Example 5 Silvey, *Statistical Inference*

An airline has the option of buying 10 second-hand aircraft.

θ of these 10 will give 1000 hours flying time without major breakdown.

\$1000p profit on each plane that doesn't break down.

Airline loses \$1000q on each plane that does break down within 1000 hrs.

One plane may be tested at a cost of \$1000r.

The two elements in $\mathcal{Y}$ are y_1, the tested plane is satisfactory, and y_2, the tested plane is unsatisfactory.

$$f(y_1|\theta) = \frac{\theta}{10} \qquad f(y_2|\theta) = 1 - \frac{\theta}{10}$$

The action space is $\mathcal{A} = \{a_1, a_2\}$, where

$$\begin{aligned} a_1 &= \text{"purchase ten planes"} \\ a_2 &= \text{"don't purchase the planes."} \end{aligned}$$

The loss function is defined as

$$\begin{aligned} L(\theta, a_1) &= r - \theta p + (10 - \theta)q, \\ L(\theta, a_2) &= r \quad \forall\ \theta. \end{aligned}$$

Now consider four decision rules:

(a) $\delta_1(y) \equiv a_1$. $R(\theta, \delta_1) = r - \theta p + (10 - \theta)q$.

(b) $\delta_2(y) \equiv a_2$. $R(\theta, \delta_2) = r$ for all θ.

(c) $\delta_3(y_1) = a_1$, $\delta_3(y_2) = a_2$

$$
\begin{aligned}
R(\theta, \delta_3) &= E_\theta[L(\theta, \delta_3(Y))] \\
&= (\theta/10)[r - \theta p + (10-\theta)q] \\
&\quad +(1-\theta/10)r \\
&= r - (\theta/10)[\theta p - (10-\theta)q].
\end{aligned}
$$

(d) $\delta_4(y_1) = a_2$, $\delta_4(y_2) = a_1$

$$
\begin{aligned}
R(\theta, \delta_4) &= (1-\theta/10)[r - \theta p + (10-\theta)q] \\
&\quad +(\theta/10)r
\end{aligned}
$$

For illustration's sake, suppose $r = 1$, $p = 2$ and $q = 3$. In this case,

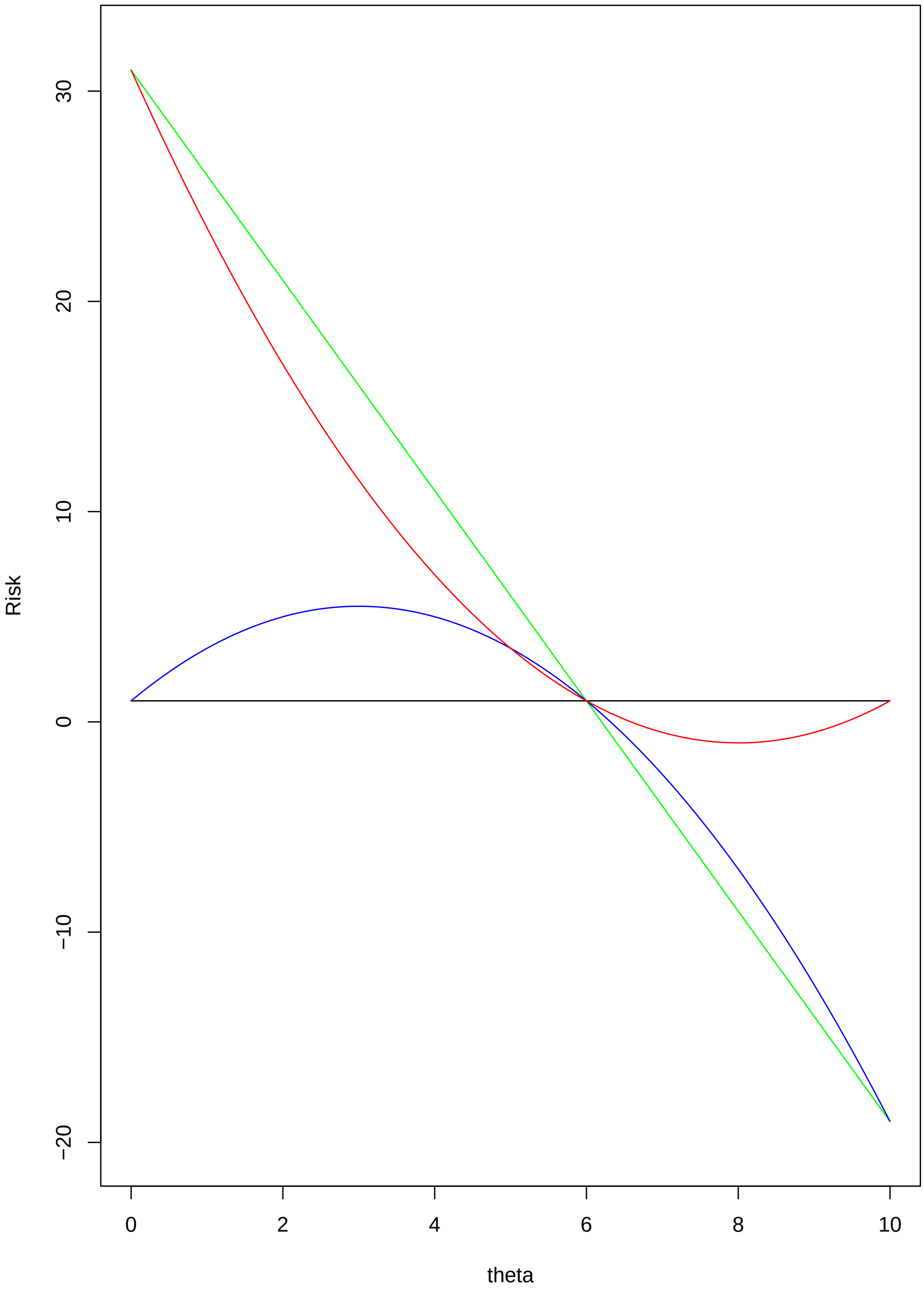
$$R(\theta, \delta_1) = 31 - 5\theta, \qquad R(\theta, \delta_2) \equiv 1$$

$$R(\theta, \delta_3) = 1 + 3\theta - \theta^2/2$$

and

$$R(\theta, \delta_4) = 31 - 8\theta + \theta^2/2.$$

Risk functions for airplane example



δ_1: green line $\quad \delta_2$: black line $\quad \delta_3$: blue line $\quad \delta_4$: red line

One can verify that when $r = 1$, $p = 2$ and $q = 3$, δ_4 is inadmissible, since $R(\theta, \delta_3) \leq R(\theta, \delta_4)$ at each $\theta = 0, 1, \ldots, 10$.

Suppose we incorporate prior information into the problem and use the Bayes principle to choose a rule. Let

$$\pi(\theta) = \frac{1}{11}, \quad \theta = 0, 1, \ldots, 10.$$

For the most general version of the problem,

$$r(\pi, \delta_1) = r + 5(q - p), \qquad r(\pi, \delta_2) = r,$$

$$\begin{aligned} r(\pi, \delta_3) &= r - 3.5(p + q) + 5q \\ &= r - 3.5p + 1.5q \end{aligned}$$

and

$$r(\pi, \delta_4) = r - 1.5p + 3.5q.$$

Note that, regardless of the values of r, p and q,

$$r(\pi, \delta_3) \leq r(\pi, \delta_4).$$

Now suppose $r = 1$, $p = 2$ and $q = 3$. Then $r(\pi, \delta_1) = 6$, $r(\pi, \delta_2) = 1$, $r(\pi, \delta_3) = -1.5$, $r(\pi, \delta_4) = 8.5$ and hence δ_3 is the Bayes rule.

While δ_4 is inadmissible in the special case above, it is not inadmissible in general. Suppose θ is somewhat large. Then according to δ_4 we are not likely to purchase the planes. This is good when q/p is sufficiently large.