Filling in Gaps in the Proof on pp. 240N-242N

We want to show that if X_t has density g, then so does X_{t+1}. If G_{t+1} is the cdf of X_{t+1}, then

$$G_{t+1}(y) = \int_{-\infty}^{\infty} F(y|x)g(x)\,dx,$$

where

$$F(y|x) = P(X_{t+1} \le y | X_t = x).$$

We have

$$F(y|x) = P(X_{t+1} \le y \cap A | X_t = x) + P(X_{t+1} \le y \cap A^c | X_t = x),$$

where A is the event that the draw from $q(\cdot|x)$ is accepted.

Let U and Y be independent random variables such that Y has density $q(\cdot|x)$ and $U \sim U(0,1)$. Then

$$\begin{aligned} P(X_{t+1} \le y \cap A | X_t = x) &= P(Y \le y, U \le \alpha(x, Y)) \\ &= \int_{-\infty}^{y} \int_{0}^{\alpha(x,\theta)} q(\theta|x)\,du\,d\theta \\ &= \int_{-\infty}^{y} q(\theta|x)\alpha(x,\theta)\,d\theta. \end{aligned}$$

Now, suppose that $y < x$. We have

$$P(X_{t+1} \le y \cap A^c | X_t = x) \le P(X_{t+1} \le y, X_{t+1} = x | X_t = x) \tag{1}$$

since the event $\{X_{t+1} \le y \cap A^c\}$ implies A^c, which implies $X_{t+1} = x$. But, it's impossible that $X_{t+1} \le y$ when $X_{t+1} = x$, and so $P(X_{t+1} \le y, X_{t+1} = x | X_t = x) = 0$. This implies that $P(X_{t+1} \le y \cap A^c | X_{t+1} = x) = 0$, because of (1).

Now let $y \ge x$. Conditional on the event $X_t = x$, the event $\{X_{t+1} \le y \cap A^c\}$ is equal to A^c. This is true because (a) $\{X_{t+1} \le y \cap A^c\} \subset A^c$, and (b) when $X_t = x$, $A^c \subset \{A^c \cap X_{t+1} = x\} \subset \{A^c \cap X_{t+1} \le y\}$. Therefore, when $y \ge x$

$$\begin{aligned} P(X_{t+1} \le y \cap A^c | X_t = x) &= P(A^c | X_t = x) \\ &= \int_{-\infty}^{\infty} P(A^c | X_t = x, Y = y) q(y|x)\,dy \\ &= \int_{-\infty}^{\infty} (1 - \alpha(x,y)) q(y|x)\,dy. \end{aligned}$$

From this point on the proof on pp. 241-242N is pretty rigorous.