Homework 6 Solution, STAT 303, Spring 1996

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1 Problem 6.22

In (a), the parameter of interest is μ, the average time it takes for all mice to get through the maze. The point of her research is to show that a loud noise makes $\mu < 18$, so this is H_a. In (b), the parameter is μ, the mean exam score of all students who could take the class from this instructor. In order to show that $\mu > 50$, he has to make this H_a. In (c), the parameter is μ, the mean score of all possible students who pass the credit examination. The language department wants to see if μ is different than 24, so H_a is $\mu \neq 24$.

2 Definition of p-value

The p-value of a test is the probability, assuming H_0 is true, of observing the data we observed or anything else less consistent with H_0. The smaller the p-value is, the more evidence there is against the null hypothesis.

3 Calculating p-values

The calculating tests of hypotheses lab will actually give you the p-value for a problem, but it is important to have some understanding of where they come from.

Each of the 11 cases ends up being the area under either the Z, t, χ^2, or F curves, with the area being found to the right of the 'calculated test statistic' for right sided tests, to the left for left sided tests, and twice the area to the left or right (whichever is smaller) for two sided tests.

4 Problem 6.32

The parameter of interest is μ, the mean calcium level of *all* pregnant women. The question concerns whether μ is different from 9.5, the mean of the general population. Thus the null and alternative hypotheses are

$$H_0: \quad \mu = 9.5$$
$$H_a: \quad \mu \neq 9.5.$$

The question basically asks could the population mean be 9.5 and still have a sample of 180 women with a mean as far away from 9.5 as 9.57 (that is, greater than or equal to 9.57 or less than or equal to 9.43 which is as far below 9.5 as 9.57 is above 9.5). We answer this by calculating

$$Z_C = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{9.57 - 9.5}{.4/\sqrt{180}} = 2.348,$$

and since we have a two sided test, the p-value is twice the area under the Z curve to the right of $Z_C = 2.348$, which the lab gives as 0.019.

Thus we reject H_0 at the 0.05 or 0.025 level of significance but not at the 0.01 level.

This is strong evidence that the mean calcium level of pregnant women in Guatamala is different than that of the general population.

The 95% confidence interval for μ is 9.5116 to 9.6284.

5 Problem 6.34

The parameter of interest is μ, the mean of *all* numbers produced by the random number generator. Since the problem is concerned with whether μ is 0.5, we have

$$H_0: \quad \mu = 0.5$$
$$H_a: \quad \mu \neq 0.5.$$

The problem basically asks could μ be 0.5 and still get a sample of 100 numbers whose mean $\bar{X}$ is as far away or farther from 0.5 as the observed value of 0.4365 (that is, less than or equal to 0.4365 or greater than or equal to 0.5635, which is the same distance above 0.5 as 0.4365 is below 0.5). We get the chance of this happening (that is, the p-value) by calculating

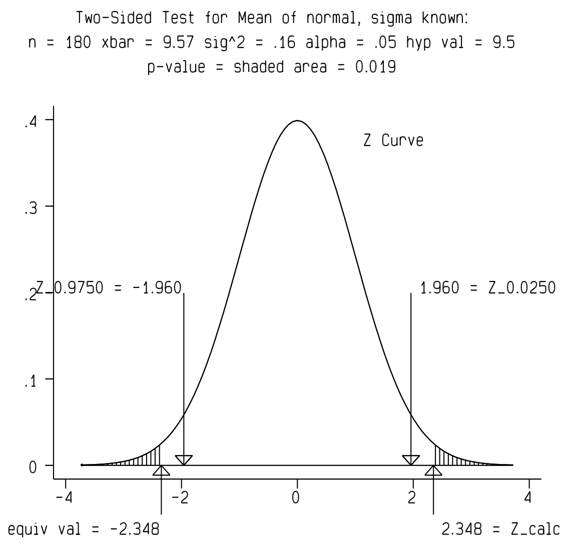

Figure 1: Graph for Problem 6.32

the Z test statistic as

$$Z_C = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{.4365 - 0.5}{0.2887/\sqrt{100}} = -2.199,$$

and since we have a two-sided test, the p-value is twice the area under the Z curve to the left of -2.199, which the lab gives as 0.028, which means we reject H_0 for $\alpha = 0.05$ but not for $\alpha = 0.01$.

Thus there is strong evidence that the random number generator is not producing the kind of numbers that it is supposed to.

6 Problem 6.24

The parameter of interest is $\mu_c - \mu_p$, where μ_c and μ_p are the means of the *populations* of blood pressures of all men getting the calcium supplement or the placebo. Since the problem talks about whether the calcium supplement *reduces* mean blood pressure, the null and alternative hypotheses are

$$\begin{aligned} H_0: &\quad \mu_c - \mu_p \geq 0 \\ H_a: &\quad \mu_c - \mu_p < 0, \end{aligned}$$

which makes the hypothesized value 0. The data consists of the difference of sample means (the value of which the problem doesn't tell us). The values of the difference of sample means that are less consistent with H_0 than the observed difference are any values *less than* the observed value.

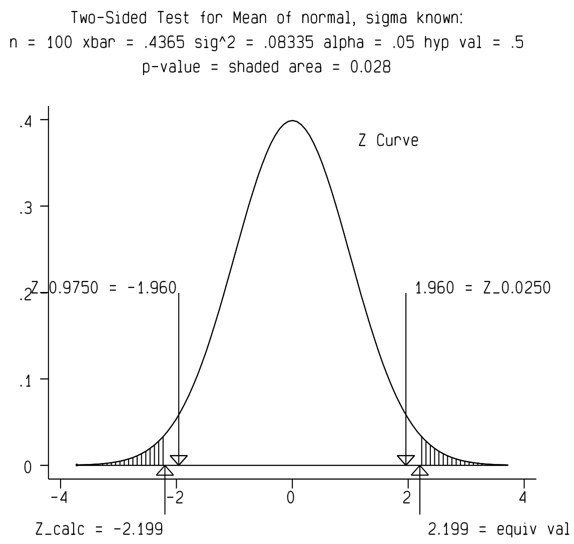

Figure 2: Graph for Problem 6.34

Thus the p-value in this problem is the probability of getting the observed difference of sample means or anything smaller if the two population means are the same. The problem tells us $p = 0.008$. Thus, if the population means are equal, there is almost no chance we would get the observed difference of sample means or anything less. This means there is overwhelming evidence against H_0 and we can conclude that the calcium supplement does reduce mean blood pressure in the population.

7 Problem 6.26

The study compares a) the average earnings of *all* men and women (μ_m and μ_f) and b) the average earnings of *all* whites and blacks (μ_w and μ_b). The conclusions talk about whether means are different, so the two alternative hypotheses are a) $H_a : \mu_m - \mu_f \neq 0$ and b) $H_a : \mu_w - \mu_b \neq 0$, and so the two p-values are the probabilities (assuming no difference in population mean earnings) of getting differences in sample means as large or larger than what were found in the study. For gender this is quite small (.038), while for race it is quite large. Thus we conclude that it is unlikely that the average earnings of men and women are the same but the data for race are quite consistent with there being no difference in average earnings between blacks and whites.

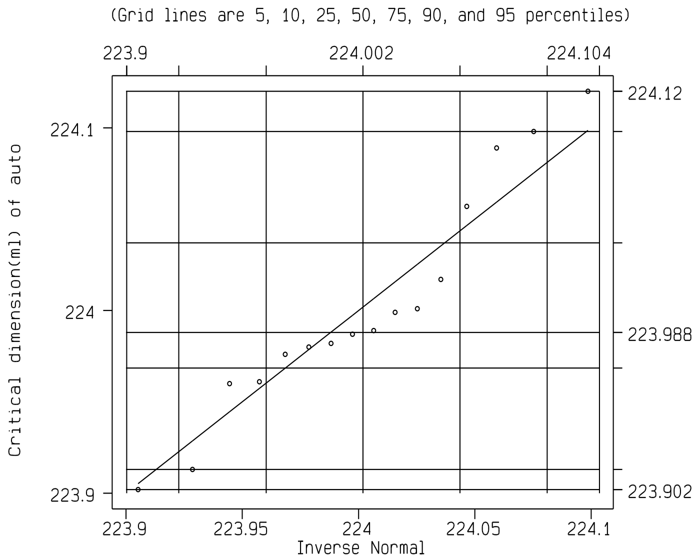

Figure 3: Normal Quantile Plot for Problem 7.14

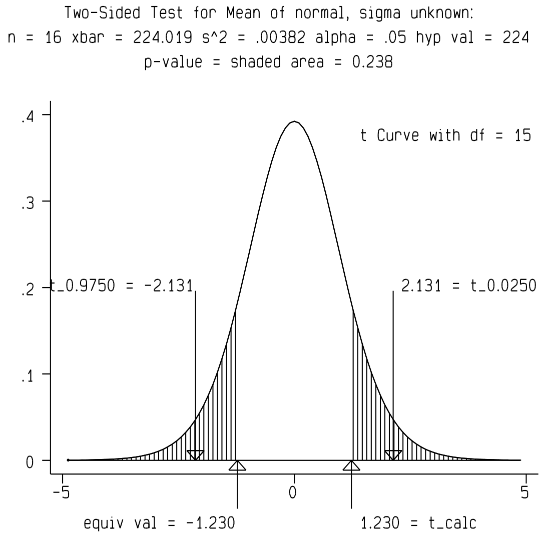

Figure 4: Graph for Problem 7.14

8 Problem 6.42

The confidence interval is (from the calculating confidence intervals lab) 59.80 to 60.20. Since 61.5 is not in this confidence interval, we would reject $H_0 : \mu = 61.5$, as we would reject the null hypothesis that $\mu = 63$.

9 Problem 7.14

Stataquest gives $\bar{X} = 224.019$ and $s = 0.0618$. The normal quantile plot, while not particularly straight, does not radically depart from a straight line so we can proceed.

The parameter of interest is μ, the mean dimension of *all* crankshafts, and since the question asks whether there is evidence that μ is *not* 224, we have

$$\begin{aligned} H_0 : &\quad \mu = 224 \\ H_a : &\quad \mu \neq 224. \end{aligned}$$

Thus we are being asked whether μ could be 224 and observe a sample mean as far away from 224 as 224.019 (that is, greater than or equal to 224.019 or less than or equal to 223.881). If we calculate (since this is Case 2)

$$t_C = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} = 1.230,$$

this question is converted to finding twice the area under the t curve (having $15 = n - 1$ degrees of freedom) to the right of 1.230, which the lab gives as a p-value of 0.238.

Thus if the population mean is in fact 224, the observed data is not unusual.

10 Problem 7.22

This is a *paired-data problem* (case 10) as the two filters are applied to the same cigarette. Thus the parameter of interest is μ_d, the mean of the differences of the two filters on *all* cigarettes. Since the question asks only how significant the difference is (and not whether it is positive or negative), we have a two tailed test, that is,

$$\begin{aligned} H_0 : &\quad \mu_d = 0 \\ H_a : &\quad \mu_d \neq 0. \end{aligned}$$

Thus the question is could the population mean difference be 0 and still get an observed mean difference in nine cigarettes of 1.32 (that is, greater than or equal to 1.32 or less than or equal to -1.32).

By calculating (Case 10)

$$t_C = \frac{\bar{d} - \mu_0}{s_d/\sqrt{n}} = 1.685,$$

the question is translated to finding twice the area to the right of 1.685 under the t curve (having $8 = n - 1$ degrees of freedom), which the lab gives as a p-value of 0.130.

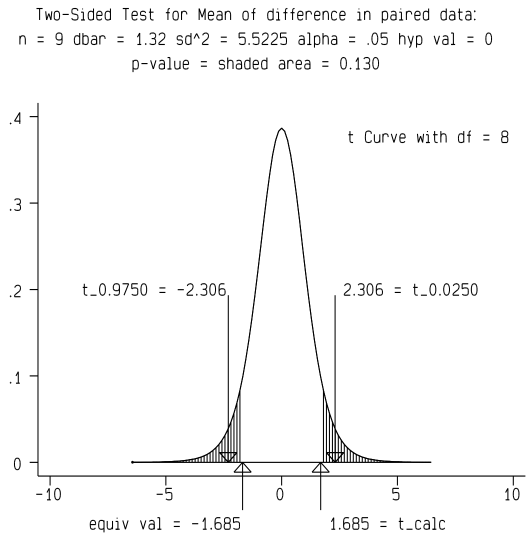

Figure 5: Graph for Problem 7.22

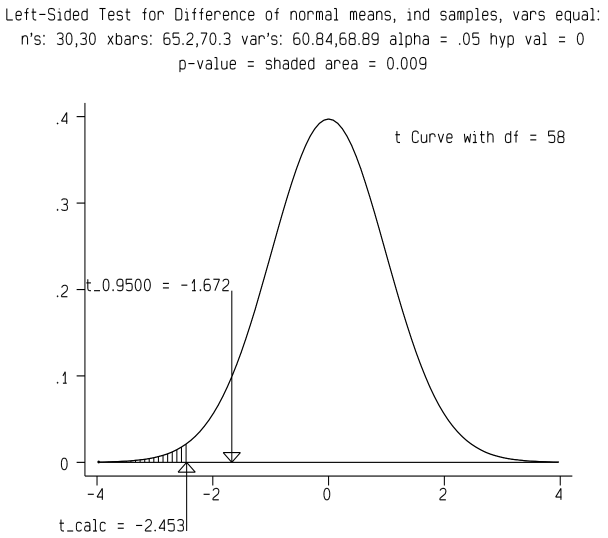

Figure 6: Graph for Problem 7.40

Thus the population mean difference could be 0 and easily obtain as large a sample mean difference as we did.

The 90% confidence interval is -0.1366 to 2.7766.

11 Problem 7.40

The parameter of interest is $\mu_b - \mu_p$, that is, the difference in means of the populations of pulse rates for people getting beta-blockers and placebo. Since the problem asks whether beta-blockers reduce mean pulse rate, we have

$$\begin{aligned} H_0: &\quad \mu_b - \mu_p \geq 0 \\ H_a: &\quad \mu_b - \mu_p < 0. \end{aligned}$$

Thus we have Case 8 or 9 depending on whether or not we can assume the two population variances are equal. The easiest way to check this is to find a 95% confidence interval for the ratio of variances (which the lab gives as 0.42 to 1.86) and see if 1 is in the interval (it is). Thus we use Case 8.

The question is could beta-blockers have no effect on mean pulse rate in the population and still observe a difference of means in samples of size 30 of as much as -5.1 (since this is a left sided test, differences more extreme than -5.1 are those *less than* -5.1).

By calculating

$$t_C = \frac{(\bar{X}_1 - \bar{X}_2) - 0}{s_P\sqrt{(1/n_1 + 1/n_2)}} = -2.543,$$

where

$$s_P^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} = 64.865,$$

the question gets translated into the area under the t curve (having $58 = n_1 + n_2 - 2$ degrees of freedom) to the left of -2.543, which the lab gives as a p-value of 0.009.

Thus if the beta-blocker has no effect in the population, there is very little chance of observing this large an effect in the samples.

Thus we conclude that beta-blockers do have an effect.

The 99% confidence for the difference of means is -10.64 to 0.44.

12 Problem 7.62

The parameter of interest is the ratio σ_1^2/σ_2^2 of variances of the *populations* of skilled and novice rowers. Since the problem only asks whether there is a difference in variances (and not whether one is bigger or smaller than the other), we have

$$\begin{aligned} H_0: &\quad \sigma_1^2/\sigma_2^2 = 1 \\ H_a: &\quad \sigma_1^2/\sigma_2^2 \neq 1, \end{aligned}$$

and thus the question is could the population variances be the same and still observe a ratio of sample

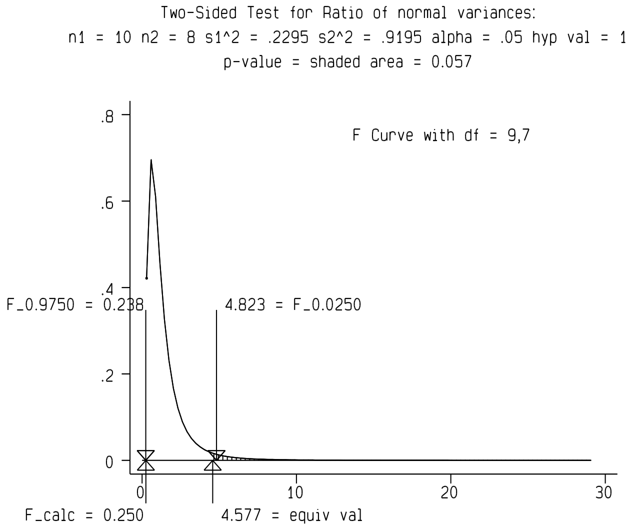

Figure 7: Graph for Problem 7.62

variances as different from 1 as (this is Case 7)

$$F_C = \frac{s_1^2}{s_2^2} = \frac{0.4791^2}{0.9589^2} = 0.25.$$

Since this is less than the expected value of 1 and we have a two sided test, the p-value is twice the area under the F curve (having $9 = n_1 - 1$ and $7 = n_2 - 1$ degrees of freedom) to the left of 0.25, which the lab gives as 0.057.

Thus if the population variances are the same, the chance of observing a ratio of sample variances as extreme or more than what we observed in the experiment is 0.057. Thus we reject H_0 for $\alpha = 0.10$ but not for $\alpha = 0.05$.

13 Problem 7.64

Again, we need to find the p-value for the null hypothesis of no difference in the variances of the populations of VOTS for children and adults. The F statistic is

$$F_C = \frac{33.89^2}{50.74^2} = 0.445,$$

and again, the p-value is twice the area to the left of 0.445 under the F curve having 9 and 19 degrees of freedom, which is 0.215, and thus we would not reject the equality of variances.

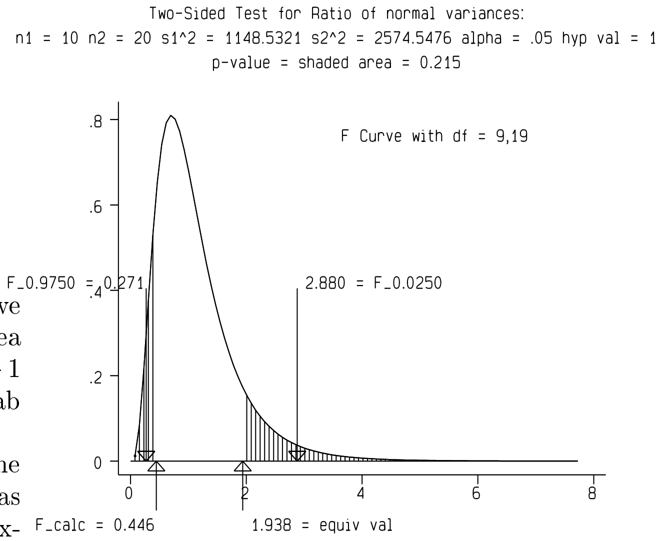

Figure 8: Graph for Problem 7.64

Confidence Intervals and Tests of Hypotheses for One and Two Samples

Situation				
Test Statistic	H_0	H_a	Rej H_0 if	p-value
Confidence Interval				

Case 1: Normal mean μ (σ^2 known) (`1 sample normal`)

Test Statistic	H_0	H_a	Rej H_0 if	p-value
$Z_C = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$	$\mu \leq \mu_0$	$\mu > \mu_0$	$Z_C > Z_\alpha$	$\Pr(Z > Z_C)$
	$\mu \geq \mu_0$	$\mu < \mu_0$	$Z_C < -Z_\alpha$	$\Pr(Z < Z_C)$
	$\mu = \mu_0$	$\mu \neq \mu_0$	$\lvert Z_C\rvert > Z_{\alpha/2}$	$2\Pr(Z > \lvert Z_C\rvert)$

$$\bar{X} \pm Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

Case 2: Normal mean μ (σ^2 unknown) (`1 sample t`)

Test Statistic	H_0	H_a	Rej H_0 if	p-value
$t_C = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}}$	$\mu \leq \mu_0$	$\mu > \mu_0$	$t_C > t_{\alpha,\nu}$	$\Pr(t > t_C)$
	$\mu \geq \mu_0$	$\mu < \mu_0$	$t_C < -t_{\alpha,\nu}$	$\Pr(t < t_C)$
	$\mu = \mu_0$	$\mu \neq \mu_0$	$\lvert t_C\rvert > t_{\alpha/2,\nu}$	$2\Pr(t > \lvert t_C\rvert)$

$$\bar{X} \pm t_{\alpha/2,\nu}\frac{s}{\sqrt{n}}, \qquad \nu = n-1$$

Case 3: Nonnormal mean μ (σ^2 unknown, n 'large') (`1 sample t`): Formulas are the same as Case 2

Case 4: Normal variance σ^2 (`1 sample variance`)

Test Statistic	H_0	H_a	Rej H_0 if	p-value
$\chi^2_C = \dfrac{(n-1)s^2}{\sigma_0^2}$	$\sigma^2 \leq \sigma_0^2$	$\sigma^2 > \sigma_0^2$	$\chi^2_C > \chi^2_{\alpha,\nu}$	$\Pr(\chi^2 > \chi^2_C)$
	$\sigma^2 \geq \sigma_0^2$	$\sigma^2 < \sigma_0^2$	$\chi^2_C < \chi^2_{1-\alpha,\nu}$	$\Pr(\chi^2 < \chi^2_C)$
	$\sigma^2 = \sigma_0^2$	$\sigma^2 \neq \sigma_0^2$	$\chi^2_C > \chi^2_{\alpha/2,\nu}$ or $\chi^2_C < \chi^2_{1-\alpha/2,\nu}$	$2\min(\Pr(\chi^2 > \chi^2_C),$ $1 - \Pr(\chi^2 > \chi^2_C))$

$$\left(\frac{(n-1)s^2}{\chi^2_{\alpha/2,\nu}}, \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,\nu}}\right), \qquad \nu = n-1$$

Case 5: Proportion π (exact CI for small n)

$$\left(\frac{XF_{1-\alpha/2,\nu_1,\nu_2}}{(n-X+1) + XF_{1-\alpha/2,\nu_1,\nu_2}}, \frac{(X+1)F_{\alpha/2,\nu_3,\nu_4}}{(n-X) + (X+1)F_{\alpha/2,\nu_3,\nu_4}}\right), \qquad X = np, \qquad \nu_1 = 2X,$$

$$\nu_2 = 2(n-X+1), \quad \nu_3 = 2(X+1), \quad \nu_4 = 2(n-X)$$

Case 6: Proportion π ($\min(n\pi, n(1-\pi)) \geq 5$) (`1 sample proportion`)

Test Statistic	H_0	H_a	Rej H_0 if	p-value
$Z_C = \dfrac{p - \pi_0}{\sqrt{\pi_0(1-\pi_0)/n}}$	$\pi \leq \pi_0$	$\pi > \pi_0$	$Z_C > Z_\alpha$	$\Pr(Z > Z_C)$
	$\pi \geq \pi_0$	$\pi < \pi_0$	$Z_C < -Z_\alpha$	$\Pr(Z < Z_C)$
	$\pi = \pi_0$	$\pi \neq \pi_0$	$\lvert Z_C\rvert > Z_{\alpha/2}$	$2\Pr(Z > \lvert Z_C\rvert)$

$$p \pm Z_{\alpha/2}\sqrt{p(1-p)/n}$$

Confidence Intervals and Tests of Hypotheses for One and Two Samples (cont)

Situation				
Test Statistic	H_0	H_a	Rej H_0 if	p-value
Confidence Interval				

Case 7: Ratio of normal variances (independent samples) (`2 sample variances`)

Test Statistic	H_0	H_a	Rej H_0 if	p-value
$F_C = \dfrac{s_1^2/s_2^2}{\delta}$	$\sigma_1^2/\sigma_2^2 \le \delta$	$\sigma_1^2/\sigma_2^2 > \delta$	$F_C > F_{\alpha,\nu_1,\nu_2}$	$\Pr(F > F_C)$
	$\sigma_1^2/\sigma_2^2 \ge \delta$	$\sigma_1^2/\sigma_2^2 < \delta$	$F_C < F_{1-\alpha,\nu_1,\nu_2}$	$\Pr(F < F_C)$
	$\sigma_1^2/\sigma_2^2 = \delta$	$\sigma_1^2/\sigma_2^2 \ne \delta$	$F_C > F_{\alpha/2,\nu_1,\nu_2}$ or $F_C < F_{1-\alpha/2,\nu_1,\nu_2}$	$2\min(\Pr(F > F_C),\ 1 - \Pr(F > F_C))$

$$\left(\frac{s_1^2/s_2^2}{F_{\alpha/2,\nu_1,\nu_2}}, \frac{s_1^2/s_2^2}{F_{1-\alpha/2,\nu_1,\nu_2}}\right), \qquad \nu_1 = n_1 - 1, \qquad \nu_2 = n_2 - 1$$

Case 8: Difference of two normal means (σ_1^2, σ_2^2 unknown but equal, independent samples) (`2 sample t`)

Test Statistic	H_0	H_a	Rej H_0 if	p-value
$t_C = \dfrac{(\bar{X}_1 - \bar{X}_2) - \delta}{\sqrt{s_P^2(1/n_1 + 1/n_2)}}$	$\mu_1 - \mu_2 \le \delta$	$\mu_1 - \mu_2 > \delta$	$t_C > t_{\alpha,\nu}$	$\Pr(t > t_C)$
	$\mu_1 - \mu_2 \ge \delta$	$\mu_1 - \mu_2 < \delta$	$t_C < -t_{\alpha,\nu}$	$\Pr(t < t_C)$
	$\mu_1 - \mu_2 = \delta$	$\mu_1 - \mu_2 \ne \delta$	$\lvert t_C\rvert > t_{\alpha/2,\nu}$	$2\Pr(t > \lvert t_C\rvert)$

$$(\bar{X}_1 - \bar{X}_2) \pm t_{\alpha/2,\nu}\sqrt{s_P^2(1/n_1 + 1/n_2)}, \qquad s_P^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1 + n_2 - 2}, \qquad \nu = n_1 + n_2 - 2$$

Case 9: Difference of two normal means (σ_1^2, σ_2^2 unknown, unequal, independent samples) (`2 sample t`)

Test Statistic	H_0	H_a	Rej H_0 if	p-value
$t_C = \dfrac{(\bar{X}_1 - \bar{X}_2) - \delta}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$	$\mu_1 - \mu_2 \le \delta$	$\mu_1 - \mu_2 > \delta$	$t_C > t_{\alpha,\nu}$	$\Pr(t > t_C)$
	$\mu_1 - \mu_2 \ge \delta$	$\mu_1 - \mu_2 < \delta$	$t_C < -t_{\alpha,\nu}$	$\Pr(t < t_C)$
	$\mu_1 - \mu_2 = \delta$	$\mu_1 - \mu_2 \ne \delta$	$t_C > \lvert t_{\alpha/2,\nu}\rvert$	$2\Pr(t > \lvert t_C\rvert)$

$$(\bar{X}_1 - \bar{X}_2) \pm t_{\alpha/2,\nu}\sqrt{s_1^2/n_1 + s_2^2/n_2}, \qquad \nu = \frac{\left(s_1^2/n_1 + s_2^2/n_2\right)^2}{\dfrac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \dfrac{\left(s_2^2/n_2\right)^2}{n_2 - 1}}$$

Case 10: Mean of differences in normal paired data (σ^2 unknown) (`paired t`)

Test Statistic	H_0	H_a	Rej H_0 if	p-value
$t_C = \dfrac{\bar{d} - \mu_0}{s_d/\sqrt{n}}$	$\mu_d \le \mu_0$	$\mu_d > \mu_0$	$t_C > t_{\alpha,\nu}$	$\Pr(t > t_C)$
	$\mu_d \ge \mu_0$	$\mu_d < \mu_0$	$t_C < -t_{\alpha,\nu}$	$\Pr(t < t_C)$
	$\mu_d = \mu_0$	$\mu_d \ne \mu_0$	$\lvert t_C >> t_{\alpha/2,\nu}$	$2\Pr(t > \lvert t_C\rvert)$

$\bar{d} \pm t_{\alpha/2,\nu}\dfrac{s_d}{\sqrt{n}}$, $\quad \nu = n - 1$, $\bar{d}$ and s_d are mean and standard deviation of differences in pairs.

Case 11: Difference $\pi_1 - \pi_2$ of proportions (independent samples, $\min(n_1\pi_1, n_1(1-\pi_1)) \ge 5$, $\min(n_2\pi_2, n_2(1-\pi_2)) \ge 5$) (`2 sample proportions`)

Test Statistic	H_0	H_a	Rej H_0 if	p-value
$Z_C = \dfrac{p_1 - p_2}{\sqrt{p(1-p)(1/n_1 + 1/n_2)}}$	$\pi_1 - \pi_2 \le 0$	$\pi_1 - \pi_2 > 0$	$Z_C > Z_\alpha$	$\Pr(Z > Z_C)$
	$\pi_1 - \pi_2 \ge 0$	$\pi_1 - \pi_2 < 0$	$Z_C < -Z_\alpha$	$\Pr(Z < Z_C)$
	$\pi_1 - \pi_2 = 0$	$\pi_1 - \pi_2 \ne 0$	$\lvert Z_C\rvert > Z_{\alpha/2}$	$2\Pr(Z > \lvert Z_C\rvert)$

$$(p_1 - p_2) \pm Z_{\alpha/2}\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}, \qquad p = \frac{n_1p_1 + n_2p_2}{n_1 + n_2}$$