So far we looked at the data that vary over space

Now we add another dimension: time

There are numerous examples and applications for space-time process since things usually change over the space and time at the same time

What we did so far is to fix a time point and analyze the spatial component of the data
When you have space-time data, you can do one of the following things:

1. Separate spatial analyses for each time point
2. Separate temporal analyses for each spatial location
3. Spatio-temporal analysis with methods for random fields in $\mathbb{R}^d \times \mathbb{R}$

In this class, we mainly take the third approach
Consider a stochastic process in the following way:

\[ \{Z(s, t) : s \in D(t) \subset \mathbb{R}^2, t \in T \subset \mathbb{R}\} \]

- Usually \( T \) is a space with integers and regularly spaced while \( D(t) \) can be either regularly or irregularly spaced.
- In many cases \( D(t) \equiv D \)
- Now we may consider \( \text{Cov}\{Z(s_1, t_1), Z(s_2, t_2)\} \)
Isotropy, stationarity...

- Suppose the mean of \( Z \) is constant over space and time.
- Then \( Z \) is stationary in time if
  \[
  \text{Cov}\{Z(s_1, t_1), Z(s_2, t_2)\} = K_1(s_1, s_2, t_1 - t_2)
  \]
  for all \( t_1 \) and \( t_2 \) and some “valid” \( K_1 \).
- We say \( Z \) is isotropic in time if
  \[
  \text{Cov}\{Z(s_1, t_1), Z(s_2, t_2)\} = K_2(s_1, s_2, |t_1 - t_2|)
  \]
  for all \( t_1 \) and \( t_2 \) and some “valid” \( K_2 \).
- If \( Z \) is stationary in both space and time, we have
  \[
  \text{Cov}\{Z(s_1, t_1), Z(s_2, t_2)\} = K_3(s_1 - s_2, t_1 - t_2)
  \]
  for all \( s_i, t_i \) and for a “valid” \( K_3 \).
- Isotropy in both space and time is defined in the same way.
Suppose we assume the process is stationary in space and time and the mean is zero. Now we consider \(\gamma(h, k) = \frac{1}{2} \text{Var}[Z(s, t) - Z(s+h, t+k)]\). The empirical spatio-temporal semivariogram estimator is given by

\[\hat{\gamma}(h, k) = \frac{1}{2|N(h, k)|} \sum_{N(h,k)} \{Z(s_i, t_i) - Z(s_j, t_j)\}^2\]

Here \(N(h, k)\) consists of the points that are about spatial distance \(h\) and time lag \(k\). Even if we assume spatial isotropy, now the picture of empirical variogram is in 3-dimensional or higher dimension.
Now we can think of three different kinds of nugget

1. One with respect to space: \(1_{(h=0)}\)
2. One with respect to time: \(1_{(k=0)}\)
3. One with respect to space and time: \(1_{(h=0, k=0)}\)

Sometimes having all the three nugget effects improve the model fit substantially, although 1 and 3 may matter more.
**Example of space-time variogram**

Fig. 4. Sample spatial and temporal variograms and their models.

Fig. 5. Sample space-time variogram surface.
How to develop space-time variogram (or covariance) model?

- You may treat you have $\mathbb{R}^{d+1}$ dimension and apply similar ideas as you did for spatial process case.
- But the distance in space and “distance” in time are different.
- A simple thing to do is to consider a “distance” in $\mathbb{R}^{d+1}$ such that

$$d((s_1, t_1), (s_2, t_2)) = \sqrt{\left[d(s_1, s_2)/\beta_1\right]^2 + \left[|t_1 - t_2|/\beta_2\right]^2}$$

- Using the above distance, you can use any isotropic model valid for $\mathbb{R}^{d+1}$, for example, a Matérn model.
- Note that you can only have a model isotropic in both space and time in this way.
Another way you can do is to consider a *separable* model.

We say a space-time covariance function is separable if it factors into purely spatial and purely temporal covariance models.

That is, \( \text{Cov}[Z(s_1, t_1), Z(s_2, t_2)] = K_s(s_1, s_2) \cdot K_t(t_1, t_2) \).

Here, \( K_s \) and \( K_t \) are valid covariance models for space and time, respectively.

Why is this method creating a valid space-time covariance model? In other words, if we simply multiply a spatial covariance function to a temporal covariance function, do we get a valid space-time covariance function?
Simple example for isotropic space-time covariance model that is separable is:

$$\text{Cov}[Z(s_1, t_1), Z(s_2, t_2)] = \exp(-|s_1 - s_2|/\beta_1) \cdot \exp(-|t_1 - t_2|/\beta_2)$$

In this case, we may call $\beta_1$ a spatial range and $\beta_2$ a temporal range parameter.

Note that the process $Z$ has same amount of smoothness both in space and time.

We may use Matérn model with different smoothness parameters for space and time to overcome this problem.
Separable model

- Separable models have many nice features
- One is computational advantage
- Another is convenient interpretation
- Another is that enables us to develop space-time covariance models easily
- However, do you see any problem with separable covariance models?
One of the biggest limitation of separable model is that there is no space-time interaction.

This is related to what we call space-time asymmetry:

- Suppose you have a wind blowing from one location to the other over time.
- In particular, suppose you have a westerly wind blowing from location $A$ to $B$ ($B$ is east of $A$).
- Then, if we think of an air pollutant concentration process (with relatively long life time) $Z$, and consider two correlations $\text{Cor}[Z(A, t), Z(B, t + s)]$ and $\text{Cor}[Z(B, t), Z(A, t + s)]$ for $s > 0$, are they equal or one should be bigger than the other?

Can separable covariance models create space-time asymmetry?
Space-time asymmetry is a very common characteristic of space-time data in geophysical and environmental problems.

Many people find space-time asymmetry in many real data sets. For example, in Irish wind data including Gneiting (2002), JASA and Stein (2005), JASA.

Separable model is definitely too limited to model such data sets.
Irish wind data

(source: Haslett and Raftery (1989), *Applied Statistics*)
Figure 2. Variograms for $\Delta Z(x, t+1)$ for Irish Wind Data. In each figure, + indicates a variogram value for coastal sites, $\times$ indicates a coastal and an inland site, and $\circ$ indicates two inland sites. The $\times$ indicates a fitted value under model (15) and $\nabla$ indicates (18). In (b), horizontal offsets within each day are proportional to latitude, with more northerly sites to the right. For improved legibility, (b) omits the temporal variogram at lag 0, which is necessarily 0 for both empirical and fitted variograms. (c) and (f) plot asymmetries for all pairs of sites for which $y$ is east of $x$.

(source: Stein (2005), JASA)
We will see hypothesis tests on separability and examples of nonseparable space-time covariance models