell=0}^i \binom{i}{\ell} d^\ell \Delta_n(f_\theta) \frac{d^{\ell-i}}{d\theta^{\ell-i}} D_{\infty,n}(f_\theta) X_n + R_{i,n}(\theta)
\]
\[
\quad = O_p(n^{-1})
\]

where
\[
R_{i,n}(\theta) = \frac{1}{n} X_n' F_n^* \sum_{\ell=0}^i \binom{i}{\ell} d^\ell \Delta_n(f_\theta) \frac{d^{\ell-i}}{d\theta^{\ell-i}} (D_n(f_\theta) - D_{\infty,n}(f_\theta)) X_n.
\]
and by using Lemma [C.4] we have

\[ (E|R_{i,n}(\theta)|^2)^{1/2} \leq Cn^{-K+1/2} \tag{C.15} \]

for \( 1 \leq i \leq 3 \). Therefore if \( K > 3/2 \), the remainder \( R_{i,n}(\theta) \) will be of order less than \( O(n^{-1}) \), which we show below is negligible with respect to the other terms. Using that

\[
\mathcal{L}_n(\theta) = K_n(\theta) + \frac{1}{n} \sum_{k=1}^{n} \frac{\tilde{J}_n(\omega_{k,n})\bar{J}_n(\omega_{k,n}, f_{\theta})}{f_{\theta}(\omega_{k,n})} = O_p(n^{-1})
\]

\[
W_{p,n}(\theta) = K_n(\theta) + \frac{1}{n} \sum_{k=1}^{n} \frac{\tilde{J}_n(\omega_{k,n})\bar{J}_n(\omega_{k,n}, f_p)}{f_{\theta}(\omega_{k,n})} \quad O_p(n^{-1})
\]

we obtain the following expansions

\[
E \left( \frac{d^2 \mathcal{L}_n(\theta)}{d\theta^2} \right), E \left( \frac{d^2 W_{p,n}(\theta)}{d\theta^2} \right) = E \left( \frac{d^2 K_n(\theta)}{d\theta^2} \right) + O(n^{-1})
\]

\[
\text{cov} \left( \frac{d\mathcal{L}_n(\theta)}{d\theta}, \frac{d^2 \mathcal{L}_n(\theta)}{d\theta^2} \right), \text{cov} \left( \frac{dW_{p,n}(\theta)}{d\theta}, \frac{d^2 W_{p,n}(\theta)}{d\theta^2} \right) = \text{cov} \left( \frac{dK_n(\theta)}{d\theta}, \frac{d^2 K_n(\theta)}{d\theta^2} \right) + O(n^{-2})
\]

\[
\text{var} \left( \frac{d\mathcal{L}_n(\theta)}{d\theta} \right), \text{var} \left( \frac{dW_{p,n}(\theta)}{d\theta} \right) = \text{var} \left( \frac{dK_n(\theta)}{d\theta} \right) + O(n^{-2})
\]

\[
E \left( \frac{d^3 \mathcal{L}_n(\theta)}{d\theta^3} \right), E \left( \frac{d^3 W_{p,n}(\theta)}{d\theta^3} \right) = E \left( \frac{d^3 K_n(\theta)}{d\theta^3} \right) + O(n^{-1})
\]

\[
\text{cov} \left( \left( \frac{d\mathcal{L}_n(\theta)}{d\theta} \right)^2, \frac{d^3 \mathcal{L}_n(\theta)}{d\theta^3} \right), \text{cov} \left( \left( \frac{dW_{p,n}(\theta)}{d\theta} \right)^2, \frac{d^3 W_{p,n}(\theta)}{d\theta^3} \right) = \text{cov} \left( \left( \frac{dK_n(\theta)}{d\theta} \right)^2, \frac{d^3 K_n(\theta)}{d\theta^3} \right) + O(n^{-2})
\]

\[+ O(n^{-3}) = O(n^{-2}). \]

Next we focus on \( K_n(\theta) \). Using straightforward arguments it can be shown that

\[
\text{cov} \left( \frac{dK_n(\theta)}{d\theta}, \frac{d^2 K_n(\theta)}{d\theta^2} \right) = n^{-1} T \left( \frac{df_{\theta}^{-1}}{d\theta}, \frac{d^2 f_{\theta}^{-1}}{d\theta^2} \right) + O(n^{-2})
\]

\[
\text{var} \left( \frac{dK_n(\theta)}{d\theta} \right) = n^{-1} T \left( \frac{df_{\theta}^{-1}}{d\theta}, \frac{df_{\theta}^{-1}}{d\theta} \right) + O(n^{-2}).
\]
Substituting the above into (C.13) (and ignoring the first term) gives

\[
2^{-1}U(\theta)^{-3} \left\{ \mathbb{E}\left( \frac{dL_n(\theta)}{d\theta} \right)^2 \mathbb{E}\left( \frac{d^3 L_n(\theta)}{d\theta^3} \right) + U(\theta)^{-2} \text{cov}\left( \frac{dL_n(\theta)}{d\theta}, \frac{d^2 L_n(\theta)}{d\theta^2} \right) \right. \\
+ 2^{-1}U(\theta)^{-3} \text{var}\left( \frac{dL_n(\theta)}{d\theta} \right) \mathbb{E}\left( \frac{d^3 L_n(\theta)}{d\theta^3} \right) + 2^{-1}U(\theta)^{-3} \text{cov}\left( \left( \frac{dL_n(\theta)}{d\theta} \right)^2, \frac{d^3 L_n(\theta)}{d\theta^3} \right) \\
= n^{-1}I(\theta)^{-2}V\left( \frac{df_\theta^{-1}}{d\theta}, \frac{d^2 f_\theta^{-1}}{d\theta^2} \right) + 2^{-1}n^{-1}I(\theta)^{-3}V\left( \frac{df_\theta^{-1}}{d\theta}, \frac{df_\theta^{-1}}{d\theta} \right) J\left( \frac{df_\theta^{-1}}{d\theta} \right) + O(n^{-2}) \\
= n^{-1}G(\theta) + O(n^{-2}).
\]

Finally by substituting (C.14) and the above into (C.13) we obtain the result. □