Definition 5.13 Given an optimisation problem with domain $\Omega \subseteq \mathbb{R}^n$,

$$\boxed{\begin{array}{lll} \text{minimise} & f(\mathbf{w}), & \mathbf{w} \in \Omega, \\ \text{subject to} & g_i(\mathbf{w}) \leq 0, & i = 1, \ldots, k, \\ & h_i(\mathbf{w}) = 0, & i = 1, \ldots, m, \end{array}}$$

we define the *generalised Lagrangian function* as

$$\begin{aligned} L(\mathbf{w}, \boldsymbol{\alpha}, \boldsymbol{\beta}) &= f(\mathbf{w}) + \sum_{i=1}^{k} \alpha_i g_i(\mathbf{w}) + \sum_{i=1}^{m} \beta_i h_i(\mathbf{w}) \\ &= f(\mathbf{w}) + \boldsymbol{\alpha}' \mathbf{g}(\mathbf{w}) + \boldsymbol{\beta}' \mathbf{h}(\mathbf{w}). \end{aligned}$$

We can now define the Lagrangian dual problem.

Definition 5.14 The *Lagrangian dual problem* of the primal problem of Definition 5.1 is the following problem:

$$\boxed{\begin{array}{ll} \text{maximise} & \theta(\boldsymbol{\alpha}, \boldsymbol{\beta}), \\ \text{subject to} & \boldsymbol{\alpha} \geq \mathbf{0}, \end{array}}$$

where $\theta(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \inf_{\mathbf{w} \in \Omega} L(\mathbf{w}, \boldsymbol{\alpha}, \boldsymbol{\beta})$. The value of the objective function at the optimal solution is called the *value of the problem.*

We begin by proving a theorem known as the weak duality theorem, which gives one of the fundamental relationships between the primal and dual problems and has two useful corollaries.

Theorem 5.21 *(Kuhn–Tucker) Given an optimisation problem with convex domain* $\Omega \subseteq \mathbb{R}^n$,

$$\begin{array}{lll} \textit{minimise} & f(\mathbf{w}), & \mathbf{w} \in \Omega, \\ \textit{subject to} & g_i(\mathbf{w}) \leq 0, & i = 1, \ldots, k, \\ & h_i(\mathbf{w}) = 0, & i = 1, \ldots, m, \end{array}$$

with $f \in C^1$ convex and g_i, h_i affine, necessary and sufficient conditions for a normal point $\mathbf{w}^*$ to be an optimum are the existence of $\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*$ such that

$$\begin{aligned} \frac{\partial L(\mathbf{w}^*, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)}{\partial \mathbf{w}} &= \mathbf{0}, \\ \frac{\partial L(\mathbf{w}^*, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)}{\partial \boldsymbol{\beta}} &= \mathbf{0}, \\ \alpha_i^* g_i(\mathbf{w}^*) = 0,\ i &= 1, \ldots, k, \\ g_i(\mathbf{w}^*) \leq 0,\ i &= 1, \ldots, k, \\ \alpha_i^* \geq 0,\ i &= 1, \ldots, k. \end{aligned}$$

Remark 5.22 The third relation is known as Karush–Kuhn–Tucker complementarity condition. It implies that for active constraints, $\alpha_i^* \geq 0$, whereas for inactive constraints $\alpha_i^* = 0$.