Topic 2 – Probability

• Basic probability
• Conditional probability and independence
• Bayes rule
• Basic reliability
• **Random process**: a process whose outcome can not be predicted with certainty
  – Examples: rolling a die, picking a card from a deck

• **Sample space (S)**: the collection of all possible outcomes to a random process
  – Examples: With one die, all possible outcomes are 1-6
  – Examples: With a deck of cards, all possible outcomes are 2-Ace of any suit

• **Event (A,B)**: a collection of possible outcomes
  – **Simple**: one outcome
  – **Compound**: more than one outcome

• **Probability**: a number between 0 and 1 (inclusive) indicating the likelihood that the event will occur
What are the probabilities relating to...

- Rolling a single die?

<table>
<thead>
<tr>
<th>x</th>
<th>p(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

- Rolling two dice?

<table>
<thead>
<tr>
<th>x</th>
<th>p(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
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<td>6</td>
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<td>7</td>
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<td>8</td>
<td></td>
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<td>9</td>
<td></td>
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<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>
Set theory review

- **Union** \((A \cup B)\) – all elements in \(A\) or \(B\)

- **Intersection** \((A \cap B)\) - all elements in \(A\) and \(B\)

- **Complement** \((A^c)\) – all elements **not** in the set
  - Also denoted as \(A’\)

- **Mutually exclusive** – two events are mutually exclusive if they have no outcomes in common
  - Example: Pregnancy and being a man
  - Example: Rolling an even or an odd number
Elementary probability

• The proportion of times that an event $A$ occurs out of $n$ trials converges to the actual probability of $A$, $P(A)$, as the number of repetitions becomes large.

• Long range frequency

• StatCrunch “coin flip simulation”. Show long range frequency of 1000 runs repeatedly for demonstration of convergence to what the actual value for “p” should be.
Probability with equally likely outcomes

- If a sample space is composed of \( k \) equally likely outcomes with \( m \) outcomes contained in \( A \), then \( P(A) = \frac{m}{k} \).
- Two way classification table for the salary data:

<table>
<thead>
<tr>
<th></th>
<th>Education</th>
<th>Engineering</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Female</td>
<td>856</td>
<td>232</td>
<td>1088</td>
</tr>
<tr>
<td>Male</td>
<td>220</td>
<td>924</td>
<td>1144</td>
</tr>
<tr>
<td>Total</td>
<td>1076</td>
<td>1156</td>
<td>2232</td>
</tr>
</tbody>
</table>

- **Randomly** select a student from the survey group, what is the probability that
  - the student is female?
  - the student is an engineer?
  - the student is a female engineer?
  - the student is a female or an engineer?
Properties of Probability

• Axioms:
  • $P(A) \geq 0$
  • $P(S) = 1$, this is really important
  • If $A$ and $B$ are mutually exclusive, $P(A \cup B) = P(A) + P(B)$

• Other properties that may not be apparent:
  • $0 \leq P(A) \leq 1$
  • $P(\emptyset) = 0$
  • $P(A^c) = 1 - P(A)$
  • $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Back to the salary example

- Consider the following two way classification table between major and gender for the salary data:

<table>
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- If we **randomly** select a student from this group, what is the probability that
  - the student is not a male educator?
  - the student is a female or an engineer?
Conditional Probability and Independence

• The **conditional probability** of $A$ given $B$ is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

if $P(B) > 0$.

• $A$ and $B$ are **independent** if $P(A \cap B) = P(A)P(B)$.

• If $P(A) > 0$ and $P(B) > 0$, then $A$ independent of $B$ means
  – $P(A \mid B) = P(A)$.

• If the events are independent, knowing $B$ occurs does not change the probability that $A$ occurs.
A and B are independent if $P(A \cap B) = P(A)P(B)$.

<table>
<thead>
<tr>
<th></th>
<th>Event A1</th>
<th>Event A2</th>
<th>Event A3</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Event B1</td>
<td>.02</td>
<td>.06</td>
<td>.12</td>
<td>.20</td>
</tr>
<tr>
<td>Event B2</td>
<td>.08</td>
<td>.24</td>
<td>.48</td>
<td>.80</td>
</tr>
<tr>
<td>Totals</td>
<td>.10</td>
<td>.30</td>
<td>.60</td>
<td>1.00</td>
</tr>
</tbody>
</table>
What are some examples of events that are independent?

• Rolling a fair die

• Flipping a fair coin

• Defective product (note difference in view here between theoretical production and some real world applications)

• HW2, #2 discussion with example.

• Sampling with replacement
Back to the examples

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- For the salary example, is the gender of the student selected independent of the student’s major?
  - If I tell you that the student is female, does that alter the probability that they are an engineer?

- Reliability example. What’s the probability that a system of two components works, if they are linked in a series and work independently of each other?
  - If the probability that each works is 80%, the probability that two in a series working is 80% x 80% or 64%. This is a common assumption in reliability.
Independence & Probability

• Independence is an important concept in the calculation of probabilities.

• The probability of a “series” of independent events is the product of their independent probabilities.

• Examples
  – Flipping a coin.
  – Rolling a die.
  – X defectives in a row, given a defect %.
  – Drives the probability calculation in many situations.
Law of total probability and Bayes rule

- If $A_1, \ldots, A_n$ are mutually exclusive, each event has positive probability and one of the events must occur, then

$$P(B) = \sum_{j=1}^{n} P(B \mid A_j)P(A_j)$$

- If $P(B) > 0$, the above result implies Bayes rule:

$$P(A_k \mid B) = \frac{P(A_k)P(B \mid A_k)}{\sum_{j=1}^{n} P(A_j)P(B \mid A_j)}$$
Voltage regulator example

- In a batch of voltage regulators, 60% came from supplier 1, 30% from supplier 2 and 10% from supplier 3.
- 95% of regulators from supplier 1 work
- 60% of regulators from supplier 2 work
- 50% of regulators from supplier 3 work
- If a regulator randomly selected from the batch works, what is the probability it came from supplier 1?
Bayes Rule, by formulation

• Assuming we want to calculate \( P(S_1 | W) \). From the table, it’d be \( .57 / .80 \).
• Based on the formula, it’d be

\[
P(S_1 | W) = \frac{P(S_1)P(W | S_1)}{P(S_1)P(W | S_1) + P(S_2)P(W | S_2) + P(S_3)P(W | S_3)}
\]

or

\[
P(S_1 | W) = \frac{.60(.95)}{.60(.95) + .30(.60) + .10(.50)} = .57 / .80
\]
Additional Baye’s – Medical testing

• Suppose that there is an AIDS test whose results are 99% positive when you have the disease (sensitivity) and 99% negative when you don’t (specificity).

• Is that good enough? Do you accept the positive result? Does that mean that there’s a 99% probability that you do have AIDS?

• Let’s run the numbers under various assumptions.
Medical Testing (2)

• Assume that 6 in 1,000 in the general population have AIDS (prevalence). The numbers fall out like this.

<table>
<thead>
<tr>
<th></th>
<th>AIDS YES</th>
<th>AIDS NO</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test +</td>
<td>5.94</td>
<td>9.94</td>
<td>15.88</td>
</tr>
<tr>
<td>Test -</td>
<td>0.06</td>
<td>984.06</td>
<td>984.12</td>
</tr>
<tr>
<td></td>
<td>6.00</td>
<td>994.00</td>
<td>1000.00</td>
</tr>
</tbody>
</table>

• What’s the actual probability, given that you got a positive response, that you do have AIDS?
Medical Testing (3)

- Assume that 100 in 1,000 in “high risk” population have AIDS. The numbers fall out like this.

<table>
<thead>
<tr>
<th></th>
<th>AIDS YES</th>
<th>AIDS NO</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test +</td>
<td>99.00</td>
<td>9.00</td>
<td>108.00</td>
</tr>
<tr>
<td>Test -</td>
<td>1.00</td>
<td>891.00</td>
<td>892.00</td>
</tr>
<tr>
<td></td>
<td>100.00</td>
<td>900.00</td>
<td>1,000.00</td>
</tr>
</tbody>
</table>

- What’s the actual probability, given that you got a positive response, that you do have AIDS?
Medical Testing (final)

• “Population” designed for and tested against has an effect on the accuracy of the results.

• If in fact you want to have a high confidence that someone has AIDS when you get a positive test, or more importantly, that they don’t when you get a negative.....that % accuracy (both sensitivity & specificity) must be higher.

• Economics (production vs. litigation)
Reliability of systems

• The **reliability** of a device or system is the overall probability that it works.

• Suppose an electronic device is composed of two components and the probability that each of the individual components work is 0.8.

• The overall reliability of the system is dependent on how that system is constructed.
  – In a series; one component after another, or
  – In parallel; with “stacked” components.
Reliability of Systems

• **Reliability** of a device is the probability the device works.

• Suppose our device is composed of $k$ **independent** components each with their own reliability

• $A_i$ - event that component $i$ works

• The reliability of our device depends on how it is made up in terms of its components

• A **series system** will work only if all its components work.

• A **parallel system** will work if any one of its components work.
Reliability of a series system

- \( R = P(\text{device works}) = P(\text{all components work}) \)
  - The operation is only as reliable as it’s weakest link.

- What are the reliability properties of a series system?
  - The probability that the system works has to be between 0 and 1.
  - The reliability of the system MUST be less than the reliability of the least reliable component.
  - Adding more components MUST decrease overall reliability.
Example of a series system

• Old fashioned Christmas tree lights

0.9 ——— 0.9 ——— 0.3 ——— 0.9

• What’s the probability that the system works?

• What’s the probability of the system working, if we add another bulb to the system, with a reliability of 0.9?
Reliability of a parallel system

- \[ R = P(\text{device works}) = P(\text{at least one comp. works}) \]
  - \[ 1 - P(\text{all components fail}) \]
  ...... complement’s rule

- What are the reliability properties of a parallel system?
  - More reliable than its most reliable component.
  - Adding components makes it more reliable.
  - Much better than a series system.
Example of a parallel system

• Redundant computer back ups
A more complex example

Please read additional file for Topic 2 for more information and problems.