Stat689: Statistical Aspects of Machine Learning and Data Mining, Fall 2009;

Homework 1 (09/22/09)

Instructor: Marc G.Genton
Student: Ganggang Xu

2.4 (a). For 2 and 3 dimensions, one has

$$Vol_r(A) = \begin{cases} \pi A^2 & r = 2; \\ \frac{4}{3}\pi A^3 & r = 3; \end{cases}$$

(b). Let ϵ be any positive number less than A, then

$$\frac{Vol_r(A) - Vol_r(A-\epsilon)}{Vol_r(A)} = \frac{A^r - (A-\epsilon)^r}{A^r} = 1 - (1-\epsilon/A)^r \to 1$$

as $r \to \infty$ for any ϵ. Therefore, almost all the volume inside the sphere tends to be concentrated along a "thin shell" closer to the surface of the sphere than to the center.

2.5

$$\frac{Vol_{sphere,r}}{Vol_{cube,r}} = \frac{S_r A^r/r}{(2A)^r} = \frac{\pi^{r/2}}{2^r\Gamma(r/2+1)}$$

Note that

$$\frac{\pi^{r/2}}{2^r} \to 0 \quad \text{as} \quad r \to \infty$$

and $\Gamma(r/2+1) \to \infty$, one has

$$\frac{Vol_{sphere,r}}{Vol_{cube,r}} \to 0$$

as $r \to \infty$.

3.6 Since any linear combination of random variables is still normal, it surffices to find the expectation and covariance matrix of Y,

$$E(Y) = E(AX+b) = A(EX) + b = A\mu + b$$

and

$$Var(Y) = Var(AX+b) = AVar(X)A^T = A\Sigma A^T$$

therefore

$$Y \sim N_s(A\mu + b, A\Sigma A^T)$$

3.9 (a). Obviously, $Y - \Sigma_{YX}\Sigma_{XX}^{-1}X$ is normal, and

$$E(Y - \Sigma_{YX}\Sigma_{XX}^{-1}X) = EY - \Sigma_{YX}\Sigma_{XX}^{-1}(EX) = \mu_Y - \Sigma_{YX}\Sigma_{XX}^{-1}\mu_X$$

and

$$\begin{aligned} Var(Y - \Sigma_{YX}\Sigma_{XX}^{-1}X) &= Var(Y) - 2\Sigma YX\Sigma_{XX}^{-1}Cov(X,Y) + \Sigma_{YX}\Sigma_{XX}^{-1}Var(X)\Sigma_{XX}^{-1}\Sigma_{XY} \\ &= \Sigma_{YY.X} \end{aligned}$$

therefore, $Y - \Sigma_{YX}\Sigma_{XX}^{-1}X \sim N_s\left(\mu_Y - \Sigma)YX\Sigma_{XX}^{-1}\mu_X, \Sigma_{YY.X}\right)$.

(b). Since $Y|X = (Y - \Sigma_{YX}\Sigma_{XX}^{-1}X)|X + \Sigma_{YX}\Sigma_{XX}^{-1}X$, by the property of normal random variables, we have $Y|X$ is normally distributed with

$$E(Y|X) = \mu_Y - \Sigma)YX\Sigma_{XX}^{-1}\mu_X + \Sigma_{YX}\Sigma_{XX}^{-1}X = \mu_Y + \Sigma_{YX}\Sigma_{XX}^{-1}(X - \mu_X)$$

and

$$Var(Y|X) = \Sigma_{YY.X}$$

that is , $Y|X \sim N_s\left(\mu_Y + \Sigma_{YX}\Sigma_{XX}^{-1}(X - \mu_X), \Sigma_{YY.X}\right)$.

(c). If Σ_{XX} is singular, then it is easily to show that

$$E(Y - \Sigma_{YX}\Sigma_{XX}^{-}X) = \mu_Y - \Sigma_{YX}\Sigma_{XX}^{-}\mu_X$$

and

$$\begin{aligned} Var(Y - \Sigma_{YX}\Sigma_{XX}^{-}X) &= \Sigma_{YY} - \Sigma_{YX}\Sigma_{XX}^{-}\Sigma_{XX}\Sigma_{XX}^{-}\Sigma_{XY} \\ &= \Sigma_{YY} - \Sigma_{YX}\Sigma_{XX}^{-}\Sigma_{XY} \end{aligned}$$

let $\Sigma_{YY.X}^{*} = \Sigma_{YY} - \Sigma_{YX}\Sigma_{XX}^{-}\Sigma_{XY}$, one has $Y - \Sigma_{YX}\Sigma_{XX}^{-}X \sim N_s\left(\mu_Y - \Sigma)YX\Sigma_{XX}^{-}\mu_X, \Sigma_{YY.X}^{*}\right)$. Following the same arguments in part (b), we can get

$$Y|X \sim N_s\left(\mu_Y + \Sigma_{YX}\Sigma_{XX}^{-}(X - \mu_X), \Sigma_{YY.X}^{*}\right)$$

3.20 Suppose R is a $n \times n$ matrix, then

$$det(R) = (1 + (n-1)\rho)(1-\rho)^{n-1}$$

and notice that $J = \mathbf{1}_{n\times 1}\mathbf{1}_{n\times 1}^T$, one has

$$R^{-1} = [(1-\rho)I + \rho J]^{-1} = \frac{1}{1-\rho}I - \frac{\rho}{(1-\rho)[1+(n-1)\rho]}J$$