A very short note on B-splines

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The present note clarifies some of the underlying facts which are used in the calculation of the basis functions of B-spline using `R`. Suppose we want to construct the basis functions for the cubic B-spline for a given value of x, a set of inner knot points, and boundary knot points. Let the inner knot points be `c(-0.5, 0, 0.5)` and the boundary knot points be `c(-4, 4)` , then the command in `R` to generate the spline basis functions is

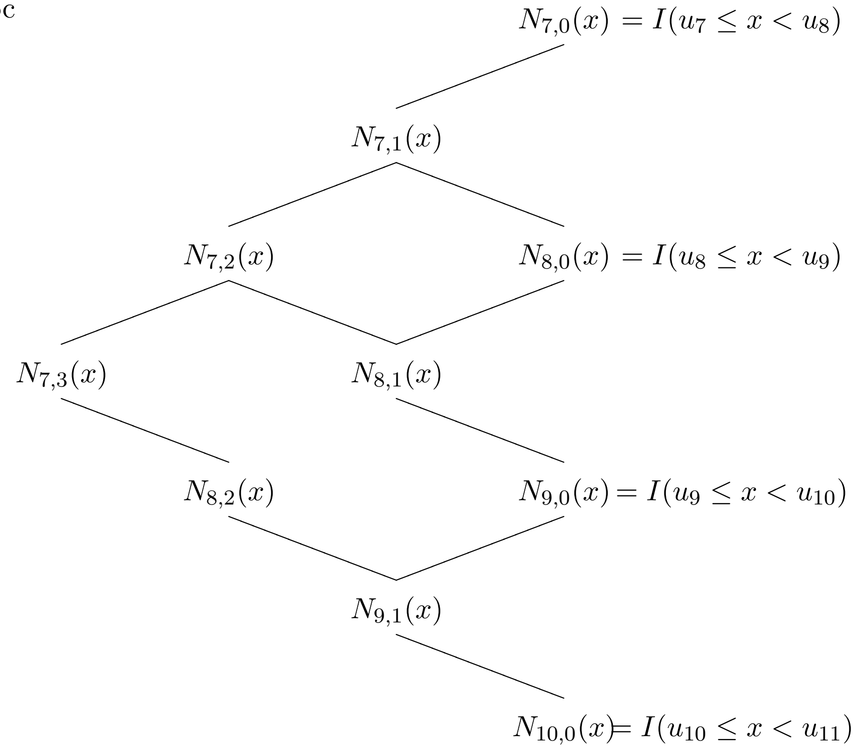
```
>library(splines)
>x=rnorm(1, 0, 1)
> x
[1] -0.2355063
>bs(x, knots=c(-0.5, 0, 0.5), Boundary.knots=c(-4, 4), degree=3, intercept=T)
```

The above command will produce 7 basis functions. Note the set of all knot points are `c(-4, -4, -4, -4, -0.5, 0, 0.5, 4, 4, 4, 4)` and the number of basis functions is $m - n - 1$, where m is the total number of knot points which is 11 in this example, and n is the degree, and for this example it is 3. Note that the knots points are ordered, and in `R` the entire set of knots are obtained by adding $(n + 1)$ lower boundary knot and $(n + 1)$ upper boundary knot with the inner knot points. Let the cubic spline basis functions be $N_{7,3}(x)$, $N_{6,3}(x), \cdots, N_{1,3}(x)$, where the second subscript denotes the degree of the splines. Each of the basis can be constructed through the following recursive formula. Thus to construct the basis functions of degree 3 one needs to compute all the basis functions of degree lower than 3. The recursive formula is given below.

$$\begin{aligned}
N_{i,0}(x) &= I(u_i \leq x < u_{i+1}) \\
N_{i,p}(x) &= \frac{x - u_i}{u_{i+p} - u_i} N_{i,p-1}(x) + \frac{u_{i+p+1} - x}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(x)
\end{aligned}$$

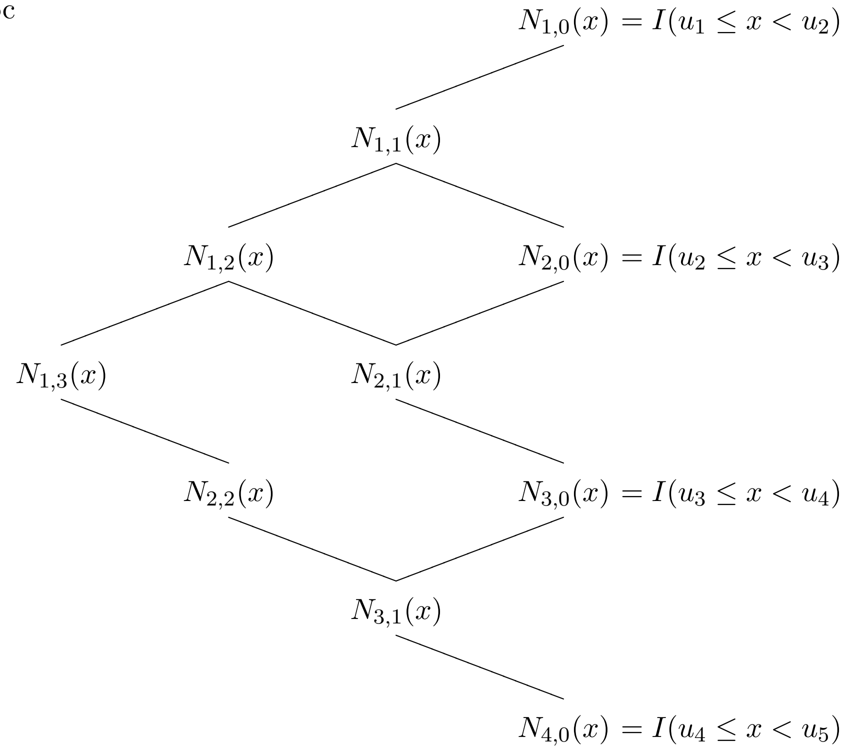
where u_i's are the ordered knot points, and the degree of the spline, p, will take values 1, 2, and 3. For the above computation we define 0/0 as 0. The following figure shows the necessary splines needed to compute before we get $N_{7,3}(x)$.

pc



Similarly to obtain the value of $N_{1,3}(x)$ one has to compute the basis which come across in the following figure.

pc



Note that when we write

```
>bs(x, knots=c(-0.5, 0, 0.5), Boundary.knots=c(-4, 4), degree=3, intercept=F)
```

then `R` will return 6 basis functions $N_{7,3}(x)$, $N_{6,3}(x), \cdots, N_{2,3}(x)$. Following is a simple `R` code to generate basis function for given inner knots and the boundary knots.

```
newbs=function(x, degree, inner.knots, Boundary.knots) {
 Boundary.knots=sort(Boundary.knots);
knots=c(rep(Boundary.knots[1], (degree+1)), sort(inner.knots),
rep(Boundary.knots[2], (degree+1)));

np=degree+length(inner.knots)+1
 s=rep(0, np)
if(x==Boundary.knots[2]) {s[np]=1} else {for( i in 1: np)
s[i]=basis(x, degree, i, knots)}

 return(s)}

 basis=function(x, degree, i, knots)
{ if(degree==0){ if((x<knots[i+1])&(x>=knots[i])) y=1 else
y=0}else{
 if((knots[degree+i]-knots[i])==0) {temp1=0} else {temp1=
 (x-knots[i])/(knots[degree+i]-knots[i])};
 if((knots[i+degree+1]-knots[i+1])==0) {temp2=0} else {temp2=
 (knots[i+degree+1]-x)/(knots[i+degree+1]-knots[i+1])}
  y= temp1*basis(x, (degree-1), i, knots) +temp2*basis(x, (degree-1),
 (i+1), knots)}
return(y)}
>  newbs(2, degree=3, inner.knots=c(-0.25, -0.5, 0, 0.25, 0.5),
+ Boundary.knots=c(-4, 4))
 [1] 0.0000000 0.0000000 0.0000000
0.0000000 0.0000000 0.1523810 0.4252154 0.3436864 0.0787172
>
```

Following is the Fortran code to generate B-spline basis function.

```
         subroutine splinebasis(d, n, m, m1, k, x, innerknots,
     *   boundaryknots, basis)
 C        This subroutine generates Bspline basis functions.
 C        x(n) is a n by 1 input vector for which B-spline basis
 C        function will be evaluated.
 C        innerknots(m1) set of m1 innerknot points.
 C        newknots is the entire set of knots, of length m=m1+2(d+1)
 C        where d is the degree of the splines.
 C        k=number of spline basis=m1+d+1
         IMPLICIT NONE
         integer*4 d, k, m,  m1, n
         double precision x(n), innerknots(m1), boundaryknots(2)
         double precision newknots(m), basis(n, k), result
         external b
         integer*4 i1, i, j
         do i1=1, (d+1)
          newknots(i1)=boundaryknots(1)
         end do
         do i1=(d+2), (m1+d+1)
          newknots(i1)=innerknots(i1-d-1)
         end do
         do i1=(m1+d+2), m
```

```
          newknots(i1)=boundaryknots(2)
         end do
         do i=1, n
          if(x(i).eq.boundaryknots(2)) then
           basis(i, k)=1.d0
           do j=1, (k-1)
            basis(i, j)=0.d0
           end do
          else
           do j=1, k
            call b(m, j, (d+1), x(i), newknots, result, b)
            basis(i, j)=result
           end do
          endif
         end do
         return
         end
C ----------------
         subroutine b(i1, i2, i3, y, newknots, result, dumsub)
 C        This subroutine calculates i2 th basis of spline of
 C        degree (i3-1).
         IMPLICIT NONE
         integer*4 i1, i2, i3
         double precision y, newknots(i1), temp1, temp2, result,
     *   result1, result2
         external dumsub
         if(i3.eq.1) then
          if((y.ge.newknots(i2)).and.(y.lt.newknots(i2+1))) then
           result=1.d0
          else
           result=0.d0
          endif
         else
          call dumsub(i1, i2, (i3-1), y, newknots, result1, dumsub)
          temp1=(y-newknots(i2))*result1/(newknots(i2+i3-1)-
     *    newknots(i2))
          if(temp1.ne.temp1) temp1=0.d0
          call dumsub(i1, (i2+1), (i3-1), y, newknots, result2, dumsub)
          temp2=(newknots(i2+i3)-y)*result2/(
     *    newknots(i2+i3)-newknots(i2+1))
          if(temp2.ne.temp2) temp2=0.d0
          result=temp1+temp2
         endif
         return
         end
```

If one wants to call this subroutine from R following is an example code. We assume that the above Fortran subroutines were saved in a file named “spline.f”.

```
 >        dyn.load("spline.so")
 >         n=10;
 >         m1=3;
 >         d=3;
 >         m=m1+2*(d+1);
```

```
>          knots=c(-0.25, 0.0, 0.25)
>          boundaryknots=c(-3, 3)
>          x=rnorm(n)
>          k=d+m1+1;
>
>          basis=matrix(0, nrow=n, ncol=k);
>          storage.mode(basis)<-"double"
>          f1=.Fortran("splinebasis", d=as.integer(d), n=as.integer(n),
+          m=as.integer(m), m1=as.integer(m1), +
+          k=as.integer(k),x=as.double(x), knots=as.double(knots),
+          boundaryknots=as.double(boundaryknots), output=basis)
>
>
```