Conjugate Families of Priors are Not Unique

The following idea comes from

> Diaconis, P. and Ylvisaker, D. (1979). Conjugate priors for exponential families. *Annals of Statistics* **7** 269-281.

Let $\mathcal{P}$ be a conjugate family of priors for the likelihoods $\mathcal{F} = \{f(\boldsymbol{y}|\boldsymbol{\theta}) : \boldsymbol{\theta} \in \Theta\}$. This means that for each $\pi \in \mathcal{P}$ and for every $\boldsymbol{y}$ in the sample space, $\pi(\cdot|\boldsymbol{y}) \in \mathcal{P}$, where

$$\pi(\boldsymbol{\theta}|\boldsymbol{y}) = \frac{f(\boldsymbol{y}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})}{m(\boldsymbol{y})}.$$

Now, let h be a positive, continuous function of $\boldsymbol{\theta}$ that is bounded over Θ. For each $\pi \in \mathcal{P}$, define

$$\pi^*(\boldsymbol{\theta}) = \frac{h(\boldsymbol{\theta})\pi(\boldsymbol{\theta})}{\int_\Theta h(\boldsymbol{t})\pi(\boldsymbol{t})\,d\boldsymbol{t}}.$$

Define $\mathcal{P}^* = \{\pi^* \propto h\pi : \pi \in \mathcal{P}\}$. Is $\mathcal{P}^*$ a conjugate family for $\mathcal{F}$? Yes!

Let $\pi^* \in \mathcal{P}^*$, $\boldsymbol{y}$ be any element of the sample space, and consider

$$\begin{aligned} \pi^*(\boldsymbol{\theta}|\boldsymbol{y}) &\propto f(\boldsymbol{y}|\boldsymbol{\theta})\pi^*(\boldsymbol{\theta}) \\ &\propto f(\boldsymbol{y}|\boldsymbol{\theta})h(\boldsymbol{\theta})\pi(\boldsymbol{\theta}) \\ &= h(\boldsymbol{\theta})m(\boldsymbol{y})\pi(\boldsymbol{\theta}|\boldsymbol{y}). \end{aligned}$$

Now, $\pi(\,\cdot\,|\boldsymbol{y}) \in \mathcal{P}$, which means that $h\pi(\,\cdot\,|\boldsymbol{y})$ is proportional to an element of $\mathcal{P}^*$, which implies that $\pi^*(\,\cdot\,|\boldsymbol{y}) \in \mathcal{P}^*$.

Therefore, $\mathcal{P}^*$ is a conjugate family for $\mathcal{F}$.

So, conjugate families are decidedly *not* unique. Given one conjugate family, we can generate uncountably many more conjugate families using the technique above.