STAT 604 Exam 2 Solutions **Fall, 1998 Dr. Newton**

GRADES:

```
10| 0
 9| 0 3 3 4 8
 8| 0 0 1 3 5 5 5 6 8
 7| 0 0 1 4 4
 6| 5
 5|
 4|
 3|
 2| 3
```

1. (10 pts) How would you TEX the following?

$$
\Sigma=\left[\matrix{\sigma_1^2&\rho\sigma_1\sigma_2\cr \rho\sigma_1\sigma_2&\sigma_2^2\cr}\right]
$$

SOLUTION:

```
$$
\Sigma=\left[\matrix{
  \sigma_1^2&\rho\sigma_1\sigma_2\cr
  \rho\sigma_1\sigma_2&\sigma_2^2\cr
}\right]
$$
```

---

2. (10 pts) What would be the result of the first splitting of

```
19 13 1 2 12 16 18 4 14 9 5 15 8 7 17 20 10 6 11 3
```

in quicksort?

SOLUTION:

- Sorting 1st, 10th, and 20th and then swapping new 10th (9, the splitting element) and new 1st (3) gives:

```
9 13  1  2 12 16 18  4 14  3  5 15  8  7 17 20 10  6 11  19
  --                                               -
```

- Starting from 2nd and going right we find the 13 is greater than 9, starting with 19th and going left we find 6 is less than 9 so we swap the 13 and 6 giving:

```
9  6  1  2 12 16 18  4 14  3  5 15  8  7 17 20 10  13 11  19
           --                          -
```

- Now repeat going right from the 1 and left from the 10 and end up swapping the 12 and 7, giving:

```
9 6 1 2 7 16 18  4 14  3  5 15  8 12 17 20 10  13 11  19
          --                    -
```

- Next swap 16 and 8:

```
9 6 1 2 7 8 18  4 14  3  5 15  16 12 17 20 10  13 11  19
            --            -
```

- Then 18 and 5:

```
9 6 1 2 7 8 5 4 14  3  5 18  16 12 17 20 10  13 11  19
                --  -
```

- Then 14 and 3:

```
9 6 1 2 7 8 18  4 3 14  5 15  16 12 17 20 10  13 11  19
                  -
```

- At which point the pointer is at the 3 which we swap with the 9 (since 3 is not greater than 9), giving finally:

```
3 6 1 2 7 8 5  4 9 14  18 15  16 12 17 20 10  13 11  19
                 -
```

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3. (10 pts) How many reps (Bernoulli trials) must one perform in a simulation in order to be at least 95% sure that one is estimating a population proportion within ± 0.01?

SOLUTION:

The 95% confidence interval for a proportion p based on n Bernoulli trials is

$$\hat{p} \pm 1.96\sqrt{\frac{p(1-p)}{n}},$$

and so we want

$$0.01 = 1.96\sqrt{\frac{p(1-p)}{n}},$$

and solving for n gives

$$n = 196^2 p(1-p),$$

and since the most $p(1-p)$ can be is $1/4$ (which is when $p = 1/2$), we know that $n = 196^2/4 = 9604$ will give us at least 95% confidence that we know p within ± 0.01,

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4. (10 pts) A megaflops is defined to be one million double precision floating point operations per second. On a two megaflops computer, how long does it take to multiply two 100 by 100 matrices 50 times?

SOLUTION:

$$\frac{50(100)^3}{2,000,000} = 25 \text{ seconds}$$

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5. (20 pts) Given a sample of size n from the lognormal distribution

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-(\log x - \mu)^2/2\sigma^2}, \quad x > 0,\ \mu \in \mathcal{R},\ \sigma > 0,$$

find the gradient and Hessian matrices to be used in minimizing via Newton–Raphson the negative of the log likelihood of μ and σ^2.

SOLUTION:

(Note: The original question had the minus sign in the exponent inside the parentheses instead of outside. This solution assumes the correct expression for f.)

We have

$$\begin{aligned}
L(\theta) = L\left(\begin{pmatrix}\mu \\ \sigma^2\end{pmatrix}\right) &= -\sum_{i=1}^{n} \log f(x_i) \\
&= -\sum_{i=1}^{n}\left[-\frac{1}{2}\log 2\pi - \frac{1}{2}\log\sigma^2 - \log x_i - (\log x_i - \mu)^2/2\sigma^2\right] \\
&= \frac{n}{2}\log 2\pi + \frac{n}{2}\log\sigma^2 + \sum_{i=1}^{n}\log x_i + \frac{1}{2\sigma^2}\sum_{i=1}^{n}(\log x_i - \mu)^2,
\end{aligned}$$

and so

$$g = \begin{pmatrix} \dfrac{\delta L}{\delta \mu} \\ \dfrac{\delta L}{\delta \sigma^2} \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{\sigma^2}\displaystyle\sum_{i=1}^{n}(\log x_i - \mu) \\ \dfrac{n}{2\sigma^2} - \dfrac{1}{2\sigma^4}\displaystyle\sum_{i=1}^{n}(\log x_i - \mu)^2 \end{pmatrix}$$

and

$$H = \begin{bmatrix} \dfrac{\delta^2 L}{\delta \mu^2} & \dfrac{\delta^2 L}{\delta\mu\delta\sigma^2} \\ \text{sym} & \dfrac{\delta^2 L}{\delta\left(\sigma^2\right)^2} \end{bmatrix} = \begin{bmatrix} \dfrac{n\mu}{\sigma^2} & \dfrac{1}{\sigma^4}\displaystyle\sum_{i=1}^{n}(\log x_i - \mu) \\ \text{sym} & -\dfrac{n}{2\sigma^4} + \dfrac{1}{\sigma^6}\displaystyle\sum_{i=1}^{n}(\log x_i - \mu)^2 \end{bmatrix}.$$

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6. (20 pts) How would you use the U(0,1) distribution in the rejection method to generate random numbers from the beta distribution having pdf

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad x \in (0,1),\ \alpha > 0,\ \beta > 0,$$

for $\alpha = \beta$ for integer values of $\alpha > 1$? For a given value of α, what is the probability of rejection? What is this probability for $\alpha = 3$?

SOLUTION:

To use the rejection method we must find a constant c satisfying

$$c = \max_{x \in (0,1)} \frac{f(x)}{g(x)},$$

or, since $g(x) = 1$ over $x \in (0, 1)$, we must find the maximum value of $f(x)$. But we can use the first and second derivatives of f to show that the max of f occurs at $x = 1/2$ and so

$$c = \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)4^{\alpha-1}} = \frac{(2\alpha - 1)!}{[(\alpha - 1)!]^2\, 4^{\alpha-1}},$$

since α is assumed to be an integer. For $\alpha = 3$, $c = 15/8$, which gives a rejection probability of $1 - 1/c = 7/15$. The rejection method is then

1. Generate U_1 from U(0,1)
2. Generate $Z = cU_2$, where U_2 is another U(0,1) independent of U_1.
3. If $Z < f(U_1)$, return $X = U_1$, else return to (1).

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7. (20 pts) Write an Splus function `ptnorm(sig1,sig2)` that will superimpose plots of the N(0,σ^2) pdf for 101 equally spaced values of σ in [`sig1,sig2`], with each pdf plotted at 301 equally spaced points from the .001 to .999 quantile of the distribution. The function must call `plot` to set up the axes and then do a `for` loop that will call `lines` to draw each pdf.

SOLUTION:

The following version of `ptnorm` is slightly more general as it has arguments corresponding to number of pdf's to draw and the number of points to use. The important thing in the function is setting up the axes using `plot`. We know the pdf with the largest variance is the most spread out while the one with the smallest variance is the tallest. These facts are used in the `plot` call below. The graph on the last page of this solution set is obtained by `ptnorm(1,2,11,101)`.

```
ptnorm <- function(sig1=1,sig2=2,nplts=101,nxs=301)
{

    plot(c(qnorm(.001,0,sig2),qnorm(.999,0,sig2)),c(0,dnorm(0,0,sig1)),
          type='n',xlab='x',ylab='f(x)',
          main=paste(nplts, ' N(0,sigma^2) pdfs for sigma in [',
            round(sig1,2),',',round(sig2,2),']'))

    for(i in 1:nplts)
    {
        sig <- sig1+(i-1)*(sig2-sig1)/(nplts-1)
        lines(x <- seq(qnorm(.001,0,sig),qnorm(.999,0,sig),length=nxs),
              dnorm(x,0,sig))
    }

}
```

---

11 N(0,sigma^2) pdfs for sigma in [1 , 2]