Solutions to Selected Problems from Assignment # 1

6.9(c) As a function of $\theta$, $f(x|\theta)/f(y|\theta)$ is proportional to

$$\frac{\prod_{i=1}^{n} [1 + e^{-(y_i - \theta)}]^2}{\prod_{i=1}^{n} [1 + e^{-(x_i - \theta)}]^2}.$$ 

The last quantity is constant for all $\theta$ iff

$$\prod_{i=1}^{n} \left(1 + e^{\theta e^{-y_i}}\right) = c \prod_{i=1}^{n} \left(1 + e^{\theta e^{-x_i}}\right) \quad (*)$$

for all $\theta$ and some constant $c$. $(*)$ is equivalent to

$$\prod_{i=1}^{n} \left(1 + e^{\theta e^{-y_{(i)}}}\right) = c \prod_{i=1}^{n} \left(1 + e^{\theta e^{-x_{(i)}}}\right).$$

Define the two polynomials

$$g_1(z) = \prod_{i=1}^{n} \left(1 + ze^{-y_{(i)}}\right) \quad \text{and} \quad g_2(z) = \prod_{i=1}^{n} \left(1 + ze^{-x_{(i)}}\right).$$

Now, $g_1(z) = cg_2(z)$ for all $z > 0$ iff the zeroes of $g_1$ are the same as the zeroes of $g_2$. Clearly, the zeroes of $g_1$ are

$$-e^{y(n)} \leq -e^{y(n-1)} \leq \cdots \leq -e^{y(1)}$$

and those of $g_2$ are

$$-e^{x(n)} \leq -e^{x(n-1)} \leq \cdots \leq -e^{x(1)}.$$

It follows that $(*)$ holds for all $\theta$ iff $x_{(i)} = y_{(i)}$, $i = 1, \ldots, n$, and so the order statistics are minimal sufficient.

6.9(d) The order statistics are minimal sufficient. The proof is similar to the one in 6.9(c).

6.9(e) $f(x|\theta)/f(y|\theta)$ is constant for all $\theta$ iff

$$\sum_{i=1}^{n} |y_{(i)} - \theta| - \sum_{i=1}^{n} |x_{(i)} - \theta|$$
is constant for all $\theta$. It’s not difficult to show that the last quantity is constant for all $\theta$ iff
\[
\sum_{\{i: y(i) > \theta\}} y(i) - \sum_{\{i: x(i) > \theta\}} x(i) + (n_x - n_y)\theta
\] (**) is constant in $\theta$, where $n_x$ and $n_y$ are, respectively, the number of $x_i$’s and $y_i$’s larger than $\theta$.

Certainly, if $x(i) = y(i)$ for each $i$, then (**) is 0 for all $\theta$. Now suppose that (**) is constant for all $\theta$, but $x(i) \neq y(i)$ for some $i$. Let $j$ be the largest integer in the set $\{i : x(i) \neq y(i)\}$, and without loss of generality suppose that $x(j) < y(j)$. Let $z$ be the largest $y(i)$ such that $y(i) < y(j)$. If there is no such $y(i)$ define $z = -\infty$. For $\max(x(j), z) < \theta < y(j)$,
\[
n_x - n_y = -1
\]
and
\[
\sum_{\{i: y(i) > \theta\}} y(i) - \sum_{\{i: x(i) > \theta\}} x(i)
\]
is constant. It follows that (**) is a straight line with nonzero slope on the interval $(\max(x(j), z), y(j))$, and hence nonconstant. But this is a contradiction to our assumption that (**) is constant, and so we have proven that the order statistics are minimal sufficient.