Statistics & Probability Letters 43 (1999) 123–130

Applications of a formula for the variance function of a stochastic process

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Received July 1997; received in revised form January 1999

Abstract

This paper uses Itô's formula to obtain a representation of the variance function of a class of stochastic processes having right continuous paths with left limits. The representation allows one to generalize recent results of Ball and Faddy concerning over- and under-dispersion of pure birth processes. An application to a cumulative damage model in reliability illustrates the generalization. For many well-known jump and diffusion processes, the representation yields an ordinary differential equation that can be explicitly solved for the variance function. © 1999 Elsevier Science B.V. All rights reserved

MSC: primary 60J25

Keywords: Compensator; Cumulative damage model; Diffusion process; Itô's formula; Martingale; Over- and under-dispersion; Point process

1. Introduction

In a homogeneous Poisson process N_t with constant intensity λ, the mean $E(N_t)$ and variance $\mathrm{Var}(N_t)$ both equal λt for all $t \geqslant 0$. Furthermore, the successive interarrival times $T_1, T_2, \ldots$ are independent and exponentially distributed with common mean λ^{-1}. Faddy (1994) conjectured that if instead the T_i are independent and exponentially distributed with different means λ_i^{-1} satisfying $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ or $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$, then the process N_t is over-dispersed in the sense that $\mathrm{Var}(N_t) \geqslant E(N_t)$ or under-dispersed in the sense that $\mathrm{Var}(N_t) \leqslant E(N_t)$, respectively. Using conventional arguments that do not easily generalize, Ball (1995) was able to establish Faddy's conjecture.

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[1] Research supported in part by USPHS grant GM53275.

[2] Supported by 1996 BGSU Faculty Research Committee basic grant.

PII: S0167-7152(98)00198-9

In seeking to extend the Faddy–Ball result, it is helpful to think first in terms of the infinitesimal conditional means and variances of a stochastic process X_t. These are defined by

$$E(\Delta X_t|X_t = x) = \mu(t,x)\Delta t + o(\Delta t)$$
$$\mathrm{Var}(\Delta X_t|X_t = x) = \sigma^2(t,x)\Delta t + o(\Delta t) \tag{1}$$

for Δt a small positive time increment and $\Delta X_t = X_{t+\Delta t} - X_t$. In the case of a pure birth process, $\mu(t, x) = \sigma^2(t, x) = \lambda(x)$. Taking expectations in the first expression of (1) and cumulating the results up to time t suggests the known integral equation

$$E(X_t) = E(X_0) + \int_0^t E[\mu(s,X_s)]\,\mathrm{d}s \tag{2}$$

for the mean $E(X_t)$. A similar but more complicated integral equation can be derived heuristically for the variance by noting that

$$\begin{aligned}\mathrm{Var}(X_{t+\Delta t}) &= E[\mathrm{Var}(X_t + \Delta X_t \mid X_t)] + \mathrm{Var}[E(X_t + \Delta X_t \mid X_t)]\\ &= E[\sigma^2(t,X_t)\Delta t + o(\Delta t)] + \mathrm{Var}[X_t + \mu(t,X_t)\Delta t + o(\Delta t)]\\ &= E[\sigma^2(t,X_t)]\Delta t + \mathrm{Var}(X_t) + 2\mathrm{Cov}[X_t, \mu(t,X_t)]\Delta t + o(\Delta t).\end{aligned}$$

Cumulating these results up to time t suggests that

$$\mathrm{Var}(X_t) = \mathrm{Var}(X_0) + \int_0^t E[\sigma^2(s,X_s)]\,\mathrm{d}s + 2\int_0^t \mathrm{Cov}[X_s, \mu(s,X_s)]\,\mathrm{d}s. \tag{3}$$

In the Faddy–Ball example with $X_t = N_t$, substitution in the mean and variance formulas (2) and (3) when $X_0 = 0$ yields

$$\begin{aligned}\mathrm{Var}(X_t) &= \int_0^t E[\lambda(X_s)]\,\mathrm{d}s + 2\int_0^t \mathrm{Cov}[X_s, \lambda(X_s)]\,\mathrm{d}s\\ &= E(X_t) + 2\int_0^t \mathrm{Cov}[X_s, \lambda(X_s)]\,\mathrm{d}s.\end{aligned}$$

Because $\lambda(x)$ is a monotone function of x, the Faddy–Ball inequalities now follow from the fact that $\mathrm{Cov}[f(Y), g(Y)] \geqslant 0$ for any random variable Y and increasing functions $f(y)$ and $g(y)$ (Barlow and Proschan, 1981; Liggett, 1985). The equality $\mathrm{Var}(X_t)=E(X_t)$ holds if and only if the set $Q \equiv \{s\in\Re : \mathrm{Cov}[X_s, \lambda(X_s)] \neq 0\}$ has Lebesgue measure 0. A lemma, stated and proved in Section 3, implies that Q is a null set if and only if $\lambda(n) = \lambda_0$ for all $n\geqslant 0$ and some $\lambda_0 > 0$. In particular, in the setting of Faddy and Ball, $\mathrm{Var}(X_t) = E(X_t)$ if and only if $\{X_t\colon t\geqslant 0\}$ is a homogeneous Poisson process.

In this paper, we rigorously derive via Itô's formula the variance representation (3) for a class of stochastic processes having right continuous paths with left limits. Jump processes and diffusion processes are special cases of this class. We then extend the Faddy–Ball inequalities to more general models arising in biostatistics and reliability theory. Finally, we show how the ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}E(X_t) = E[\mu(t, X_t)]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}(X_t) = E[\sigma^2(t, X_t)] + 2\mathrm{Cov}[X_t, \mu(t, X_t)]$$

derived from the integral equations (2) and (3) can be explicitly solved for several well-studied stochastic processes.

2. Main result

Assume that $(\Omega, \mathscr{F}, P)$ is a complete probability space with a right-continuous increasing family $(\mathscr{F}_t)_{t\geqslant 0}$ of sub σ-fields of $\mathscr{F}$, each containing all P-null sets. Let B_t be a standard Brownian motion. For functions $\beta(s,x)$ and $\alpha(s,x)$, consider the following stochastic differential equation

$$X_t = X_0 + \int_0^t \beta(s,X_s)\,\mathrm{d}B_s + \int_0^t \alpha(s,X_s)\,\mathrm{d}s + N_t, \tag{4}$$

where X_0 is an $\mathscr{F}_0$-measurable random variable that is independent of B_t, N_t is a pure jump $\mathscr{F}_t$-adapted integrable process with finite square variation in the sense of

$$E\left[\sum_{s\leqslant t}(N_s - N_{s-})^2\right] < \infty$$

for all finite t, and X_t is an $\mathscr{F}_t$-adapted process having sample paths that are right continuous with left limits. The main goal of this section is to derive the variance formula (3).

The stochastic differential equation (4) will possess a unique solution X_t provided the coefficients satisfy the Lipschitz condition

$$|\beta(t,x) - \beta(t,y)|^2 + |\alpha(t,x) - \alpha(t,y)|^2 \leqslant c_1(T)|x-y|^2$$

for all $x, y \in R$ and $t \in [0,T]$ and the growth condition

$$|\beta(t,x)|^2 + |\alpha(t,x)|^2 \leqslant c_2(T)(1+x^2) \tag{5}$$

for all $x \in R$, $t \in [0,T]$, and finite $T>0$, where $c_1(T)$ and $c_2(T)$ are positive constants depending only on T. The proof of this result is omitted because of its similarity to the proof of Theorem 9.1 of Chapter IV in Ikeda and Watanabe (1989). Øksendal (1995) treats the case of a pure diffusion process. We assume that the solution X_t satisfies $E(X_t^2) < \infty$ for all $t \geqslant 0$. Indeed, growth condition (5) guarantees that X_t is square integrable by a similar argument to Theorem 2.4 of Chapter IV in Ikeda and Watanabe (1989).

Let $\Delta N_t = N_t - N_{t-}$ denote the jump of N_t at t and 1_A the indicator function of event A. It is helpful to decompose N_t into the difference $N_t = N_t^+ - N_t^-$ of the sums

$$N_t^+ = \sum_{s\leqslant t} \Delta N_s 1_{\{\Delta N_s > 0\}}, \qquad N_t^- = \sum_{s\leqslant t} |\Delta N(s)| 1_{\{\Delta N_s < 0\}} \tag{6}$$

of the positive and negative jumps. According to the Doob–Meyer decomposition theorem, there exist adapted integrable predictable increasing processes A_t^+ and A_t^- such that $M_t^+ = N_t^+ - A_t^+$ and $M_t^- = N_t^- - A_t^-$ are square integrable martingales with $M_0^+ = M_0^- = 0$. We further assume that there exist nonnegative predictable processes $\lambda^+(t,X_s)$ and $\lambda^-(t,X_s)$ such that

$$A^+(t) = \int_0^t \lambda^+(s,X_s)\,\mathrm{d}s, \quad A^-(t) = \int_0^t \lambda^-(s,X_s)\,\mathrm{d}s. \tag{7}$$

Note here the implicit Markovian assumption that the processes $\lambda^{\pm}(t,X_t)$ depend on $\mathscr{F}_t$ only through X_t. Sufficient conditions for the existence of the $\lambda^{\pm}(t,X_t)$ have been given by Karr (1986, pp. 52–71). Indeed, the processes $\lambda^+(t,X_t)$ and $\lambda^-(t,X_t)$ can be interpreted as stochastic intensity rates and defined via

$$\lambda^{\pm}(t,X_t) = \lim_{h\downarrow 0} \frac{1}{h} \Pr\left\{N_{t+h}^{\pm} - N_t^{\pm} \geqslant 1 | \mathscr{F}_t\right\}.$$

A more intuitive way of viewing $\lambda^{\pm}(t,X_t)$ involves the waiting times T_n^+ and T_n^- until the nth random positive and negative jumps, respectively. Let $f_n^{\pm}(\cdot)$ be versions of the conditional densities of the interarrival times $T_{n+1}^{\pm} - T_n^{\pm}$ given $\mathscr{F}_{T_n^{\pm}}$. Associated with these conditional densities are the survivor functions $\overline{F}_n^{\pm}(t) = \int_t^{\infty} f_n^{\pm}(u)\,\mathrm{d}u$ and the intensity (hazard) functions $\lambda_n^{\pm}(t) = f_n^{\pm}(t - T_n^{\pm})/\overline{F}_n^{\pm}(t - T_n^{\pm})$. The stochastic intensity rates of $N^{\pm}$ are then defined by

$$\lambda^{\pm}(t,X_t) = \lambda_n^{\pm}(t) \tag{8}$$

for $T_n^{\pm} \leqslant t < T_{n+1}^{\pm}$.

To illustrate, suppose that $N_t^{\pm}$ are inhomogeneous Poisson processes with intensity functions $\xi^{\pm}(t)$. Then

$$f_n^{\pm}(t - T_n^{\pm}) = \xi^{\pm}(t)\exp\left\{-\int_{T_n^{\pm}}^{t} \xi^{\pm}(u)\,\mathrm{d}u\right\}$$

$$\overline{F}_n^{\pm}(t - T_n^{\pm}) = \exp\left\{-\int_{T_n^{\pm}}^{t} \xi^{\pm}(u)\,\mathrm{d}u\right\}$$

are versions of the conditional density and survivor functions of $T_{n+1}^{\pm} - T_n^{\pm}$ given $\mathscr{F}_{T_n^{\pm}}$. It follows from Eq. (8) that $\lambda^{\pm}(t,X_t) = \xi^{\pm}(t)$.

If we now let $A_t = A_t^+ - A_t^-$ and define $\mu(t,x) = \alpha(t,x) + \lambda^+(t,x) - \lambda^-(t,x)$, then (4) can be rewritten as

$$X_t = X_0 + \int_0^t \beta(s,X_s)\,\mathrm{d}B_s + \int_0^t \mu(s,X_s)\,\mathrm{d}s + M_t, \tag{9}$$

where $M_t = N_t - A_t$ is a square-integrable martingale. Taking expectations in (9) immediately yields Eq. (2), from which we also deduce, via the fundamental theorem of calculus, the relation

$$\begin{aligned}\int_0^t E(X_s)E[\mu(s,X_s)]\,\mathrm{d}s &= \int_0^t E(X_s)\,\mathrm{d}E(X_s)\\ &= \tfrac{1}{2}E(X_t)^2 - \tfrac{1}{2}E(X_0)^2.\end{aligned} \tag{10}$$

Applying a general version of Itô's formula (Jacod and Shiryayev, 1987) to Eq. (9) with transformation function $f(x) = x^2$, we obtain

$$\begin{aligned}X_t^2 &= X_0^2 + 2\int_0^t X_s\beta(s,X_s)\,\mathrm{d}B_s + 2\int_0^t X_s\mu(s,X_s)\,\mathrm{d}s\\ &\quad +2\int_0^t X_{s-}\,\mathrm{d}M_s + \int_0^t \beta^2(s,X_s)\,\mathrm{d}s + \sum_{s\leqslant t}[X_s^2 - X_{s-}^2 - 2X_{s-}\Delta X_s]\\ &= X_0^2 + 2\int_0^t X_s\beta(s,X_s)\,\mathrm{d}B_s + 2\int_0^t X_s\mu(s,X_s)\,\mathrm{d}s\\ &\quad +2\int_0^t X_{s-}\,\mathrm{d}M_s + \int_0^t \beta^2(s,X_s)\,\mathrm{d}s + \sum_{s\leqslant t}\Delta N_s^2.\end{aligned}$$

Therefore, the second moment is

$$E(X_t^2) = E(X_0^2) + 2\int_0^t E[X_s\mu(s,X_s)]\,\mathrm{d}s + \int_0^t E[\beta^2(s,X_s)]\,\mathrm{d}s + E\left[\sum_{s\leqslant t}\Delta N_s^2\right]. \tag{11}$$

We are now in a position to recover formula (3). Eqs. (10) and (11) together imply

$$\mathrm{Var}(X_t)-\mathrm{Var}(X_0)=\int_0^t E[\beta^2(s,X_s)]\,\mathrm{d}s+E\left[\sum_{s\leqslant t}\Delta N_s^2\right]+2\int_0^t \mathrm{Cov}[X_s,\mu(s,X_s)]\,\mathrm{d}s. \tag{12}$$

Because $\{\sum_{s\leqslant t}\Delta N_s^2\}_{t\geqslant 0}$ is an $\mathscr{F}_t$-adapted integrable increasing process, i.e., a submartingale, there exists an $\mathscr{F}_t$-adapted integrable increasing process L_t such that $\{\sum_{s\leqslant t}\Delta N_s^2-L_t\}_{t\geqslant 0}$ is a martingale. Suppose $L_t=\int_0^t \gamma(s,X_s)\,\mathrm{d}s$ for some nonnegative measurable function $\gamma(s,x)$. Then $E(\sum_{s\leqslant t}\Delta N_s^2)=\int_0^t E[\gamma(s,X_s)]\,\mathrm{d}s$, and the variance formula (3) follows immediately from Eq. (12) upon defining $\sigma^2(s,x)=\beta^2(s,x)+\gamma(s,x)$.

3. Application to jump processes

Consider a jump process $X_t=N_t$ with $N_0=0$, absolute jumps of unit size, and positive and negative jump parts N^+ and N^- as defined in (6). Assume that the compensators A_t^+ and A_t^- have densities $\lambda^+(t,n)$ and $\lambda^-(t,n)$ satisfying (7). Because the jumps are of unit magnitude, Eq. (3) becomes

$$\mathrm{Var}(N_t)=\int_0^t E[\lambda^+(s,N_s)+\lambda^-(s,N_s)]\,\mathrm{d}s+2\int_0^t \mathrm{Cov}[N_s,\lambda^+(s,N_s)-\lambda^-(s,N_s)]\,\mathrm{d}s. \tag{13}$$

As in the Faddy–Ball model, it now follows that

$$\mathrm{Var}(N_t)\geqslant\int_0^t E[\lambda^+(s,N_s)+\lambda^-(s,N_s)]\,\mathrm{d}s$$

when $\lambda^+(s,n)-\lambda^-(s,n)$ is increasing in n for each fixed $s\geqslant 0$. The reverse inequality holds when $\lambda^+(s,n)-\lambda^-(s,n)$ is decreasing in n for each fixed s. If negative jumps are impossible ($\lambda^-(s,n)\equiv 0$), then $E(N_t)=\int_0^t E[\lambda^+(s,N_s)]\,\mathrm{d}s$, and we have sufficient conditions for over- and under-dispersion.

A sufficient condition for the equality

$$\mathrm{Var}(N_t)=\int_0^t E[\lambda^+(s,N_s)+\lambda^-(s,N_s)]\,\mathrm{d}s \tag{14}$$

to hold is that N_t and $\lambda^+(t,N_t)-\lambda^-(t,N_t)$ are independent for all $t\geqslant 0$. If the difference $\lambda^+(t,n)-\lambda^-(t,n)$ is monotone in n, then equality (14) is true if and only if the difference $\lambda^+(t,N_t)-\lambda^-(t,N_t)$ is constant for all $t\geqslant 0$. This fact is an obvious consequence of equality (13) and the following lemma.

Lemma. *Let Z be a random variable and let f and g be real-valued measurable nondecreasing functions defined on the range of Z and satisfying $E[f(Z)^2]<\infty$ and $E[g(Z)^2]<\infty$. If f is strictly increasing, then* $\Pr\{g(Z)=E[g(Z)]\}=1$ *is a necessary and sufficient condition for* $\mathrm{Cov}[f(Z),g(Z)]=0$.

Proof. The sufficiency of the condition is clear. We use a coupling argument (Liggett, 1985) to establish necessity. Let Z^* be independent of Z and share its distribution. Because $[f(Z)-f(Z^*)][g(Z)-g(Z^*)]\geqslant 0$, a brief calculation shows that

$$\begin{aligned}0&\leqslant E\{[f(Z)-f(Z^*)][g(Z)-g(Z^*)]\}\\&=2\mathrm{Cov}[f(Z),g(Z)].\end{aligned}$$

If equality holds in this inequality, then $\Pr\{[f(Z)-f(Z^*)][g(Z)-g(Z^*)]=0\}=1$, from which we deduce $\Pr\{g(Z)-g(Z^*)=0\}=1$ using the fact that f is strictly increasing. If $g(Z)$ is not constant, then $c\in\Re$

exists such that $\Pr\{g(Z)\leqslant c\}>0$ and $\Pr\{g(Z)>c\}>0$. Since Z^* is an independent copy of Z, it follows that

$$\Pr\{g(Z)\leqslant c, g(Z^*)>c\}=\Pr\{g(Z)\leqslant c\}\Pr\{g(Z^*)>c\}>0,$$

contradicting $\Pr\{g(Z)-g(Z^*)=0\}=1$. This contraction proves that $g(Z)$ is constant. □

4. Application to a cumulative damage model

The cumulative damage model (Barlow and Proschan, 1981) has important applications in reliability theory. The basic model postulates that a system encounters shocks that occur according to a point process N_t. When the system suffers its ith shock, a nonnegative random damage Y_i accrues. The damage random variables Y_i form an i.i.d. sequence independent of N_t and having distribution function F, mean θ, and variance τ^2. At time $t\geqslant 0$, the cumulative damage to the system is

$$D_t=\sum_{i=1}^{N_t} Y_i.$$

The system fails at time $S=\inf\{t\geqslant 0\colon D_t\geqslant c\}$, where c is some damage threshold. If N_t is Poisson with intensity λ, then the distribution of S has the increasing failure rate average (IFRA) property regardless of the choice of F (Barlow and Proschan, 1981). Furthermore in this setting, the identities $E(D_t)=\theta\lambda t$ and $\mathrm{Var}(D_t)=(\theta^2+\tau^2)\lambda t$ imply that

$$\mathrm{Var}\left\{\frac{D_t}{\theta}\right\}-\left(1+\frac{\tau^2}{\theta^2}\right)E\left\{\frac{D_t}{\theta}\right\}=0. \tag{15}$$

We now extend this formula to the case where N_t is a general nondecreasing point process with jumps of unit size.

In conjunction with Eqs. (2) and (3), standard arguments show that

$$E(D_t)=\theta E(N_t)$$

$$\begin{aligned}\mathrm{Var}(D_t)&=\theta^2\mathrm{Var}(N_t)+\tau^2E(N_t)\\&=(\theta^2+\tau^2)E(N_t)+2\theta^2\int_0^t \mathrm{Cov}[N_s,\lambda^+(s,N_s)]\,\mathrm{d}s,\end{aligned}$$

where $\lambda^+(t,n)$ is the density of the compensator A_t^+ of N_t. It follows that

$$\mathrm{Var}\left\{\frac{D_t}{\theta}\right\}-\left(1+\frac{\tau^2}{\theta^2}\right)E\left\{\frac{D_t}{\theta}\right\}=2\int_0^t \mathrm{Cov}[N_s,\lambda^+(s,N_s)]\,\mathrm{d}s. \tag{16}$$

We can now assert that the left hand side of Eq. (16) is $\geqslant 0$ when $\lambda^+(s,n)$ is increasing in n for fixed s and $\leqslant 0$ when $\lambda^+(s,n)$ is decreasing in n for fixed s, with strict inequality whenever $\lambda^+(t,n)$ is not constant in n.

5. Some other examples

To illustrate the calculations of $E(X_t)$ and $\mathrm{Var}(X_t)$ using formulas (2) and (3), we consider a jump process and a diffusion process. Table 1 then summarizes $E(X_t)$ and $\mathrm{Var}(X_t)$ for other well-known diffusion processes.

Table 1
Mean and variance of some diffusion processes

Process	$\mu(t,x)$	$\sigma^2(t,x)$	Mean $E(X_t)$	Variance $\mathrm{Var}(X_t)$
Brownian drift	α	β^2	$E(X_0)+\alpha t$	$\mathrm{Var}(X_0)+\beta^2 t$
Ornstein–Uhlenbeck	$-\alpha x$	β^2	$E(X_0)\mathrm{e}^{-\alpha t}$	$\mathrm{Var}(X_0)\mathrm{e}^{-2\alpha t}+\dfrac{\beta^2}{2\alpha}(1-\mathrm{e}^{-2\alpha t})$
Black–Scholes	αx	$\beta^2 x^2$	$E(X_0)\mathrm{e}^{\alpha t}$	$[E(X_0^2)\mathrm{e}^{\beta^2 t}-E(X_0)^2]\mathrm{e}^{2\alpha t}$
Squared Bessel	n	$4x$	$E(X_0)+nt$	$\mathrm{Var}(X_0)+4E(X_0)t+2nt^2$
Brownian bridge	$\dfrac{x}{t-1}$	1	0	$t(1-t)$

Example 5.1 (*Linear birth–death–immigration process*). As observed in Section 3 and in Karlin and Taylor (1975), we can express the infinitesimal mean and variance of a birth–death–immigration process as

$$\mu(t,x)=\lambda^+(t,x)-\lambda^-(t,x),$$

$$\sigma^2(t,x)=\lambda^+(t,x)+\lambda^-(t,x).$$

If the rates $\lambda^+(t,x)=\lambda_0^+ +\lambda_1^+ x$ and $\lambda^-(t,x)=\lambda_0^- +\lambda_1^- x$ are linear functions of the number of existing particles $X_t=x$, then the mean and variance equations (2) and (3) become

$$E(X_t)=E(X_0)+\int_0^t[\mu_0+\mu_1 E(X_s)]\,\mathrm{d}s,$$
$$\mathrm{Var}(X_t)=\mathrm{Var}(X_0)+\int_0^t[\sigma_0+\sigma_1 E(X_s)]\,\mathrm{d}s+2\int_0^t\mu_1\mathrm{Var}(X_s)\,\mathrm{d}s,$$

where $\mu_i=\lambda_i^+-\lambda_i^-$ and $\sigma_i=\lambda_i^++\lambda_i^-$ for $i=0,1$. Straightforward calculus shows that

$$E(X_t)=\left[E(X_0)+\frac{\mu_0}{\mu_1}\right]\mathrm{e}^{\mu_1 t}-\frac{\mu_0}{\mu_1},$$
$$\mathrm{Var}(X_t)=\left[\mathrm{Var}(X_0)+\frac{\sigma_0}{2\mu_1}+\frac{E(X_0)\sigma_1}{\mu_1}+\frac{\mu_0\sigma_1}{2\mu_1^2}\right]\mathrm{e}^{2\mu_1 t}-\left[\frac{E(X_0)\sigma_1}{\mu_1}+\frac{\mu_0\sigma_1}{\mu_1^2}\right]\mathrm{e}^{\mu_1 t}-\frac{\sigma_0}{2\mu_1}+\frac{\mu_0\sigma_1}{2\mu_1^2}.$$

For birth–death–immigration processes in which the coefficients $\lambda_i^+(t)$ and $\lambda_i^-(t)$ depend on t, the integrals defining $E(X_t)$ and $\mathrm{Var}(X_t)$ cannot always be explicitly evaluated. However, these integrals are always amenable to numerical quadrature methods.

Example 5.2 (*Squared Bessel bridge*). A Bessel process R_t is defined as the Euclidean distance from the origin to an n-dimensional Brownian motion $B_t, n\geqslant 2$. The tied-down Bessel process produced by conditioning on the event $R_0=R_1=0$ is called the Bessel bridge. The square X_t of the Bessel bridge satisfies the stochastic differential equation (Revuz and Yor, 1991),

$$X_t=2\int_0^t\sqrt{X_s}\,\mathrm{d}B_s+\int_0^t\left(n-\frac{2X_s}{1-s}\right)\mathrm{d}s.$$

The corresponding integral equations for the mean and variance functions are

$$E(X_t) = nt - 2\int_0^t \frac{E(X_s)}{1-s}\,\mathrm{d}s,$$

$$\mathrm{Var}(X_t) = 4\int_0^t E(X_s)\,\mathrm{d}s - 4\int_0^t \frac{\mathrm{Var}(X_s)}{1-s}\,\mathrm{d}s.$$

These give rise to the differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}E(X_t) = n - \frac{2E(X_t)}{1-t},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}(X_t) = 4E(X_t) - \frac{4\mathrm{Var}(X_t)}{1-t}$$

with solutions $E(X_t) = nt(1-t)$ and $\mathrm{Var}(X_t) = 2nt^2(1-t)^2$. Note here that the $\beta(t,x) = 2\sqrt{x}$ in Eq. (4) is not linear in x, but $\sigma^2(t,x) = 4x$ is. □

Acknowledgements

The authors would like to thank a referee for his helpful comments and suggestions.

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