Topic 3 - Discrete distributions

- Basics of discrete distributions
- Mean and variance of a discrete distribution
- Binomial distribution
- Poisson distribution and process
• A **random variable** is a function which maps each element in the sample space of a random process to a numerical value.

• A **discrete random variable** takes on a finite or countable number of values.

• We will identify the distribution of a discrete random variable $X$ by its **probability mass function (pmf)**,

\[ f_X(x) = P(X = x). \]

• Requirements of a pmf:

\[ f(x) \geq 0 \text{ for all possible } x \]
\[ \sum_{\text{all } x} f(x) = 1 \]
Cumulative Distribution Function

• The **cumulative distribution function (cdf)** is given by

\[ F(x) = P(X \leq x) = \sum_{\text{all } t \leq x} f(t) \]

• An increasing function starting from a value of 0 and ending at a value of 1.

• When we specify a pmf or cdf, we are in essence choosing a **probability model** for our random variable.
Reliability example

• Consider the series system with three independent components each with reliability \( p \).

• Let \( X_i \) be 1 if the \( i \)th component works (S) and 0 if it fails (F). This is called a Bernoulli random variable.

• Let \( f_{X_i}(x) = P(X_i = x) \) be the pmf for \( X_i \).

\[ - f_{X_i}(1) = \]

\[ - f_{X_i}(0) = \]
Reliability example continued

- Let $X = \sum_{i=1}^{3} X_i$ be the number of comps. that work

- What is the pmf for $X$?

<table>
<thead>
<tr>
<th>Outcome</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X$</th>
<th>Probability</th>
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<tbody>
<tr>
<td>SSS</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>SSF</td>
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<td>SFS</td>
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<td>FFF</td>
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<tr>
<td>SFF</td>
<td></td>
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</tbody>
</table>
Reliability example continued

• Plot the pmf for $X$ for $p = 0.5$.

• Plot the cdf for $p = 0.5$. 
Reliability example continued

• What is the probability there are at most 2 working components if $p = 0.5$?

• What is the probability the device works if $p = 0.5$? (Remember, it’s a series)
Mean of a discrete random variable

- $\mu_X = E(X)$, mean of $X$ or expected value of $X$
  - Given an array of discrete outcomes, the summation of the individual outcomes times their respective probability of occurring.
  - The weighted average outcome. $\sum x(p(x))$

- $E(h(X)) = \sum_{\text{all } x} h(x)f(x)$, expected value of $h(X)$
  - the expected value of a function is the result of the function, given $x$, times the corresponding probability of that occurrence.
  - The weighted average value of the function.
Variance of a discrete random variable

- $\sigma_x^2 = E[(X - \mu_x)^2]$, variance of $X$

- Show $\sigma_x^2 = E(X^2) - \mu_x^2$

  $E[(X - \mu_x)^2] = E[X^2 - 2\mu X + \mu^2]$

  $= \sum X^2 p(X) - 2\mu \sum Xp(X) + \mu^2 \sum p(X)$

  $= E(X^2) - (2\mu \ast \mu) + (\mu^2 \ast 1)$

  $= E(X^2) - \mu^2$  

  The "shortcut" formula
Reliability example continued

• What is the mean of $X$ if $p = 0.5$?

• What is the variance of $X$ if $p = 0.5$?

<table>
<thead>
<tr>
<th>x</th>
<th>p(x)</th>
<th>x * p(x)</th>
<th>x^2</th>
<th>x^2 * p(x)</th>
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<td>0.000</td>
<td>0</td>
<td>0.000</td>
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<td>0.375</td>
<td>0.375</td>
<td>1</td>
<td>0.375</td>
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<td>4</td>
<td>1.500</td>
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<tr>
<td>3</td>
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<td>0.375</td>
<td>9</td>
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</tr>
<tr>
<td>1.000</td>
<td>1.500</td>
<td>3.000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
What’s the mean and variance of $X$?

The mean of $X^2$ is an example of $E(h(x))$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$p(x)$</th>
<th>$x*p(x)$</th>
<th>$x^2$</th>
<th>$(x^2)*p(x^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.125</td>
<td>0.000</td>
<td>0</td>
<td>0.000</td>
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<td>1.125</td>
</tr>
</tbody>
</table>

$E(h(x)) = \sum[x^2*p(x)] = 3$

$Var (x) = E(x^2) − E(x)^2 = 3 − 1.5^2 = 0.75$

$Std dev (x) = 0.75^{.5} = 0.86603$
Binomial distribution

• Bernoulli trials:
  – Each trial can result in one of two outcomes (S or F)
  – Trials are independent
  – The probability of success, \( P(S) \), is a constant \( p \) for all trials

• Suppose \( X \) counts the number of successes in \( n \) Bernoulli trials.

• The random variable \( X \) is said to have a **Binomial distribution**
  with parameters \( n \) and \( p \).

• \( X \sim \text{Binomial}(n,p) \)

• The \( X \) from the reliability example falls into this category.
Binomial pmf

• What is the probability of any outcome sequence from $n$ Bernoulli trials that contains $x$ successes and $n-x$ failures?

• How many ways can we arrange the $x$ successes and $n-x$ failures?
  – Combinations of $n$ things taken $k$ at a time
  – Remember the number of ways we could come up with 2 successes out of the 3 trials in the reliability example?
  – SSF, SFS or FSS

\[
 f(x) = P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \quad x = 0, \ldots, n
\]
How the two components work together
Binomial properties

• The Binomial is valid (all probabilities sum to one and all are at least 0).
  
  \[ E(X) = u_x = np \quad \text{and} \quad \sigma_x^2 = np(1 - p) \]

• Example: Look at the reliability problem we did.

• Based on \( p=0.5 \), the mean was \( np \) or \( 3 \times 0.5 \) or 1.5.

• Based on \( p=0.5 \), the variance was \( npq \) or \( 3 \times 0.5 \times 0.5 = 0.75 \)

• Change values for \( p \) and the mean, variance and shape of the resulting distribution changes. In addition, some information on “proofs” for the Binomial are included in the Moment Generating Functions file under Topic 3.
Nurse employment case

• Contract requires 90% of records handled timely

• 32 of 36 sample records handled timely, she was fired!

• Can each sample record be considered as a Bernoulli trial?

• If the proportion of all records handled timely is 0.9, what is the probability that 32 or fewer would be handled timely in a sample of 36?

Binomial calculator
Nurse employment case, calculations by hand

\[ P(X \leq 32) = 1 - P(X \geq 33) \]
\[ P(X \geq 33) = P(X = 33) + P(X = 34) + P(X = 35) + P(X = 36) \]

\[ P(X = 33) = \binom{36}{33} (0.9)^{33} (0.1)^{3} = (7140)(0.0309031544)(0.0010) = 0.22065 \]

\[ P(X = 34) = \binom{36}{34} (0.9)^{34} (0.1)^{2} = (630)(0.027812839)(0.01) = 0.17522 \]

\[ P(X = 35) = \binom{36}{35} (0.9)^{35} (0.1)^{1} = (36)(0.025031555)(0.1) = 0.09011 \]

\[ P(X = 36) = \binom{36}{36} (0.9)^{36} (0.1)^{0} = (1)(0.0225284)(1) = 0.02253 \]

\[ P(X \leq 32) = 1 - P(X \geq 33) = 1 - 0.22065 - 0.17522 - 0.09011 - 0.02253 = 0.49159 \]
Horry county murder case

- 13% of the county is African American

- Only 22 of 295 summoned were African American

- Can a summoned juror be considered as a Bernoulli trial?

- If the prop. of African Americans in the jury pool is 0.13, what is the probability that 22 or fewer would be African American in a sample of 295?

[Binomial calculator]
Horry county murder case – by hand

\[ P(X \leq 22 \mid X \sim Binomial(295, 0.13)) \]

\[ = P(X = 22) + P(X = 21) + \ldots + P(X = 0) \]

\[ P(X = 22) = \binom{n}{x} p^x (1 - p)^{n-x} = \binom{295}{22} 0.13^{22} (1 - 0.13)^{295-22} \]

\[ = \frac{295!}{22!273!} (0.13)^{22} (0.87)^{273} \]

\[ P(X = 22) = 0.00085541 \]
\[ P(X = 21) = 0.00045965 \]
\[ P(X = 20) = 0.00023490 \]
\[ \vdots \]
\[ P(X = 0) = 1.4394e-18 \implies \text{summation} = 0.00175346 \]
Poisson distribution

• The **Poisson distribution** is used as a probability model for the number of events occurring in an interval where the expected number of events is proportional to the length of the interval.

• Examples
  – # of computer breakdowns per week
  – # of customers entering a business per hour
  – # of telephone calls per hour
  – # of imperfections in a foot long piece of wire
  – # of bacteria in a culture of a certain area

• \[ f(x) = P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!} \quad x = 0, 1, \ldots \]
Poisson properties

• The Poisson is a valid distribution (all probabilities are at least 0 and sum to 1. The mean and variance of the Poisson are both equal to $\lambda$.

• An example of a Poisson process with calculations for $\lambda = 3$ is contained in the file “Nuts and Bolts Poisson” in Topic 3. In addition, the Moment Generating Function “proof” of this for the mean are contained in the Moment Generating Functions file.
Poisson example

- My car breaks down once a week on average.

- Using a Poisson model, what is the probability the car will break down at least once in a week?

- What is the probability it breaks down more than 52 times in a year?

Poisson calculator
Poisson calculation by hand

Car breaks down 1 time per week on average. What’s the probability that it breaks down at least once in a week?

\[ P(X \geq 1) = 1 - P(X < 1) = 1 - P(X=0) \]

\[ P(X = 0) = \frac{e^{-1}1^0}{0!} = e^{-1} = 0.3679 \]

\[ 1 - 0.3679 = 0.6321 \]

Car breaks down 52 times per year. What’s the probability that it breaks down more than 52 times in a year?

There’s no quick calculation to this type of problem, other than using a Stat package, like StatCrunch.
Additional Poisson – by hand

Assume that the average number of help calls on our computer site in a 10-minute period for a specific time of day is 4. We need to reboot our machine due to updates and will be down for 10 minutes. What’s the probability that we have at most 1 phone call in that 10-minute period?

\[ P(X \leq 1 \mid X \sim \text{Poisson}(\lambda = 4)) \]

\[ P(X \leq 1) = P(X = 0) + P(X = 1) \]

\[ P(X = 0) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-4} 4^0}{0!} = e^{-4} = 0.01832 \]

\[ P(X = 1) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-4} 4^1}{1!} = 0.07326 \]

\[ P(X \leq 1) = P(X = 0) + P(X = 1) = 0.01832 + 0.07326 = 0.09158 \]
Other distributions

• Discrete uniform
  – Finite number of identifiable outcomes, all with the same probability of occurrence. Examples

• Hypergeometric
  – Sampling without replacement from a specified sized group, with a known number of successes.
  – Not independent probabilities (changes from sample to sample).

• Negative Binomial
  – Similar to a straight Binomial, but the number of trials stops when a certain number of successes is reached.
  – If we believe p% of the population is a success, how many should we have to sample to obtain x successes?