ON MODELS FOR THE PROBABILITY OF FATIGUE FAILURE OF A STRUCTURE

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INTRODUCTION

N.J. Hoff[1] has summarized the arguments in favor of the view that the safety of an aircraft needs to be established by means of probabilistic considerations, utilizing statistical data on the loads to be encountered by the aircraft, and on the strength values of the material of which it is composed. Only by use of the theory of probability can the airplane designer develop a design philosophy which will enable him to take into account static failure due to a rare excessive load, fatigue failure, and failure due to creep deformations.

This paper represents an attempt by a person trained in probability theory to survey some of the problems involved in evaluating structural safety. Part I of this Report is a review of the probabilistic considerations involved in evaluating the strength of materials, and the construction of so called S-N curves. In Part II is briefly advanced a probabilistic model for the life before fatigue failure of a structure. The ideas involved in Part II are rather complex and only an outline of the ideas involved are presented.

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PART I

REVIEW OF PROBABILISTIC CONSIDERATIONS INVOLVED

1. THE STATISTICAL NATURE OF THE FATIGUE LIFE OF MATERIALS

Let a specimen of a material be subjected to a large number of applications of a load at a given level of stress (the force per unit of surface is called stress). *The fatigue life of a material at a stress level* s *is defined to be the number* N(s) *of applications of stress* s *that has to be applied to the material in order for it to break.* That, for a given stress level s, N(s) is a random variable is an experimentally very well determined fact; apparently identical specimens, tested under carefully controlled conditions, do not break after exactly the same number of load applications but rather at widely scattered numbers of load applications. Alternately phrased, if one were to build a number of airplane wings to the same specifications and fly them under identical conditions the number of miles flown before failure of the wings would not be the same for all the airplane wings. The reader should note that the units in which the fatigue life of an item is expressed are at the disposal of the investigator; the fatigue life may just as well be defined as the number of miles flown or the number of hours in service, as the number of load applications at a certain stress level.

Let us consider here the number N(s) of load applications at stress level s that a specimen of material sustains before breaking. To obtain the probability distribution of N(s), one should test a large number of specimens at stress level s and obtain a histogram such as is sketched in Figure 1. The x-axis in Figure 1 is divided into a number of intervals (centered at values x_1, $x_2, \ldots$) over which one draws a line representing the fraction of specimens tested whose observed fatigue life N(s) lies in the interval. If we denote by $\hat{f}(x_i)$ the fraction of specimens whose observed fatigue lives are contained in the interval centered at x_i then $\hat{f}(x_i)$ may be taken as an estimate of the probability that the random variable N(s) will have an observed value in the interval centered at x_i.

At this point it may be worthwhile to introduce some notation. The *distribution function* of a random variable X is a function F(x) defined by

$$F(x) = \text{Prob}[X \leqslant x]$$

In words, F(x) is the probability that an observed value of X will be less than or equal to the assigned number x. In fatigue studies one also considers the *survival function* of a random variable X which is a function L(x) defined by

$$L(x) = 1 - F(x) = \text{Prob}[X > x]$$

In words, L(x) is the probability that an observed value of X will exceed x. A random variable X is said to be continuous if there is a function f(x), called the probability density function of X, such that

$$F(x) = \int_{-\infty}^{x} f(x').dx'$$

$$\text{Prob}[x < X \leqslant (x+dx)] = f(x).dx$$

The problem before us is that of obtaining the probability law of the random variable N(s), *either by specifying its distribution function* F(x) *or its probability density function* f(x). From a large number of observations of N(s) one may estimate the true probability density function f(x) by drawing a smooth function f(x) which 'fits' the observed histogram $\hat{f}(x_i)$. This procedure does not lead to a closed analytical expression for the probability density function. To avoid this difficulty, one usually attempts to estimate the true probability density function f(x) of N(s) by first, on the basis of theoretical considerations, assuming that the true probability density function belongs to a specified family of probability density functions, which is indexed by a finite number of parameters. One then estimates the true probability density function by estimating the parameters.

The method of estimating the probability law by means of estimating the parameters of a pre-assigned family of probability density functions is to be especially preferred to merely fitting a curve to the observed histogram in cases where our interest is in determining the values of N(s) which have extremely small probability (10^{-3} to 10^{-6}) of not being attained. The number of observations used to construct the histogram may only be of the order of 10^2. By curve fitting alone one is not justified in extrapolating the histogram so as to find the values of N(s) which have extremely small probabilities of being attained.

In the next section, we discuss various models which have been proposed for the probability law of the number of load cycles to failure.

To conclude this section, we define some statistical terminology which will be employed in this Report. Given a random variable X with probability density function f(x), the mode, median, and mean are measures of the location of the values of the random variable which are most likely to be observed, while the variance is a measure of the spread of the interval in which the observed value of the random variable is most likely to fall. The precise mathematical definitions of these concepts are as follows: the mode m_0 is the point (if it exists) at which the probability density function f(x) achieves its maximum, the median m_e is the point at which the distribution function F(x) is equal to 1/2, the mean m is defined by the integral

$$m = \int_{-\infty}^{\infty} x\, f(x).dx,$$

and the variance σ^2 is defined by the integral

$$\sigma^2 = \int_{-\infty}^{\infty} (x - m)^2\, f(x).dx$$

2. SOME POSSIBLE PROBABILITY LAWS FOR THE NUMBER OF LOAD CYCLES TO FAILURE

Several families of probability density functions have been suggested for the theoretical probability law of the random variable N(s), the number of load cycles to failure of a material specimen at stress level s. The most important of these are

1) the lognormal (or logarithmic-normal) distribution

2) the extreme value distribution

3) the Weibull (or third asymptotic extreme value) distribution

4) the Gamma (or χ^2) distribution.

All these distributions have the property that they are skewed in a manner similar to that of the histogram in Figure 1.

A random variable X which takes only positive values is defined to have a lognormal distribution with parameters μ and σ (where μ and σ are real constants such that $-\infty < \mu < \infty$ and $\sigma > 0$) if its probability density function is given by

$$f(x) = \left.\begin{cases} \dfrac{1}{x\sigma\sqrt{2\pi}} \exp\left[\dfrac{-1}{2\sigma^2}\{\log x - \mu\}^2\right] & x > 0 \\ 0 & x < 0 \end{cases}\right\} \quad (1)$$

The lognormal distribution derives its name from the fact that a random variable X is lognormal if and only if logX is normally distributed with mean μ and variance σ^2. A random variable X which is lognormal with parameters μ and σ has

$$\left.\begin{aligned} \text{mode} &= e^{\mu-\sigma^2} \\ \text{median} &= e^{\mu} \\ \text{mean} &= e^{\mu+\frac{1}{2}\sigma^2} \\ \text{variance} &= e^{2\mu+\sigma^2}[e^{\sigma^2} - 1] \end{aligned}\right\} \quad (2)$$

For a proof of these relationships, see Reference 2, p.8. Indeed, the reader should consult this Reference for a complete discussion of the properties of the lognormal distribution.

A random variable X is defined to have an extreme value distribution with parameters u and β (where u and β are constants such that $-\infty < u < \infty$ and $\beta > 0$) if its distribution function is given by

$$F(x) = 1 - \exp\left\{-e^{-\left(\frac{x-u}{\beta}\right)}\right\} \quad (3)$$

or if its probability density function is given by

$$f(x) = \exp\left\{-\left(\frac{x-u}{\beta}\right) - e^{-\left(\frac{x-u}{\beta}\right)}\right\} \tag{4}$$

The parameters u and β of the extreme value distribution play the role of location and scale parameters in much the same way that the mean m and variance σ^2 of the normal distribution are respectively measures of location and scale. The extreme value distribution with parameters u and β has

$$\left.\begin{aligned} \text{mode} &= u \\ \text{median} &= u + (0.36651)\beta \\ \text{mean} &= u + (0.57722)\beta \\ \text{variance} &= \frac{\pi}{\sqrt{6}}\beta \end{aligned}\right\} \tag{5}$$

For a proof of these relations, and a complete discussion of the extreme value distribution, the reader should consult the recent book by E.J. Gumbel[3].

A random variable X, which takes only values greater than some number ϵ, is defined to have a Weibull distribution with parameters v and k (where $v \geqslant \epsilon$ and $k > 1$) if its distribution function is given by

$$\left.\begin{aligned} F(x) &= 1 - \exp\left\{-\left(\frac{x-\epsilon}{v-\epsilon}\right)^k\right\} & x \geqslant \epsilon \\ &= 0 & x < \epsilon \end{aligned}\right\} \tag{6}$$

or if its probability density function is given by

$$\left.\begin{aligned} f(x) &= \frac{k}{v-\epsilon}\left(\frac{x-\epsilon}{v-\epsilon}\right)^{k-1}.\exp\left\{-\left(\frac{x-\epsilon}{v-\epsilon}\right)^k\right\} & x > \epsilon \\ &= 0 & x < \epsilon \end{aligned}\right\} \tag{7}$$

The Weibull distribution has (see Reference 3, pp.280-1)

$$\left.\begin{aligned} \text{mode} &= \epsilon + (v-\epsilon)\left(1 - \frac{1}{k}\right)^{1/k} \\ \text{median} &= \epsilon + (v-\epsilon)(\log 2)^{1/k} \\ \text{mean} &= \epsilon + (v-\epsilon)\Gamma\left(1 + \frac{1}{k}\right) \end{aligned}\right. \tag{8}$$

$$\text{variance} = (v-\epsilon)^2\left[\Gamma\left(1+\frac{2}{k}\right) - \Gamma^2\left(1+\frac{1}{k}\right)\right] \qquad (8)$$

where $\Gamma(x)$ is the Gamma function (defined in Equation (10) below). For ϵ and v fixed, the mode, median, and mean of a Weibull distribution all tend to v as k tends to ∞. Consequently, the parameter v may be taken as the parameter of location of Weibull distribution in preference to the usual characteristics of mode, median, or mean.

A random variable X, which takes only positive values, is defined to have a Gamma distribution (or Pearson Type III distribution) with parameters r and λ (where r = 1, 2, ... and λ is a positive constant) if its probability density function is given by

$$\left.\begin{aligned} f(x) &= \frac{\lambda}{\Gamma(r)}(\lambda x)^{r-1}.e^{-\lambda x} && x > 0 \\ &= 0 && x < 0 \end{aligned}\right\} \qquad (9)$$

where $\Gamma(r)$ is the Gamma function defined by

$$\Gamma(r) = \int_0^\infty x^{r-1}.e^{-x}.dx \qquad (10)$$

Note that $\Gamma(r) = (r-1)!$, if r is a positive integer. A random variable which is Gamma distributed with parameters r and λ has

$$\left.\begin{aligned} \text{mode} &= \frac{r-1}{\lambda} \\ \text{median} &= \frac{c(r)}{\lambda} \\ \text{mean} &= \frac{r}{\lambda} \\ \text{variance} &= \frac{r}{\lambda^2} \end{aligned}\right\} \qquad (11)$$

where c(r) is the solution of

$$\int_0^{c(r)} y^{r-1}.e^{-y}.dy = \frac{1}{2}\Gamma(r) \qquad (12)$$

3. MODELS FOR PROBABILITY LAWS

We next discuss various theoretical probabilistic schemes which lead to the probability distributions just described.

Various probabilistic models which give rise to the lognormal distribution are summarized in Reference 2, Chapter 3. The most important of these models is the *theory of proportionate effect*, first advanced by Kapteyn in 1903. Let $X_1, X_2, \ldots, X_n$, be a sequence of random variables which represent the magnitude at successive times of the size of a biological organism, a geological specimen, or a crack arising as a result of fatigue. Suppose that each magnitude X_n is related to its predecessor by the following rule:

$$X_n - X_{n-1} = \epsilon_n X_{n-1} \tag{13}$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots$ are independent random variables. In words, (13) says that the change $X_n - X_{n-1}$ at each step in the development of the process under study is a random proportion ϵ_n of the previous value X_{n-1}.

From (13) it follows that

$$X_n = (1 + \epsilon_n)X_{n-1} = (1 + \epsilon_n)(1 + \epsilon_{n-1}) \ldots (1 + \epsilon_2)X_1 \tag{14}$$

From (14), one sees that for large n, X_n is the product of a large number of independent factors. Therefore its logarithm log X_n is the sum of a large number of independent terms, no one of which is dominant. The central limit theorem of probability theory then applies to assert that log X_n is approximately normally distributed, from which it follows that X_n is lognormal.

The lognormal distribution has been advanced by Freudenthal and Gumbel[4] (pp.317-8) as the distribution function of the extent of progressive damage. Let consecutive stress cycles $S_1, S_2, \ldots, S_n, \ldots$ be applied to a specimen and let A_n be the disrupted area within the specimen after the application of the first n load cycles $S_1, S_2, \ldots, S_n$. If one assumes the law of proportionate effect

$$A_k - A_{k-1} = S_k A_{k-1} \tag{15}$$

then the extent A_n of damage after n stress cycles is approximately lognormal for large values of n. However, if one is interested in the number N(s) of stress cycles producing failure, it appears that the probability distribution of N(s) is, insofar as theoretical considerations are concerned, more likely to be described by an extreme value or Weibull distribution rather than a lognormal distribution.

The extreme value distribution arises in the following way. Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent observations of a random variable X. Let $Y_1 = X_1$, Y_2 = maximum $(X_1, X_2), \ldots, Y_n$ = maximum $(X_1, X_2, \ldots, X_n), \ldots$ and let $Z_1 = X_1$, Z_2 = minimum $(X_1, X_2), \ldots, Z_n$ = minimum $(X_1, X_2, \ldots, X_n), \ldots$. The random variables Y_n and Z_n are the extreme values (respectively the maximum and the minimum) of the sample of independent observations $X_1, X_2, \ldots, X_n$. The asymptotic distribution (i.e., the distribution for large values of n) of the extreme values Y_n and Z_n can be shown to be one of several types, depending on the behaviour of the distribution function of the parent random variables $X_1, X_2, \ldots, X_n, \ldots$. A complete discussion of the asymptotic distribution of extreme values is given in Gumbel's book[3]. What we have called here the extreme value distribution is more precisely called by Gumbel the Type I asymptotic distribution of extreme values. What we have called here the Weibull distribution is more precisely called by Gumbel the Type III asymptotic distribution of extreme values.

The extreme value distribution is the asymptotic distribution of the minimum of a large sample of independent observations $X_1, X_2, \ldots, X_n$ of a random variable X whose distribution function F(x) behaves, as x tends to $-\infty$, exponentially in the sense that

$$\lim_{x \to -\infty} \left[\frac{d}{dx} \frac{F(x)}{f(x)} \right] = 0 \tag{16}$$

where f(x) is the probability density function of X.

The Weibull distribution is the asymptotic distribution of the minimum of a large number of independent observations $X_1, X_2, \ldots, X_n$ of a random variable X which cannot take values less than some lower limit ϵ.

It should be pointed out that if one knows the exact distribution of the independent observations $X_1, X_2, \ldots, X_n$, then one can write down an exact expression for the distribution of their extreme values. The virtue of the theory of extreme values is that it provides an approximate method for evaluating, under a minimum of assumptions, the distribution of the maximum and minimum values in a sample.

Epstein[5] and Freudenthal and Gumbel[6] have stated the physical assumptions under which one may expect the breaking strength of material to possess either an extreme value distribution or a Weibull distribution. Roughly speaking, the assumptions are that the strength of specimen is determined by the worst flaw among the large number of flaws present in the specimen. Flaws are assumed to be distributed randomly throughout the material. The size of flaws is assumed to obey a probability distribution of type suitable for the application of the asymptotic theory of extreme values.

The extreme value and Weibull distributions can also be derived from another point of view, based on the notion of failure rate. Let T be a random variable representing the service life to failure of a specified item of equipment. Let F(x) be the distribution function of T, and let f(x) be its probability density function. We define a function $\mu(x)$, called the *intensity function, or hazard function, or conditional rate of failure function,* by

$$\mu(x) = \frac{f(x)}{1 - F(x)} \tag{17}$$

In words, $\mu(x).dx$ is the conditional probability that the item will fail between x and x + dx, given that it has survived a time T greater than x.

For a given hazard function $\mu(x)$ the corresponding distribution function is

$$1 - F(x) = [1 - F(x_0)] e^{-\int_{x_0}^{x} \mu(z).dz} \tag{18}$$

where x_0 is an arbitrary value of x, since (17) can be rewritten

$$\frac{d}{dx} \ln[1 - F(x)] = -\mu(x) \tag{19}$$

If $F(\epsilon) = 0$, then

$$1 - F(x) = e^{-\int_{\epsilon}^{x} \mu(z).dz} \qquad x > \epsilon \tag{20}$$

The extreme value distribution corresponds to the hazard function

$$\mu(t) = \frac{1}{\beta} e^{\left(\frac{t-u}{\beta}\right)} \qquad \epsilon = -\infty \tag{21}$$

The Weibull distribution corresponds to the hazard function

$$\mu(t) = k\left(\frac{t-\epsilon}{v-\epsilon}\right)^{k-1} \tag{22}$$

Birnbaum and Saunders[7] have shown how to modify the foregoing scheme so as to obtain a model which leads to the Gamma distribution.

4. REMARK ON THE PROBLEM OF ESTIMATION OF PARAMETERS

At this point it would be natural to discuss the problem of estimation of the parameters of the probability laws described in the foregoing sections. However, this discussion is omitted, and the reader is referred to the books of Aitchison and Brown[2], Gumbel[3], and Cramér[8].

5. THE SURVIVORSHIP P-S-N DIAGRAM

Having determined the probability distribution of the number N(s) of load cycles to failure at stress s, for various levels of stress, it is most convenient to present this information graphically on a diagram, such as Figure 2, which we may call the survivorship P-S-N diagram.

The survivorship P-S-N diagram is obtained in the following way. For a given stress level s and number P such that $0 \leqslant P \leqslant 1$, define the quantity N(s,P) to be the number of load cycles at stress level s which a specimen has probability P of surviving; in symbols,

$$\text{Prob}\left[N(s) > N(s,P)\right] = P \tag{23}$$

Now for various values of P (say for $1 - P = 0, 10^{-5}, 10^{-3}, 10^{-2}, 10^{-1}, 0.5$) plot the function N(s,P) as a function of s. In this way one obtains the survivorship P-S-N curve.

The survivorship P-S-N curve for a given value of P can be estimated from empirial test data, by letting N(s,P) be the number of load cycles at stress level s survived by a proportion P of the specimens tested. However, for values of P very close to 0 or 1, and small sample sizes, N(s,P) can only be obtained by assuming that the probability law of N(s) belongs to some finite parameter family whose parameters are estimated from the observations.

PART II

PROBABILISTIC MODEL FOR FATIGUE LIFE OF A STRUCTURE

1. A DEFINITION OF CUMULATIVE DAMAGE

The problem in practice is not the behavior of materials under repeated applications of loads of a given stress level, but rather the behavior of materials under a large number of applications of loads of varying magnitude. A *cumulative damage* hypothesis is a concept which attempts to relate the behavior of a specimen under load cycles of varying stress amplitude to its behavior under load cycles of constant stress amplitude.

Various cumulative damage hypotheses have been advanced by various authors (see References 9-11).

In this section we discuss a cumulative damage hypothesis which makes use of the ideas of a recently developed part of probability theory called *renewal theory*. (An excellent review of renewal theory is given by W.L. Smith in Reference 11). It is a large sample theory which avoids making any assumptions about the probability law of N(s), the number of cycles to failure, and makes use only of the mean and variance of N(s). The material of this section is based partly on ideas of M.V. Johns, Jr. In deriving the crucial design formulas below we will make various interpretations of the mathematical operations we employ. It should be noted that the mathematical derivation given is capable of several possible physical interpretations other than the one given here.

Let s be a given stress level. We assume that the damage done to a given structure by a load application at stress level s is a random variable, denoted by Z(s). If the structure is subjected to n repeated loads at stress level s, the total damage done is assumed to be the sum of n independent random variables $Z_1(s)$, $Z_2(s), \ldots, Z_n(s)$, each identically distributed as Z(s), where $Z_j(s)$ denotes the damage done by the j^{th} load application.

Let us regard the strength of a structure as being a given number T. Let $N_T(s)$ be the number of cycles to failure if a structure of strength T is subjected to repeated load applications at stress level s. Then $N_T(s)$ may be defined as the smallest integer n such that

$$Z_1(s) + Z_2(s) + \ldots + Z_n(s) > T \tag{1}$$

It can be shown that one can obtain the probability law of Z(s) from a knowledge, for all positive values of T, of the mean $E[N_T(s)]$ of $N_T(s)$ for all T by the following formulas for the Laplace transforms. Define

$$\varphi(u) = \int_0^\infty e^{-ux} P[Z(s) < x].dx \tag{2}$$

$$\psi(u) = \int_0^\infty e^{-uT} E[N_T(s)].dT \tag{3}$$

Then, for all $u > 0$

$$\varphi(u) = \frac{\psi(u)}{1 + u\psi(u)} \tag{4}$$

It should be noted that to obtain the probability law of the damage Z(s) done by a single load application it is not necessary to know the probability law of the number $N_T(s)$ of load applications to failure.

For our purposes here, it will suffice to determine the mean and variance of Z(s) from the mean and variance of $N_T(s)$ which presumably can be measured in laboratory tests. The following limit theorems constitute basic results of renewal theory:

$$\lim_{T\to\infty}\left[\frac{E[N_T(s)]}{T}\right] = \frac{1}{E[Z(s)]} \tag{5}$$

$$\lim_{T\to\infty}\left[\frac{Var[N_T(s)]}{T}\right] = \frac{Var[Z(s)]}{E^3[Z(s)]} \tag{6}$$

From (5) and (6) it follows that for very large values of T,

$$E[Z(s)] = \frac{T}{E[N_T(s)]} \tag{7}$$

$$Var[Z(s)] = \frac{Var[N_T(2)]}{E^3[N_T(s)]} T^2 \tag{8}$$

We next discuss how to use (7) and (8) to evaluate the cumulative damage to a structure due to a large number of load applications at varying stress amplitudes.

Let us consider a structure which has been subjected to a large number (say M) load cycles at varying stress amplitudes. More precisely, let $s_1, s_2, \dots, s_M$ be the successive stress amplitudes. Let Z_j denote the damage done to the structure by the j^{th} load application (which has stress s_j). The total damage D done to the structure is assumed to be the sum

$$D = Z_1 + Z_2 + \dots + Z_M \tag{9}$$

which has mean

$$E[D] = \sum_{i=1}^{M} E[Z_i] \tag{10}$$

and variance

$$\mathrm{Var}[D] = \sum_{i=1}^{M} \mathrm{Var}[Z_i] \tag{11}$$

assuming the random variables $Z_1, \ldots, Z_M$ to be independent.

The number M of load cycles to which a structure is subjected in service is a random variable. In particular, let us assume that there are a finite number of stress amplitudes $s_1, s_2, \ldots, s_r$ (written in increasing order) to which the structure is subject, and let $M(s_1), M(s_2), \ldots, M(s_r)$ denote respectively the number of load cycles endured by the specimen at each of these stress amplitudes, where

$$M(s_1) + \ldots + M(s_r) = M \tag{12}$$

Define D(s) to be the damage due to stresses of amplitude s. Since the number M(s) of load applications at stress level s is a random variable, the mean and variance of D(s) are given by (see Feller[6], p.276)

$$E[D(s)] = E[M(s)].E[Z(s)] \tag{13}$$

$$\mathrm{Var}[D(s)] = E[M(s)].\mathrm{Var}[Z(s)] + \mathrm{Var}[M(s)].E^2[Z(s)] \tag{14}$$

Further, one may show

$$\mathrm{Cov}[D(s_1), D(s_2)] = \mathrm{Cov}[M(s_1), M(s_2)].E[Z(s_1)].E[Z(s_2)] \tag{15}$$

Assume that the total damage is the sum

$$D = D(s_1) + \ldots + D(s_r) \tag{16}$$

One may show that D has mean and variance

$$E[D] = \sum_i E[M(s_i)].E[Z(s_i)] \tag{17}$$

$$\mathrm{Var}[D] = \sum_i E[M(s_i)].\mathrm{Var}[Z(s_i)] + \mathrm{Var}[\Sigma\, M(s_i).E[Z(s_i)]] \tag{18}$$

In view of Equations (17), (18), (7) and (8) we finally obtain that

$$E[D] = T\left[\sum_i \frac{E[M(s_i)]}{E[N_T(s_i)]}\right] \tag{19}$$

$$\mathrm{Var}[D] = T^2\left[\sum_i E[M(s_i)]\frac{\mathrm{Var}[N_T(s_i)]}{E^3[N_T(s_i)]}\right] + T^2\left[\mathrm{Var}\left[\sum_i \frac{M(s_i)}{E[N_T(s_i)]}\right]\right] \tag{20}$$

In order to use these expressions for the mean and variance of the damage D, we now turn our attention to the problem of determining the mean and variance of M(s).

2. A DEFINITION OF STRESS HISTORY OF A STRUCTURE

The notion is now well established that airplane behavior in rough air can be studied by means of the theory of stationary stochastic processes. In particular, it has been shown that gust velocities (encountered at a given time and place) may be considered a Gaussian stationary process. An excellent discussion of how the theory of Gaussian processes may be used to determine the stress history of a structure is given by Press[13]. Here we only sketch the ideas involved.

The statistical properties of the peaks of a Gaussian process have been studied by Rice[14] and Cartwright and Longuet-Higgins[15]. One may use these results to obtain a formula for E[M(s)], the expected number of load applications at stress amplitude s that an airplane endures per unit time due to gust loads. For a stationary Gaussian process y(t) with mean 0 and power spectrum $\Phi(\omega)$, the expected number of maxima of height y per unit time is given by

$$\nu(y) = N_1 \frac{y}{\sigma} e^{-y^2/2\sigma^2} \tag{21}$$

where

$$N_1 = \frac{1}{2\pi}\left[\frac{\int_0^\infty \omega^4 \Phi(\omega).d\omega}{\int_0^\infty \omega^2 \Phi(\omega).d\omega}\right]^{1/2} \tag{22}$$

and

$$\sigma^2 = \int_0^\infty \Phi(\omega).d\omega = E[y^2(t)] \tag{23}$$

is the mean square value of y(t). Equation (21) is valid if the gust velocity spectrum is narrow. This assumption can be relaxed, using the work of Cartwright and Longuet-Higgins[15].

If we regard the successive stress amplitudes to which an airplane is subject as being proportional to the heights of the successive peak gust velocities encountered by the airplane, then the expected number of load applications of stress amplitude s *endured by the plane in the course of* L *units of flying time is*

$$E[M(s)] = LN_1 \frac{s}{\sigma} e^{-y^2/2\sigma^2} \tag{24}$$

where now σ^2 is equal to the mean square of the gust velocity y(t) multiplied by the proportionality factor which converts gust velocities into stresses.

Unfortunately it is more difficult to obtain expressions for the variance Var[M(s)] and the covariances Cov[$M(s_1)$, $M(s_2)$]. Consequently we will assume that these terms do not contribute decisively to Equation (20) of Section 1, and can be accounted for by, say, doubling the first term in that equation.

3. A DEFINITION OF PROBABILITY OF FATIGUE FAILURE AND DESIGN LIFETIME

In view of the foregoing considerations, the relative damage endured by a structure after L time units of flight, defined as the ratio D/T of the damage endured by it to its strength T, has mean and variance

$$E\left[\frac{D}{T}\right] = LN_1\int_0^\infty ds\,\frac{s}{\sigma}\,e^{-s^2/2\sigma^2}\cdot\frac{1}{E[N(s)]} \tag{25}$$

$$\mathrm{Var}\left[\frac{D}{T}\right] = 2LN_1\int_0^\infty ds\,\frac{s}{\sigma}\,e^{-s^2/2\sigma^2}\cdot\frac{\mathrm{Var}[N(s)]}{E^3[N(s)]} \tag{26}$$

where N_1 is the expected number of peak gust velocities per unit time, σ^2 is the mean square stress per unit time (defined in the preceding section), and E[N(s)] and Var[N(s)] are respectively the mean and variance of the number N(s) of load applications to failure at stress amplitude s.

Assuming asymptotic normality of the relative damage, the *probability of failure of the structure,* defined as

$$\mathrm{Prob}\left[\frac{D}{T} \geqslant 1\right] = \frac{1}{\sqrt{2\pi}}\int_\eta^\infty e^{-\frac{1}{2}y^2}\,.\,dy$$

$$\eta = \frac{1 - E\left[\frac{D}{T}\right]}{\left(\mathrm{Var}\left[\frac{D}{T}\right]\right)^{1/2}} \tag{27}$$

can be computed, from a knowledge of the mean and variance given by (25) and (26).

The probability in Equation (25) depends on the strength of the material composing the airplane, as measured in S-N tests, on the gust history which it is expected the airplane will encounter, and on the design lifetime L of the airplane. Given any pre-assigned probability of fatigue failure which it is designed to guarantee, the designer can choose the design lifetime L accordingly.

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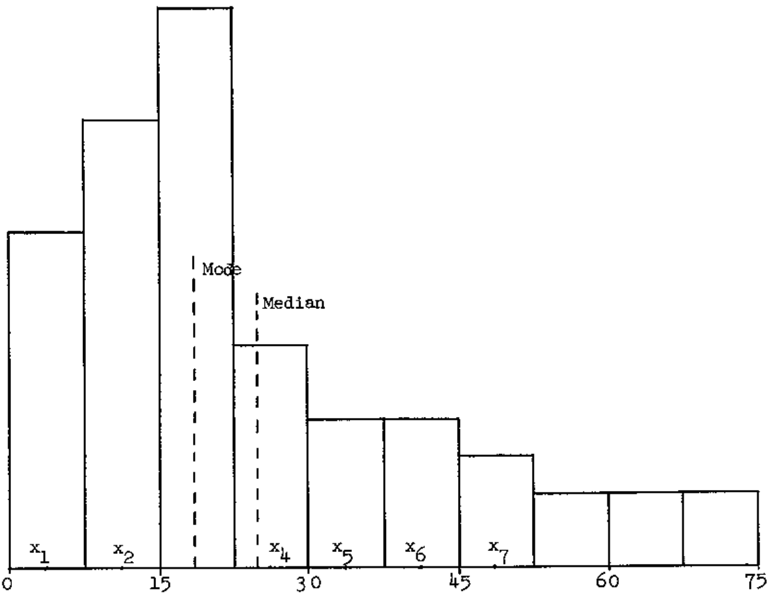

Fatigue lives at ±30,000 lb/in² in 10^6 cycles

Fig.1 Frequency histogram for fatigue test results on 75 S-T aluminum

(taken from A.G. Pugsley[16], plotted from data in Reference 4)

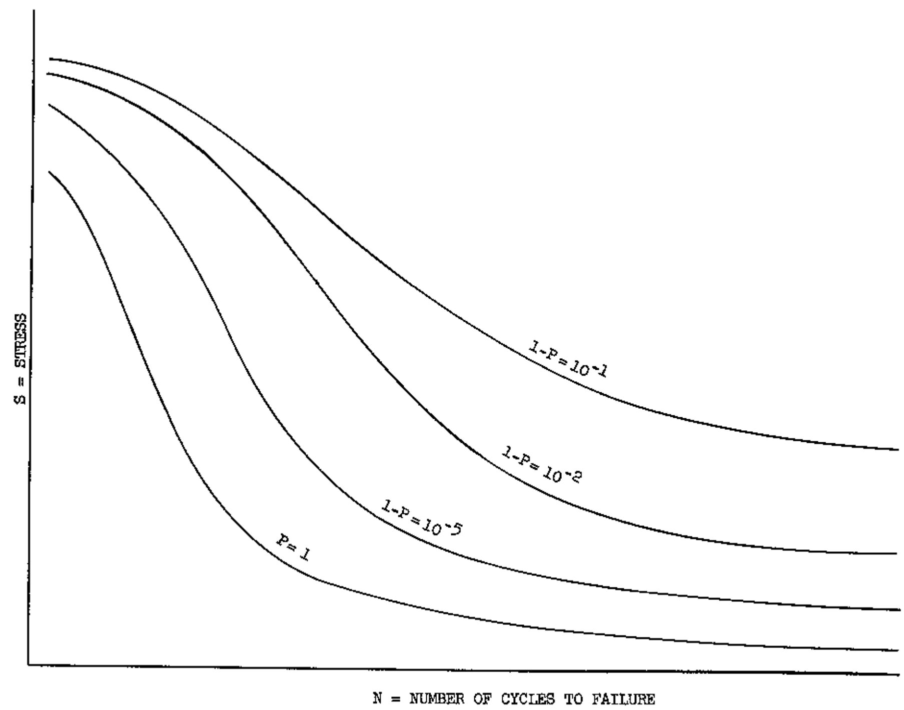

Fig.2 Schematic P-S-N diagrams