STAT 604 Final Exam Solutions **Fall, 1997 Dr. Newton**

GRADES:

```
9| 0 0 1 2 5
8| 2 2 5 8
7| 7
6| 5 6
5| 1 1
```

1. (10 pts) The Chebychev polynomials are defined by

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n > 1.$$

What is $T_4(x)$?

SOLUTION:

$T_4(x) = 8x^4 - 8x^3 + 1$

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2. (10 pts) What is the square root of the matrix

$$A = \begin{bmatrix} 6 & 3 & 2 \\ 3 & 5 & 1 \\ 2 & 1 & 8 \end{bmatrix}$$

SOLUTION:

$$A^{1/2} = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ \sqrt{6}/2 & \sqrt{7/2} & 0 \\ \sqrt{6}/3 & 0 & \sqrt{22/3} \end{bmatrix}.$$

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3. (10 pts) What is the estimate of σ^2 obtained by jackknifing s^2 for a sample of size n?

SOLUTION:

Show

$$s^2_{(i)} = \frac{1}{n-2}\left[(n-1)s^2 - \frac{n}{n-1}(X_i - \bar{X})^2\right],$$

from which it easily follows that $s^2_{JK} = s^2$.

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4. (20 pts) Which function has a sharper peak at $x = 0$, the standard normal pdf f or the standard Cauchy pdf

$$g(x) = \frac{1}{\pi(1+x^2)}?$$

SOLUTION:

$$|f''(0)| = \left|-\frac{1}{\sqrt{2\pi}}\right| = 0.3989 < 0.6366 = \left|-\frac{2}{\pi}\right| = |g''(0)|\,,$$

and so Cauchy is sharper. To see this visually, try

```
x  <- seq(-.1,.1,length=101)
fx <- dnorm(x)-dnorm(0)
gx <- dcauchy(x)-dcauchy(0)
plot(x,gx,type='n')
lines(x,fx,lty=1)
lines(x,gx,lty=2)
```

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5. (10 pts) What is the relative absolute error in approximating $\int_0^1 x^2 dx$ by the trapezoidal rule using 21 grid points.

SOLUTION:

The trapezoidal rule gives $\hat{I} = 267/800$ which gives $|\hat{I} - I|/|I| = 1/800$.

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6. (10 pts) If all you have is a N(0,1) random number generator, how would you generate χ^2_v and F_{v_1,v_2} random variables?

SOLUTION:

$$\chi^2_v = \sum_{i=1}^{v} X_i^2, \qquad F_{v_1,v_2} = \frac{\sum_{i=1}^{v_1} X_i^2/v_1}{\sum_{i=1}^{v_2} Y_i^2/v_2},$$

where the X_i's and Y_i's are iid N(0,1).

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7. (10 pts) What is the result of making the vector $y = (1,2,3,4,5)^T$ orthogonal to the vector $x = (6,4,8,3,1)^T$?

SOLUTION:

Need $z = y - \dfrac{< x, y >}{< x, x >} x = y - \dfrac{55}{126} x.$

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8. (10 pts) How would you TEX

$$g(\alpha,\beta) = n\log\beta + \frac{1}{2\beta^2}\sum_{i=1}^{n}(\log x_i - \alpha)^2?$$

SOLUTION:

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$$
g(\alpha,\beta)=n\log\beta+\dfrac{1}{2\beta^2}\sum_{i=1}^n(\log x_i-\alpha)^2?
$$
```

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9. (10 pts) What is the quantile function corresponding to the pdf

$$f(x) = \frac{1}{\alpha^2} x e^{-x^2/(2\alpha^2)}, \qquad x > 0?$$

SOLUTION:

$$F(x) = 1 - e^{-x^2/2\alpha^2}, \qquad Q(u) = \sqrt{-2\alpha^2 \log(1-u)}.$$

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10. (10 pts) If $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$, where B and E are $(r \times r)$ and $(s \times s)$, respectively, show (by first sweeping A on its first r diagonals and the on the last s diagonals) that the upper left hand $(r \times r)$ part of A^{-1} is given by

$$F = B^{-1} + B^{-1}C(E - DB^{-1}C)^{-1}DB^{-1}.$$

SOLUTION:

Using the basic facts of sweeping a matrix on either its upper left or lower right block of diagonals gives

$$G = \text{SWEEP}(1, \ldots, r)A = \begin{bmatrix} B^{-1} & B^{-1}C \\ -DB^{-1} & E - DB^{-1}C \end{bmatrix},$$

and then

$$\text{SWEEP}(r+1, \ldots, r+s)G = \begin{bmatrix} B^{-1} + B^{-1}C(E - DB^{-1}C)^{-1}DB^{-1} & -B^{-1}C(E - DB^{-1}C)^{-1} \\ (E - DB^{-1}C)^{-1}DB^{-1} & (E - DB^{-1}C)^{-1} \end{bmatrix}.$$