Factorial Experiment

Often it is of interest to study the effect of more than one factor upon the response of experimental units.

Could vary the factors one at a time. However, this is slow and doesn’t provide information about interrelations between factors.

The factorial experiment allows one to vary factors simultaneously and thereby study interrelations among factors as well as the influence of individual factors.

A *factorial experiment* is one in which each treatment (i.e., each combination of factor levels) is used at least once.

We’ll consider only the case of two factors.
Suppose a factor $A$ has $a$ levels and a factor $B$ has $b$ levels. There are thus $ab$ treatments. A single replication of a factorial experiment consists of the assignment of $ab$ experimental units at random, each to a different treatment.

Suppose we have just one replication of a factorial experiment, and let $X_{ij}$ be the response to level $i$ of factor $A$ and level $j$ of $B$.

A possible model for the data is one having the same form as in our RBD.

$$X_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}, \quad i = 1, \ldots, a, \quad j = 1, \ldots, b.$$  

$\mu$: overall mean  
$\alpha_i$: factor $A$ effect  
$\beta_j$: factor $B$ effect
Consider now the difference in average response of levels 1 and 3 of factor $A$ for a fixed level $j$ of factor $B$. This is

$$(\mu + \alpha_1 + \beta_j) - (\mu + \alpha_3 + \beta_j) = \alpha_1 - \alpha_3,$$

which is the same for any of the $b$ levels of $B$.

Graphically, this looks as follows:

![Graph showing the difference in average response for levels 1 and 3 of factor A for a fixed level j of factor B.](image-url)
**Interaction**

Often, the effects of the two factors are not additive. In other words, the effect of factor $A$ may depend on the level of factor $B$. In this case, the factors are said to **interact**. An example of an interaction is shown in the following graph.

![Graph Illustrating Interaction]

The distinguishing graphical feature of an interaction is that the lines are not parallel.
The appropriate linear model for two factors with interaction is

\[ X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}, \]

where \( \gamma_{ij} \) is the interaction term.

Now consider the difference in average response of levels 1 and 3 of factor \( A \) when factor \( B \) is fixed at level \( j \):

\[
(\mu + \alpha_1 + \beta_j + \gamma_{1j}) - (\mu + \alpha_3 + \beta_j + \gamma_{3j}) = (\alpha_1 - \alpha_3) + (\gamma_{1j} - \gamma_{3j}).
\]

This depends on \( j \), the level of \( B \).

In order to detect interaction, at least two replications of the experiment are needed. Let \( X_{ijk} \) be the \( k \)th observation from treatment \( (i, j) \), i.e., level \( i \) of factor \( A \) and level \( j \) of factor \( B \).
The observation may be decomposed as follows:

\[ X_{ijk} = \bar{X} + (\bar{X}_{i..} - \bar{X}) + (\bar{X}_{.j} - \bar{X}) + (\bar{X}_{i.j} - \bar{X}_{i..} - \bar{X}_{.j} + \bar{X}) + (X_{ijk} - \bar{X}_{ij.}) \]

\( \bar{X} \): grand mean of all data

\( \bar{X}_{i..} - \bar{X} \): deviation due to factor \( A \)

\( \bar{X}_{.j} - \bar{X} \): deviation due to factor \( B \)

\( \bar{X}_{i.j} - \bar{X}_{i..} - \bar{X}_{.j} + \bar{X} \): interaction

\( X_{ijk} - \bar{X}_{ij.} \): error
ANOVA Table for Factorial Experiment

Let $r$ be the number of replications.

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>SS</th>
<th>df</th>
<th>MS</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor A</td>
<td>$SS_A$</td>
<td>$a - 1$</td>
<td>$MS_A$</td>
<td>$F_A$</td>
</tr>
<tr>
<td>Factor B</td>
<td>$SS_B$</td>
<td>$b - 1$</td>
<td>$MS_B$</td>
<td>$F_B$</td>
</tr>
<tr>
<td>Interaction</td>
<td>$SS_{AB}$</td>
<td>$(a - 1)(b - 1)$</td>
<td>$MS_{AB}$</td>
<td>$F_{AB}$</td>
</tr>
<tr>
<td>Error</td>
<td>$SS_E$</td>
<td>$ab(r - 1)$</td>
<td>$MS_E$</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>$TSS$</td>
<td>$abr - 1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Important!

One should first test the hypothesis of no interaction. This is

\[ H_0 : \gamma_{ij} = 0 \] for all \( i \) and \( j \).

This hypothesis is rejected at level of significance \( \alpha \) if \( F_{AB} = MS_{AB}/MS_E \) is larger than \( F_{(a-1)(b-1),ab(r-1),\alpha} \).

When the interaction is significant, one should not perform the \( F \) tests for main effects. Why?

- A significant main effects test may be misleading in that levels of \( A \), say, may only differ at one level of \( B \).

- An insignificant main effects test may also be misleading. Could have \( A_1 \) higher than \( A_2 \) at level 1 of \( B \) and \( A_1 \) the same amount lower than \( A_2 \) at level 2 of \( B \).
Strategy in two-factor experiment

1. Test for interaction.

2.(a) If the test for interaction is not significant, test

\[ H_0 : \alpha_1 = \cdots = \alpha_a = 0 \]

and

\[ H_0 : \beta_1 = \cdots = \beta_b = 0 \]

using \( F_A \) and \( F_B \). Tukey’s procedure may be used to compare individual levels of each factor.

2.(b) If the test for interaction is significant, plot the treatment means and try to explain the interaction. Can use Tukey’s procedure to compare treatment means.
Effects of Planting Date and Fertilizer on Soybean Yield

An agronomic experiment was conducted to assess the effects of the date of planting (early or late) and the type of fertilizer (none, Aero, Na or K) on the yield of soybeans.

32 homogeneous experimental plots were available. The 8 treatments were randomly assigned to the plots with 4 plots assigned to each treatment.

Plot of Treatment Means for Soybean Data
Rough conclusions from graph:

- Late planting appears to be better than early except perhaps when no fertilizer is used.

- When planting early, it seems better not to use fertilizer.

- Are these rough conclusions substantiated by an ANOVA?
### ANOVA Table for Soybean Data

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>SS</th>
<th>df</th>
<th>MS</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Date</td>
<td>32</td>
<td>1</td>
<td>32</td>
<td>10.41</td>
</tr>
<tr>
<td>Fertilizer</td>
<td>16.4</td>
<td>3</td>
<td>5.47</td>
<td>1.78</td>
</tr>
<tr>
<td>Interaction</td>
<td>38.4</td>
<td>3</td>
<td>12.8</td>
<td>4.17</td>
</tr>
<tr>
<td>Error</td>
<td>73.7</td>
<td>24</td>
<td>3.07</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>160.5</td>
<td>31</td>
<td></td>
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</table>

The $F$ statistic for interaction is significant at the 0.05 level of significance. ($F_{3,24,.05} = 3.01$)

This substantiates that there is an interaction and lends credence to our conclusions on the previous page.
Somewhat more definitive conclusions can be drawn by comparing the difference between two treatment means with the LSD value:

\[ \text{LSD} = t_{ab(r-1),\alpha/2} \sqrt{\frac{MS_E}{r}} \sqrt{2} \]

If the difference between two treatment means is larger than \( \text{LSD} \), then it’s reasonable to conclude that they are significantly different.

**Note:** This procedure is ”liberal” and should be used as a rough guide. More on this shortly.

For the soybean data,

\[ \text{LSD} = t \sqrt{3.07} \sqrt{\frac{2}{4}} = 2.56. \]
Conclusions based on LSD:

- Aero, Na and K are never significantly different.

- "No fertilizer" better than any of Aero, Na or K when planting early.

- "No fertilizer" not significantly different than other three for late planting.