STAT 604 Exam 1 Solution **Fall, 1995 Dr. Newton**

1. (20 points) There are many formulas for SSR, but the definition is SSR $= \sum_{i=1}^{n} e_i^2$, where the vector $e = y - X\hat{\beta}$ (and so $e_i = y_i - i$th row of X times $\hat{\beta}$). Thus we have

```
      double precision function ssr(y,X,betah,n,m,ndim)
      double precision y(n),X(ndim,m),betah(m),ei

      ssr=0.0d0
      do 20 i=1,n

c   Get e_i:

         ei=y(i)
         do 10 j=1,m
 10      ei=ei-betah(j)*X(i,j)

 20   ssr=ssr+ei*ei

      return
      end
```

2. (10 points) The basic point is that $(LU)_{i,j} = \sum_{k=1}^{n} L_{ik}U_{kj} = \sum_{k=1}^{\min(i,j)} L_{ik}U_{kj}$, since $L_{ik} = 0$ for $k > i$ and $U_{kj} = 0$ for $k > j$. Thus the number of multiplications and additions for the (i,j)th element of the product is $\min(i,j)$ (in counting operations in an inner product, it is standard to ignore the fact that there is one less addition than multiplications) and thus the total number is

$$T_n = \sum_{i=1}^{n}\sum_{j=1}^{n} \min(i,j).$$

If we break the sum over i into 1 to $n-1$ and n by itself, and then the sum over j into 1 to i and $i+1$ to n, we get

$$T_n = \sum_{i=1}^{n-1}\left[\sum_{j=1}^{i}\min(i,j) + \sum_{j=i+1}^{n}\min(i,j)\right] + \sum_{j=1}^{n}\min(n,j) = \sum_{i=1}^{n-1}\left[\sum_{j=1}^{i} j + \sum_{j=i+1}^{n} i\right] + \sum_{j=1}^{n} j,$$

which after lots of algebra (and using the facts that the sum and sum of squares of the first n positive integers are $n(n+1)/2$ and $n(n+1)(2n+1)/6$) ends up being

$$T_n = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \sim \frac{n^3}{3},$$

as opposed to n^3 if we just multiply two $(n \times n)$ matrices. (Note that the notation $a_n \sim b_n$ means $\lim_{n\to\infty} a_n/b_n \to 1$.)

There is actually a very simple way to determine T_n. It is the sum of the elements of a matrix like the one for $n = 4$:

```
1  1  1  1
1  2  2  2
1  2  3  3
1  2  3  4
```

This matrix for n is the same as the one for $n-1$ with an extra row and column appended and the sum of elements in the new row and column is $2(n-1)n/2+n=n^2$. Thus $T_n = T_{n-1}+n^2$, and since $T_1 = 1$, we have T_n is the sum of squares of the first n positive integers which is $n(n+1)(2n+1)/6$.

3. (10 points) We have

```
735 = 512 + 128 + 64 + 16 + 8 + 4 + 2 + 1 = 0000 0010 1101 1111

537 = 512 + 16  +  8 = 1.000011001 x 2^9  = 0 10001000 000011001  14 0's

since 9+127 = 136 = 128 + 8 = 10001000

Thus -537 = 1100 0100 0000 0110 0100 0000 0000 0000
```

4. (15 points) Since there is no `ndim` in the subroutine, the first 20 elements of `x` (which is its first column) will be reshaped into a 10 by 2 matrix whose first column are 1's and second column 0's, and thus

```
x in main      x in subroutine      xtx in sub       xtx in main
---------      ---------------      -----------      -----------

   1  1            1  0               10  0          10 0 0 0 0
   1  2            1  0                0  0           0 0 0 0 0
   1  3            1  0                               0 0 0 0 0
   1  4            1  0                               0 0 0 0 0
   1  5            1  0                               0 0 0 0 0
   1  6            1  0
   1  7            1  0
   1  8            1  0
   1  9            1  0             what's written:
   1 10            1  0             ---------------
   0  0
   0  0                               10  0
   0  0                                0  0
   0  0
   .  .
   .  .
   0  0
```

5. (20 points) If we let $B = \text{SWEEP}(k)A$ and $C = \text{SWEEP}(k)B$, then using the definition of SWEEP, it is easy to show for $i \neq k$ and $j \neq k$, that $C_{kk} = A_{kk}$, $C_{ik} = A_{ik}$, $C_{kj} = A_{kj}$, and $C_{ij} = A_{ij}$.

6. (10 points) We have $\chi^2_m = \sum_{i=1}^m Z_i^2$, where $Z_1, \ldots, Z_m$ are iid N(0,1) while

$$F_{m_1,m_2} = \frac{\chi^2_{m_1}/m_1}{\chi^2_{m_2}/m_2},$$

where the numerator and denominator of F_{m_1,m_2} are independent. Thus all we need is a N(0,1) random number generator to get χ^2's and F's (there are faster ways to do these but this would work).

7. (15 points) By random point, we mean that the probability that the point falls in a certain region is proportional to the area of that region, and thus the chance that a point in the square also is in the circle is the ratio of the two areas, namely $p = \pi/4$. Thus if we generate n points in the square (by generating two $U(-1,1)$'s) and let $\hat{p} = X/n$ where X is the number of points in the circle, we can get an estimate of π as $4\hat{p}$. Thus to have π fall in an interval of width .01, we need p to fall in an interval of width .01/4. Thus from the large sample binomial confidence interval for p we need

$$\frac{.01}{4} = \frac{1.96(2)\sqrt{p(1-p)}}{\sqrt{n}} \leq \frac{1.96}{\sqrt{n}},$$

since $p(1-p) \leq 1/4$. Solving this gives $n = 614,656$.