Let us consider a stationary (or isotropic) random field $Z$ with a covariance function $K$.

- Note $\text{Var}\{Z(s_1) - Z(s_2)\} = 2K(0) - 2K(s_1 - s_2) = 2\gamma(s_1 - s_2)$
- $2\gamma(\cdot)$ is called a variogram and $\gamma(\cdot)$ is called a semivariogram
- If $Z$ is isotropic, then $\gamma(h) = \gamma_0(|h|)$
- Most of the variogram models available so far are isotropic
Nugget effect gives the variation of the process at a finer scale than the smallest distance measured
How do we plot variogram?

- Suppose you have the data points $Z(s_1), \cdots, Z(s_n)$ and assume isotropy for the data.
- First plot squared differences of the data for every possible pairs against the distances between the two locations.
- This plot is called a variogram cloud.
- Then you bin the distances and in each bean, you take average of the squared differences.
- Sometimes you may look at the quantiles instead of the mean.
Can you guess what model it is, what are the sill, range, and nugget (if there is one)?
Some isotropic variogram models (let $h = |s_1 - s_2|$):

**(A) Spherical model** (valid in $\mathbb{R}^d$, $d = 1, 2, 3$):

$$
\gamma(h) = \begin{cases} 
0, & h = 0 \\
 c_0 + c_s \left\{ \frac{3}{2} \frac{h}{a_s} - \frac{1}{2} \left( \frac{h}{a_s} \right)^3 \right\}, & 0 < h \leq a_s \\
 c_0 + c_s, & h \geq a_s 
\end{cases}
$$

**(B) Wave (hole-effect) model** (valid in $\mathbb{R}^d$, $d = 1, 2, 3$):

$$
\gamma(h) = \begin{cases} 
0, & h = 0 \\
 c_0 + c_w \left\{ 1 - a_w \sin\left( \frac{h}{a_w} \right)/h \right\}, & h > 0 
\end{cases}
$$
More examples
Some example pictures

(a) Spherical model

(b) Hole–effect model
Another example

Variogram of TOMS ozone data:
How to use variogram pictures

- First check if the process is isotropic
- If it is, you can guess what variogram model should be appropriate, what are the parameter values (roughly), and whether there should be a nugget effect in the model or not
- If not, you can plot variogram against the differences of some coordinates
- To check if the process is geometrically anisotropic or not, you can use *directional variograms*
- You can plot variograms for some parts of the data to check nonstationarity
- We will see some actual examples next week!