Joint pdf calculation

Example 1 *Consider random variables X, Y with pdf f(x,y) such that*

$$f(x,y) = \begin{cases} 6x^2y, & 0 < x < 1,\ 0 < y < 1 \\ 0, & \textit{otherwise.} \end{cases}.$$

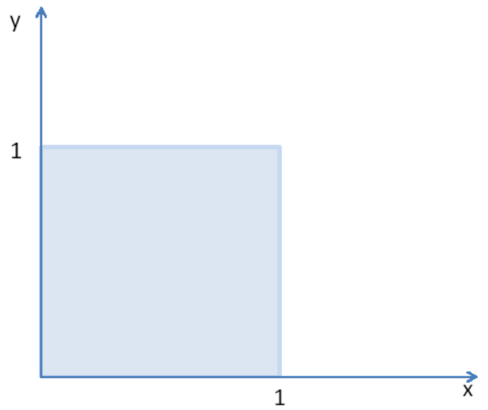

Figure1. $\{(x,y)|0 < x < 1, 0 < y < 1\}$

- Note that $f(x,y)$ is a valid pdf because

$$\begin{aligned} P(-\infty < X < \infty, -\infty < Y < \infty) &= P(0 < X < 1, 0 < Y < 1) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy \\ &= 6\int_0^1\int_0^1 x^2ydxdy \\ &= 6\int_0^1 y\left\{\int_0^1 x^2dx\right\}dy \\ &= 6\int_0^1 \frac{y}{3}dy \\ &= 1. \end{aligned}$$

- Following the definition of the marginal distribution, we can get a marginal distribution for X. For $0 < x < 1$,

$$f(x) \equiv \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 f(x,y)dy = \int_0^1 6x^2ydy = 3x^2\int_0^1 2ydy = 3x^2.$$

If $x \leq 0$ or $x \geq 1$, $f(x) = 0$ (Figure1).

- Similarly we can get a marginal distribution for Y. For $0 < y < 1$,

  $$f(y) \equiv \int_{-\infty}^{\infty} f(x,y)dx = \int_0^1 f(x,y)dx \ = \int_0^1 6x^2ydx \ = 2y\int_0^1 3x^2dx = 2y$$

  If $y \le 0$ or $y \ge 1$, $f(y) = 0$ (Figure1).

- $$E\left(X\right) \equiv \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 3x^3dx = \frac{3}{4},$$

  $$Var\left(X\right) \equiv E\left(X^2\right) - \left\{E\left(X\right)^2\right\} = \int_{-\infty}^{\infty} x^2 f(x)dx - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2.$$

- Let's calculate $P\left(X > Y\right)$. For fixed x_o, $0 < y < x_o$ (Figure 2).

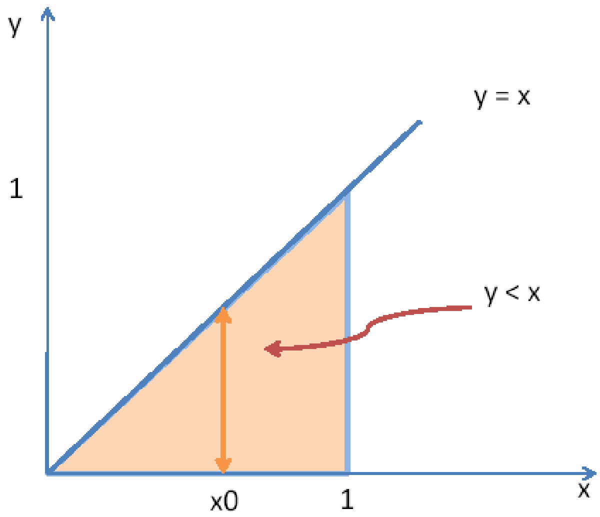

Figure 2. $\{(x,y)|y < x, 0 < x < 1, 0 < y < 1\}$

Therefore

$$\begin{aligned} P\left(X > Y\right) &= \int_0^1 \left\{\int_0^x f(x,y)dy\right\} dx \\ &= \int_0^1 \left\{\int_0^x 6x^2ydy\right\} dx \\ &= \int_0^1 3x^4dx = \frac{3}{5}. \end{aligned}$$

Example 2 *Consider random variables X, Y with pdf f(x,y) such that*

$$f(x,y) = \begin{cases} 8xy, & 0 < x < 1, \quad 0 < y < x \\ 0, & \textit{otherwise.} \end{cases}.$$

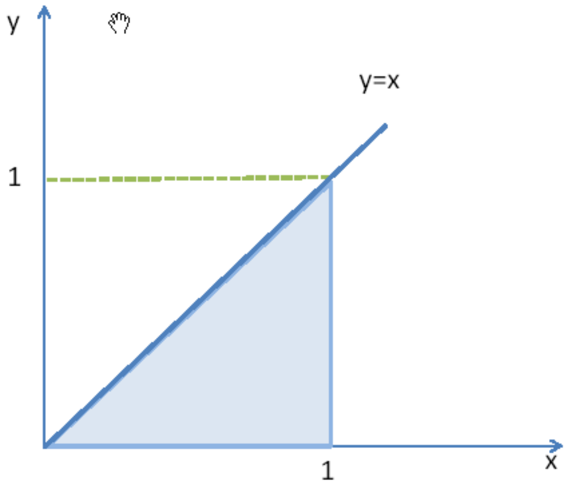

Figure 3. $\{(x, y)|0 < x < 1, 0 < y < x\}$

- *Note that given* x_o*, domain of* y *is given by* $0 < y \leq x_o$*. (Figure4)*

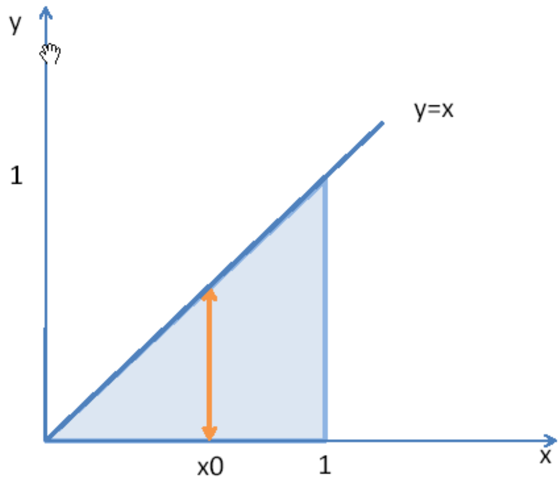

Figure 4.

$$
\begin{aligned}
P(-\infty < X < \infty, -\infty < Y < \infty) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx \\
&= \int_0^1 \left\{ \int_0^x f(x,y)dy \right\} dx \\
&= 4\int_0^1 x \left\{ \int_0^x 2ydy \right\} dx \\
&= 4\int_0^1 x^3 dx \\
&= 1.
\end{aligned}
$$

- Marginal distribution of X is, for $0 < x < 1$,

  $$f(x) \equiv \int_{-\infty}^{\infty} f(x,y)dy = \int_0^x f(x,y)dy = 4x\int_0^x 2ydy = 4x^3.$$

  If $x \le 0$ or $x \ge 1$, $f(x) = 0$ (Figure3).

- Similarly, for a given $0 < y_o < 1$, domain of x is given by $y_o \le x < 1$. For $0 < y < 1$,

  $$f(y) \equiv \int_{-\infty}^{\infty} f(x,y)dx = \int_y^1 f(x,y)dx = 4y\int_y^1 2xdx = 4y\left(1-y^2\right).$$

  If $y \le 0$ or $y \ge 1$, $f(y) = 0$ (Figure3).

- $$
\begin{aligned}
E(X) &= \int_0^1 xf(x)dx = \int_0^1 4x^4dx = \frac{4}{5}, \\
Var(X) &= E\left(X^2\right) - \left\{E\left(X\right)^2\right\} = \int_0^1 x^2 f(x)dx - \left(\frac{4}{5}\right)^2 = \frac{2}{3} - \left(\frac{4}{5}\right)^2.
\end{aligned}
$$

- Let's calculate $P(Y < \frac{1}{2}X)$ (Figure 5).

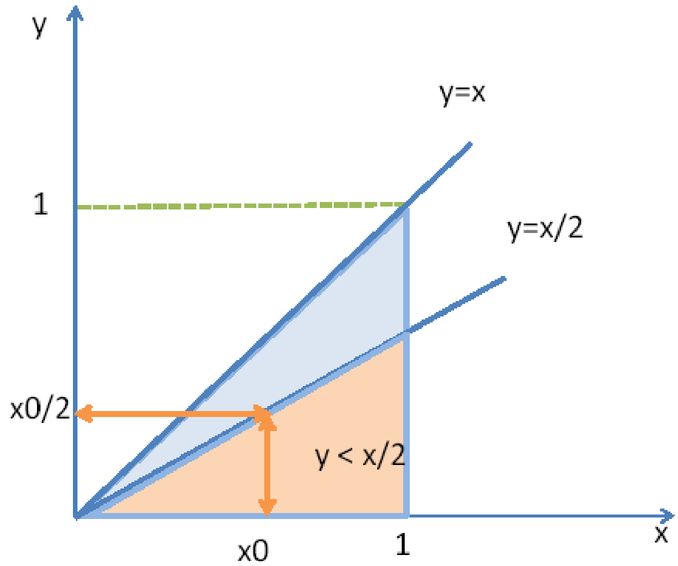

Figure 5.

$$
\begin{aligned}
P\left(Y<\frac{1}{2}X\right) &= \int_0^1\int_0^{x/2} f(x,y)dydx \\
&= \int_0^1 4\left\{\int_0^{x/2} 2ydy\right\}dx \\
&= \int_0^1 x^2dx \\
&= \frac{1}{3}.
\end{aligned}
$$

Example 3 *Consider random variables X, Y with pdf f(x,y) such that*

$$
f(x,y)=\begin{cases} 3\left(x+y\right), & 0<x<1,\ 0<y<1,\ 0<x+y<1 \\ 0, & \textit{otherwise}. \end{cases}.
$$

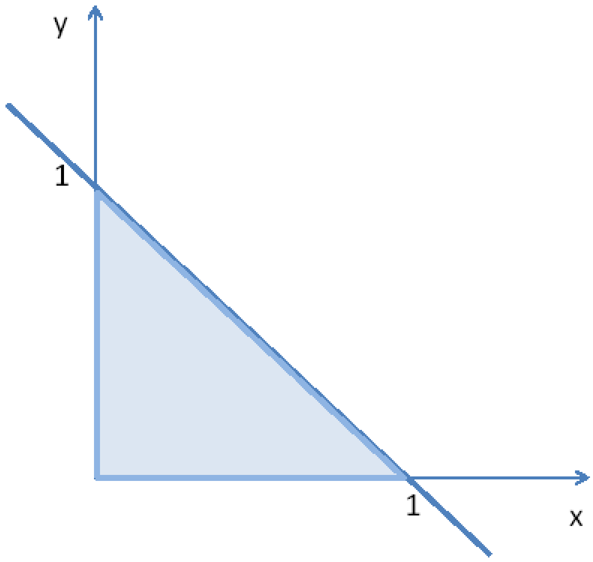

Figure 6. $\{(x,y)|0<x<1, 0<y<1, 0<x+y<1\}$

- *Note that given* x_o*, domain of* y *is given by* $0 \le y \le 1-x_o$*. Therefore,*

$$\begin{aligned} P(-\infty<X<\infty,-\infty<Y<\infty) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx \\ &= \int_0^1 \left[\int_0^{1-x} f(x,y)dy\right] dx \\ &= 3\int_0^1 \left[\int_0^{1-x} (x+y)\,dy\right] dx \\ &= 3\int_0^1 \frac{1}{2}(1-x)(1+x)\,dx \\ &= 1. \end{aligned}$$

- *Note again that given* $0 \le x_o \le 1$*, domain of* y *is given by* $0 < y \le 1-x_o$ *so that for* $0<x<1$*,*

$$f(x) \equiv \int_{-\infty}^{\infty} f(x,y)dy \int_0^{1-x} 3(x+y)\,dy = \frac{3}{2}(1-x)(1+x).$$

If $x \le 0$ *or* $x \ge 1$*,* $f(x)=0$ *(Figure6).*

- *Similarly, given* $0<y_o<1$*, domain of* x *is given by* $0 < x \le 1-y_o$

so that for $0 < y < 1$,

$$f(y) \equiv \int_{-\infty}^{\infty} f(x,y)dx = \int_{0}^{1-y} 3\left(x+y\right)dx = 3\left[\frac{1}{2}x^2 + yx\right]_0^{1-y} = \frac{3}{2}\left(1-y\right)\left(1+y\right).$$

If $y \leq 0$ *or* $y \geq 1$, $f(y) = 0$ *(Figure6).*

- $$\begin{aligned} E\left(X\right) &\equiv \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_0^1 \left\{\frac{3}{2}x\left(1-x\right)\left(1+x\right)\right\} dx \\ &= \frac{3}{2}\int_0^1 \left(x-x^3\right)dx = \frac{3}{2}\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = \frac{3}{8}, \end{aligned}$$

$$\begin{aligned} E\left(X^2\right) &\equiv \int_{-\infty}^{\infty} x^2 f(x)dx \\ &= \frac{3}{2}\int_0^1 \left(x^2-x^4\right)dx = \frac{3}{2}\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \frac{1}{5}, \end{aligned}$$

$$Var\left(X\right) \equiv E\left(X^2\right) - \left[E\left(X\right)\right]^2 = \frac{1}{5} - \left(\frac{3}{8}\right)^2.$$

- *Let's calculate* $P(X > Y)$. *(Figure7,8)*

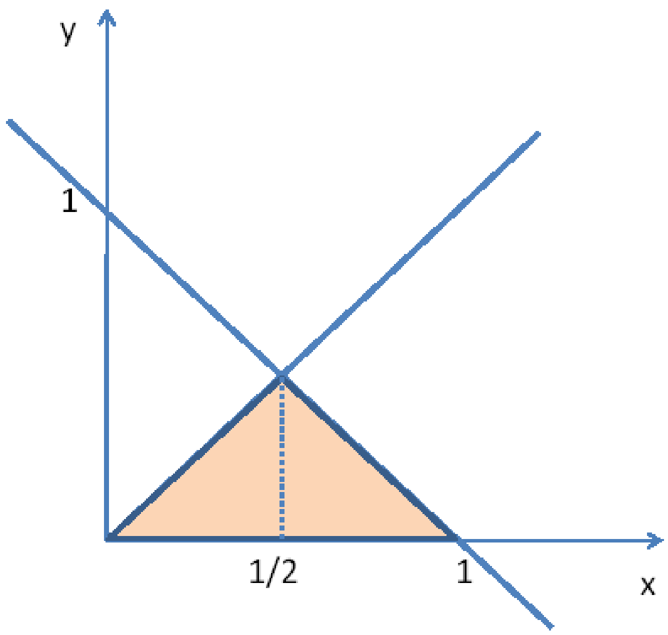

Figure 7. $\{(x,y)|0 < x < 1, 0 < y < 1, 0 < x+y < 1, x > y\}$

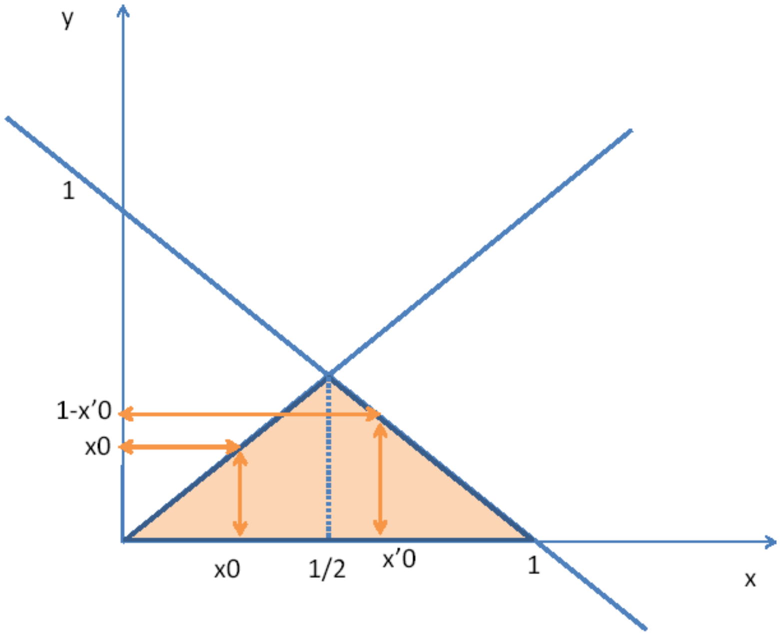

Figure 8

$$
\begin{aligned}
P\left(X>Y\right) &= \int_0^{1/2}\int_0^{x} f(x,y)dydx + \int_{1/2}^{1}\int_0^{1-x} f(x,y)dydx \\
&= \int_0^{1/2}\left\{\int_0^{x} 3\left(x+y\right)dy\right\}dx + \int_{1/2}^{1}\left\{\int_0^{1-x} 3\left(x+y\right)dy\right\}dx \\
&= \frac{3}{2}\int_0^{1/2} 3x^2dx + \frac{1}{2}\int_{1/2}^{1}\left(3-3x^2\right)dx \\
&= \frac{1}{2}.
\end{aligned}
$$