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SPECTRAL ANALYSIS OF ASYMPTOTICALLY STATIONARY TIME SERIES (1)

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1. *Introduction.*

In the theory of time series analysis and statistical communications theory, a central role is played by the notions of correlation function and spectrum. For the time series analyst, estimation of the correlation function and of the spectrum represent two of the basic tools to be used in determining the mechanism generating an observed time series. For the communication theorist, the spectrum provides the major concept in terms of which to analyze the effect of passing random processes (representing either signal or noise) through linear (and, to some extent, non-linear) devices. The question naturally arises : under what conditions can it be said that a time series possesses a spectrum. In this paper it is shown how to construct a theory of the existence, interpretation, and estimation of the spectra which seems to me to be more in accord with the manner in which physical scientists use these ideas than the widely accepted definition of the spectrum based on the notion of a stationary process.

Statisticians regard a time series $[X(t), -\infty < t < \infty]$ as being a stochastic process; the properties of time series are to be expressed as average (expectations) over the ensemble of possible realizations of the time series. A time series is said to be *weakly stationary* (see Doob [5]) if it has finite second moments and if there exists an even function, denoted $R(v)$ and called the *covariance function* of the time series such that, for all v in $-\infty < v < \infty$,

(1) $$R(v) = E[X(t)\,X(t+v)],$$

independently of t in $-\infty < t < \infty$.

The covariance function $R(v)$ is easily shown to be a function of non-negative type : for any set of complex numbers $c_1, \ldots, c_n$ and real numbers $v_1, \ldots, v_n$

(2) $$\sum_{i,j=1}^{n} c_i c_j^* \, R(v_i - v_j) \geq 0$$

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where c_j^* denotes the complex conjugate of c_j. Consequently, if it is assumed that $R(v)$ is continuous, it follows by a well-known theorem of Bochner that there exists a bounded non-decreasing function of a real variable ω, denoted $F(\omega)$, and called the *spectral distribution function*, such that

$$R(v) = \int_{-\infty}^{\infty} e^{iv\omega} dF(\omega), \quad -\infty < v < \infty. \tag{3}$$

The *spectral density function* $f(\omega)$ is defined as the derivative of the absolutely continuous part of $F(\omega)$. If $F(\omega)$ is itself absolutely continuous, then

$$R(v) = \int_{-\infty}^{\infty} e^{iv\omega} f(\omega)\, d\omega. \tag{4}$$

In words, (4) states that the covariance junction $R(v)$ is the Fourier transform of the spectral density function $f(\omega)$ from which it follows that

$$f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iv\omega} R(v)\, dv. \tag{5}$$

It should be noted that (5) holds as an absolutely convergent integral if $R(v)$ is absolutely convergent; otherwise, the integral in (5) is to be interpreted as a Cauchy principal value and (4) then holds at all frequencies ω at which $f(\omega)$ is continuous. Equations (4) and (5) are basic relations in the theory of correlation and spectra and together constitute what are usually called the *Wiener-Khintchine relations* in commemoration of the pioneering work of N. Wiener [13] and A. Khintchine [7].

The correlation function $\rho(v)$ of a weakly stationary time series is defined to be the normalized covariance function :

$$\rho(v) = \frac{R(v)}{R(0)}. \tag{6}$$

It is immediate that the correlation function possesses spectral representations of the form of (4) and (5).

In the literature of statistical communication theory the foregoing definitions of covariance function and spectrum are not completely accepted. Many writers (for example, see [3]) insist that since the observer usually has available only a single function of time which represents the time series, the definition of correlation (or covariance) function and spectrum must be based on the single record available to him. Consequently, given a time series $[X(t), t \geqslant 0]$, the covariance function $R(v)$ is defined for $v \geqslant 0$ by

$$R(v) = \lim_{T \to \infty} \frac{1}{T} \int_0^{T-v} X(t) X(t+v)\, dt \tag{7}$$

assuming that the limit in (7) exists, in some probabilistic mode of convergence, and

$R(v)$ is a continuous non-random function of v. The spectral distribution function $F(\omega)$ and spectral density function $f(\omega)$ of the time series can then be defined by (3) and (4). However, many writers do not take this approach to the definition of the spectrum; rather they define

$$f(\omega) = \lim_{T \to \infty} \frac{1}{2\pi T} \left| \int_0^T X(t) \, e^{-i\omega t} \, dt \right|^2 \tag{8}$$

The definition of the covariance function given by (7) can be justified on the grounds that if the time series $[X(t), t \geqslant 0]$ is « stationary and ergodic », then (7) will coincide with the definition given by (1). On the other hand, the notion of a « stationary and ergodic » stochastic process is not general enough to cover all the time series which one meets in practice; in particular time series such as

$$X(t) = \cos \omega_0 t \qquad t \geqslant 0 \tag{9}$$

or

$$X(t) = \cos(\omega_0 t + \varphi(t)) \qquad t \geqslant 0 \tag{10}$$

where ω_0 is a positive constant and $[\varphi(t), t \geqslant 0]$ is a suitable normal (Gaussian) process possessing correlation functions in the sense of (7). Thus there are strong grounds for accepting (7) as the definition of the covariance function. On the other hand, it is incorrect to define the spectral density function by (8), since the limit required there will in general not exist. *The only logically defensible method of defining the spectral density function is via* (4) *and* (5).

A time series $[X(t), t \geqslant 0]$ is said to posses a generalized harmonic analysis, with covariance function $R(v)$, if (7) holds. The aim of this paper is to (*i*) determine conditions for a time series to possess a generalized harmonic analysis, and (*ii*) discuss the interpretation and estimation of the spectrum of a time series possessing a generalized harmonic analysis.

2. *Conditions that a time series possess a generalized harmonic analysis.*

In this section we consider a time series $[X(t), t \geqslant 0]$ with uniformly bounded fourth moments, and such that for all $T > 0$ and $0 \leqslant v \leqslant T$ the integral

$$R_T(v) = \frac{1}{T} \int_0^{T-v} X(t) \, X(t+v) \, dt \tag{1}$$

is well defined as a limit in quadratic mean of its Riemann approximating sums. We seek to determine conditions under which there exists a function of v, to be denoted $R(v)$ and called the covariance function of the time series, such that for all $v \geqslant 0$

$$\lim_{T \to \infty} E[|R_T(v) - R(v)|^2] = 0. \tag{2}$$

We shall also mention conditions under which in addition

(3) $$\lim_{T \to \infty} R_T(v) = R(v) \text{ with probability one.}$$

We call $R_T(v)$ the *sample covariance function* of the time series, based on a sample of length T.

Since

(4) $$E[|R_T(v) - R(v)|^2] = \text{var}\,[R_T(v)] + \{E[R_T(v)] - R(v)\}^2$$

it is clear that (2) holds if and only if

(5) $$\text{var}\,[R_T(v)] \longrightarrow 0 \quad \text{as} \quad T \longrightarrow \infty$$

and

(6) $$E[R_T(v)] = \frac{1}{T}\int_0^{T-v} E[X(t)\;X(t+v)]\,dt \longrightarrow R(v) \quad \text{as} \quad T \longrightarrow \infty.$$

From (6) we obtain a conclusion of great practical significance. A time series $[X(t), t \geqslant 0]$ for which there exists a function $R(v)$ satisfying (6) could be said to possess a *covariance function* $R(v)$ and a spectrum satisfying (1.3) and (1.4). Note that (6) holds if in particular there exists a function $R(v)$ satisfying

(7) $$E[X(t)\,X(t+v)] = R(v) \quad \text{for all} \quad t$$

or satisfying

(8) $$\lim_{t \to \infty} E[X(t)\;X(t+v)] = R(v).$$

Conditions (6) and (8) provide possible extensions of the notion of weak stationarity. Time series satisfying (6) or (8) could perhaps be called *asymptotically weakly stationary*.

In order to extend the notion of an ergodic stationary process, we must state conditions under which (5) holds. Let us define for any $t > 0$ and $v \geqslant 0$

(9) $$C_t(v) = \frac{1}{t}\int_0^t \text{cov}[X(s)\,X(s+v),\; X(t)\,X(t+v)]\,ds.$$

We may then write

(10) $$\text{var}[R_T(v)] = \frac{2}{T^2}\int_0^{T-v} t\,C_t(v)\;dt$$

which follows immediately from the fact that

(11) $$\text{var}[R_T(v)] = \frac{2}{T^2}\int_0^{T-v} dt \int_0^t ds\, \text{cov}[X(s)\,X(s+v), X(t)\,X(t+v)].$$

Theorem.1. In order that (5) holds, it is necessary and sufficient that for each $v \geqslant 0$

$$\lim_{t \to \infty} C_t(v) = 0. \tag{12}$$

Proof. That (5) implies (12) follows from the fact that by Schwarz's inequality

$$|C_t(v)| \leqslant \sigma[R_t(v)]\, \sigma[X(t)\, X(t+v)] + \frac{1}{t}\int_{t-v}^{t} |\operatorname{cov}\, [X(s)\, X(s+v),\, X(t)\, X(t+v)]|\, ds. \tag{13}$$

Conversely (12) implies (5); from (10) one obtains the inequality, for $T > T' > 0$,

$$\operatorname{var}[R_T(v)] \leqslant \frac{2}{T^2}\int_{0}^{T'-v} |tC_t(v)|\, dt + \sup_{t > T'} |C_t(v)| \tag{14}$$

which tends to 0 if one lets first $T \longmapsto \infty$ and then $T' \longmapsto \infty$.

One thus sees that (2) holds for any time series satisfying (6) and (12). *These conditions thus extend the notion of a stationary ergodic process.*

To understand the meaning of condition (12) let us consider the form it takes in the case of a normal weakly stationary time series $X(t)$ with zero means. Then

$$\begin{aligned} \operatorname{Cov}[X(s)\, X(s+v), X(t)\, X(t+v)] &= \operatorname{cov}[X(s), X(t)]\, \operatorname{cov}[X(s+v),\, X(t+v)] \\ &\quad + \operatorname{cov}[X(s), X(t+v)]\, \operatorname{cov}[X(t),\, X(s+v)] \\ &= R^2(t-s) + R(t-s+v)\, R(t-s-v). \end{aligned} \tag{15}$$

From (9) and (15) it follows that

$$\begin{aligned} C_t(v) &= \frac{1}{t}\int_{0}^{t} \{R^2(t-s) + R(t-s+v)\, R(t-s-v)\}\, ds \\ &= \frac{1}{t}\int_{0}^{t} \{R^2(u) + R(u+v)\, R(u-v)\}\, du. \end{aligned} \tag{16}$$

From (16) one sees that (12) holds for all $v \geqslant 0$ if and only if

$$\lim_{t \to \infty} \frac{1}{t}\int_{0}^{t} R^2(u)\, du = 0 \tag{17}$$

which is the condition for a normal stationary process (with constant means) to be ergodic.

One can very roughly express condition (12) qualitatively as follows : in order for (12) to hold, the values $X(s)$ and $X(t)$ of the time series at two points s and t must become asymptotically uncorrelated as the distance $[s\text{-}t]$ between s and t tends to infinity.

So far we have considered conditions for (2) to hold. One can also state conditions for (3) to hold. Using the method used to prove the Strong Law of Large Numbers in Parzen [10] one can prove that (3) holds if (6) holds and if in addition, for each v, there exists a positive number q such that

$$\lim_{t \to \infty} t^q C_t(v) = 0; \tag{18}$$

in words, (18) says that $C_t(v)$ tends to 0 as some power of t. The intuitive interpretation of condition (18) is similar to that of condition (12).

3. *Existence, interpretation and estimation of the spectrum.*

Let $[X(t),\ t \geq 0]$ be a time series, with uniformly bounded fourth moments, for which one can define a function $R(v)$, defined for $v \geq 0$ and called the covariance function, satisfying (2.2). Now the sample covariance function $R_T(v)$, defined for all v in $-\infty < v < \infty$ as an even function, is a function of non-negative type. To prove this, note that one can write

$$R_T(v) = \int_{-\infty}^{\infty} e^{iv\omega} f_T(\omega)\, d\omega \qquad -\infty < v < \infty \tag{1}$$

where

$$f_T(\omega) = \frac{1}{2\pi T} \left| \int_0^T e^{-it\omega} X(t)\, dt \right|^2 \qquad -\infty < \omega < \infty \tag{2}$$

is a non-negative even function, called the *sample spectral density function.* Assuming that $R(v)$ is continuous, it then follows by the continuity theorem of probability theory that there exists a non-decreasing bounded function $F(\omega)$ satisfying (1.3).

Thus any time series satisfying (2.2) possesses a spectral distribution function $F(w)$ defined implicitly by (1.3). In order to give a physical interpretation to $F(\omega)$ we need to introduce the notion of a spectral average.

Given a bounded function $A(\omega)$ which is continuous (except for a set of points ω which has measure zero with respect to the measure induced by $F(\omega)$ on the real line) we define

$$J(A) = \int_{-\infty}^{\infty} A(\omega)\, dF(\omega) \tag{3}$$

to be a *spectral average* corresponding to the *spectral window* $A(w)$. Similarly we define

$$J_T(A) = \int_{-\infty}^{\infty} A(\omega)\, f_T(\omega)\, d\omega \tag{4}$$

to be a sample spectral average. It may be shown, using the methods used in Parzen [10] to prove the continuity theorem of probability theory, that if (2.2) holds then for every spectral window $A(\omega)$

(5) $$\lim_{T\to\infty} E[\,|J_T(A)-J(A)|^2]=0$$

while if (2.3) holds then for every spectral window $A(\omega)$

(6) $$\lim_{T\to\infty} J_T(A)=J(A) \qquad \text{with probability one.}$$

Equations (5) and (6) not only show that consistent estimates of spectral averages exist but also show us how to physically interpret sample spectral averages. If we regard the time series $[X(t),\ t\geq 0]$ as representing a current passing through a unit resistance, then the power dissipated by the current during the time interval 0 to T is defined to be

(7) $$\frac{1}{T}\int_0^T X^2(t)\,dt$$

(since, by Ohm's law, if X(t) were a constant I (amps) the power dissipated by the current passing through a resistance of R ohms would be I^2R watts). Now let $b(t)$ be an integrable function defined on $-\infty<t<\infty$ such that the spectral window $A(\omega)$ may be written

(8) $$A(\omega)=\left|\int_{-\infty}^{\infty} e^{i\omega s}\,b(s)\,ds\right|^2.$$

Next let us define

(9) $$\begin{aligned} X_T(t) &= X(t) && 0\leq t\leq T \\ &= 0 && \text{otherwise.} \end{aligned}$$

Then the sample spectral average $J_T(A)$ may be written

(10) $$\begin{aligned} J_T(A) &= \frac{1}{2\pi T}\int_{-\infty}^{\infty} d\omega \left| \int_{-\infty}^{\infty} e^{i\omega s} b(s)\,ds \int_{-\infty}^{\infty} e^{i\omega t} X_T(t)\,dt \right|^2 \\ &= \frac{1}{2\pi T}\int_{-\infty}^{\infty} d\omega \left| \int_{-\infty}^{\infty} dt\, e^{i\omega t} \int_{-\infty}^{\infty} ds\, b(s)\, X_T(t-s) \right|^2 \\ &= \frac{1}{T}\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} ds\, b(s)\, X_T(t-s) \right|^2 \end{aligned}$$

In order to understand the meaning of (10) we must introduce the notion of a *filter.* Any procedure or process (be it computational, physical, or purely conceptual) is called a time-invariant linear filter if it can be regarded as converting an input to an output and satisfies two conditions :

(*i*) The output corresponding to the superposition of two inputs is the superposition of the corresponding outputs,

(*ii*) The only effect of delaying an input by a fixed time is to delay the output by the same time.

An important class of filters is that defined by integral operators, so that if $[Y(t), -\infty<t<\infty]$ is the output function corresponding to an input function $[X(t), -\infty<t<\infty]$, then

$$(11) \qquad Y(t)=\int_{-\infty}^{\infty} b(t-s)\, X(s)\, ds = \int_{-\infty}^{\infty} b(s)\, X(t-s)\, ds .$$

The function $b(s)$ is called the impulse response function of the filter, and its Fourier transform

$$(12) \qquad B(\omega)=\int_{-\infty}^{\infty} e^{i\omega s}\, b(s)\, ds$$

is called the frequency response function of the filter. Now (10) says that if one passes a finite sample of a time series $[X(t), 0\leq t\leq T]$ into a filter with frequency response function $B(\omega)$, and if one lets $[Y_T(t), -\infty<t<\infty]$ denote the corresponding output function, then

$$(13) \qquad \frac{1}{T}\int_{-\infty}^{\infty} \left| Y_T(t) \right|^2 dt = \int_{-\infty}^{\infty} \left| B(\omega) \right|^2 f_T(\omega)\, d\omega = \int_{-\infty}^{\infty} A(\omega) f_T(\omega)\, d\omega$$

since, by (8), $A(\omega)$ is equal to the square modulus $[B(\omega)]^2$ of the frequency response function of the filter. In view of (13), $f_T(\omega)$ is a measure of the power content in the current $[X(t), 0\leq t\leq T]$ contributed by its harmonic component with frequency ω. This fact justifies calling $f_T(\omega)$ the sample spectral density function.

From (13), and (5) or (6), it follows that

$$(14) \qquad \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} \left| Y_T(t) \right|^2 dt = \int_{-\infty}^{\infty} \left| B(\omega) \right|^2 dF(\omega)$$

where the convergence in (14) is in the same mode of convergence as prevails in (5) or (6). *Equation (14) justifies a method frequently employed to estimate the spectrum of a time series possessing a generalized harmonic analysis.* One passes a record $[X(t), 0\leq t\leq T]$ of finite length T through a filter, whose output is squared and averaged. From (14) one sees that the quantity obtained in this way, which is of the form of the left-hand side of (13), is a consistent estimate of the spectral average represented by the right-hand side of (14).

A statistical interpretation of the spectral distribution function $F(\omega)$ can be obtained from (14). For any frequencies $\omega_1 < \omega_2$, the increment $F(\omega_2)$-$F(\omega_1)$ represents the contribution to the average mean square

$$(15) \qquad R(v) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left| X(t) \right|^2 dt$$

of the time series arising from its harmonic components with frequencies between ω_1 and ω_2.

The usefulness of the spectral density function in communication theory in large part derives from the fact that (14) holds. Using (14), one can determine the filtering effects of various networks through which a time series is passed. Consequently, if a time series $X(t)$ may be regarded as the sum, $X(t) = S(t) + N(t)$, of two time series $S(t)$ and $N(t)$ representing signal and noise, respectively, then from a knowledge of the respective spectral density functions of the signal and noise one may develop schemes for filtering out the signal from the noise.

4. *Examples of time series possessing a generalized harmonic analysis.*

In the next three sections we give three examples (which seem to be of some practical importance) of non-stationary time series $[X(t), t \geq 0]$ which possess a generalized harmonic analysis. In addition to the examples given here, a number of other examples can be developed (see, for example, Johns [6], Blanc-Pierre and Fortet [4], Chapter 9, and Kampé de Fériet and Frenkiel [6′].

Example I : *A Modulated Stationary Process*

Let $[X(t), t \geq 0]$ be a stationary time series with continuous covariance function $R_X(v)$:

$$(1) \qquad R_X(v) = E[X(t)\ X(t+v)].$$

Let $[g(t), t \geq 0]$ be a non-random bounded function possessing a generalized harmonic analysis; that is, the limit

$$(2) \qquad R_g(v) = \lim_{T \to \infty} \frac{1}{T} \int_0^{T-|v|} g(t)\ g(t+|v|)\ dt$$

exists for all real numbers v and *is* continuous *at* $v = 0$. In particular (2) holds for any finite sum of harmonics

$$(3) \qquad g(t) = \sum_{i=1}^{k} a_i \cos \omega_i t$$

with covariance function given by

(4) $$R_g(v) = \frac{1}{2} \sum_{i=1}^{k} |a_i|^2 \cos \omega_i v$$

Define the time series $\{Y(t), t \geq 0\}$ by

(5) $$Y(t) = X(t)\, g(t)$$

which we may call the original time series $[X(t), t \geq 0]$ modulated by the function $g(t)$. It is clear that the time series $[Y(t), t \geq 0]$ is asymptotically weakly stationary in the sense that the limit

(6) $$R_Y(v) = \lim_{T \to \infty} \frac{1}{T} \int_0^{T-v} E[Y(t)\, Y(t+v)]\, dt$$

exists. Indeed, since

(7) $$\begin{aligned} E[Y(t)\, Y(t+v)] &= g(t)\, g(t+v)\, E[X(t)\, X(t+v)] \\ &= g(t)\, g(t+v)\, R_X(v) \end{aligned}$$

it follows that

(8) $$R_Y(v) = R_X(v)\, R_g(v).$$

Equation (8) provides a simple, and indeed well-known, result for the covariance function of a modulated time series. However, in order for us to regard the notion of the covariance function of the modulated time series $[Y(t), t \geq 0]$ to be well defined, it must hold that

(9) $$\lim_{T \to \infty} \frac{1}{T} \int_0^{T-v} Y(t)\, Y(t+v)\, dt = R_Y(v)$$

where the limit in (9) holds at least as a limit in quadratic mean. It seems reasonable to suppose that if the original stationary time series is ergodic, then (9) holds. Let us show that this is the case if $[X(t), t \geq 0]$ is an ergodic stationary normal time series with zero means.

According to Theorem 1, in order for (9) to hold it is necessary and sufficient that, as $t \longmapsto \infty$,

(10) $$C_t(v) = \frac{1}{t} \int_0^t \operatorname{cov}[Y(s)\, Y(s+v),\; Y(t)\, Y(t+v)]\, ds \to 0.$$

Now

$$\begin{aligned} C_t(v) &= \frac{1}{t} \int_0^t g(s)\; g(s+v)\; g(t)\; g(t+v) \operatorname{cov}[X(s)\, X(s+v),\; X(t)\, X(t+v)\, ds \\ &= \frac{1}{t} \int_0^t g(s)\, g(s+v)\, g(t)\, g(t+v)\; \{R_X^2(t-s) + R_X(t-s+v)\, R_X(t-s-v)\}\, ds. \end{aligned}$$

Let G be an upper bound for $g(t)$; then

$$(11) \qquad |C_t(v)| \leqslant \frac{G^4}{t} \int_0^t \{R_X^2(u) + |R_X(u+v)\, R_X(u-v)|\}\, du.$$

From (11) and the ergodicity of $X(t)$, which is equivalent to (2.17), it follows that (10) holds and consequently that (9) holds.

5. Example II : *An asymptotically stationary linear process.*

For our second example of a non-stationary time series possessing a generalized harmonic analysis, we will discuss a discrete parameter time series (both for its own interest and to avoid for ease of exposition certain mathematical complexities that arise in discussing the analogous continuous parameter model).

Let $[X(t),\ t = 1, 2, \ldots]$ be a discrete parameter series given by

$$(1) \qquad X(t) = \sum_{u=1}^{t} W(t-u)\, \eta(u)$$

where $[\eta(u),\ u = 1, 2, \ldots]$ is a sequence of independent identically distributed random variables with mean zero and finite fourth moments, and $[W(u),\ u = 0, 1, 2, \ldots]$ is a sequence of constants satisfying, for some positive constants C and ρ,

$$(2) \qquad |W(u)| \leqslant Ce^{-\rho u}, \qquad u = 0, 1, 2, \ldots.$$

One important way in which a time series of the form of (1) can arise is as the solution of a stochastic difference equation :

$$(3) \qquad a_0 X(t) + a_1 X(t-1) + \ldots + a_m X(t-m) = \eta(t), \qquad t = 1, 2, \ldots$$

where it is assumed that $X(t) = 0$ for $t < 1$ and that all the roots of the characteristic equation

$$(4) \qquad a_0 z^m + a_1 z^{m-1} + \ldots + a_m = 0$$

are in modulus less than 1. The mean and covariance function of $X(t)$ is given by

$$(5) \qquad E[X(t)] = 0, \qquad \text{for all} \quad t \geqslant 1,$$

$$(6) \qquad E[X(s)\, X(t)] = \sigma^2 \sum_{u=1}^{s} W(s-u)\, W(t-u),$$

$$= \sigma^2 \sum_{u=0}^{s-1} W(u)\, W(t-s+u), \qquad \text{if} \quad t \geqslant s \geqslant 1,$$

where $\sigma^2 = E[\eta^2(u)]$. From (6) it is clear that the time series is asymptotically stationary in the sense that the limit

$$(7) \qquad \lim_{t \to \infty} E[X(t)\, X(t+v)] = R(v)$$

exists and is given by

$$\text{(8)} \qquad R(v) = \sum_{\alpha=0}^{\infty} W(\alpha)\, W(v+\alpha).$$

If instead of (1) we had written

$$\text{(1}'\text{)} \qquad X(t) = \sum_{u=-\infty}^{t} W(t-u)\, \eta(u)$$

then the time series $[X(t), t = 1, 2, ...]$ would be stationary, with covariance function given by (8). The study of time series of the form of (1′) was initiated by Bartlett (see [1]) who called them *linear processes.* For very large values of t, the time series (1) is practically equal to the time series (1′). In view of this fact, it seems justified to reduce the study of the time series (1) to that of the time series (1′) which is stationary and therefore more amenable to treatment. While it seems plausible that the large sample properties of sample averages (such as means and covariances) should be exactly the same for the series (1) as for the series (1′), this fact needs to be proved. The notion of an asymptotically stationary time series provides a framework for such a proof. It should be noted that for the solutions of stochastic difference equations such a proof has been given by Mann and Wald [9]. Consequently, this paper can be regarded as an attempt to integrate the work of Mann and Wald [9] with the theory of stationary time series.

In order for the covariance function (8) to be regarded as operationally meaningful, it must hold that (for $v = 0, 1, ...$)

$$\text{(9)} \qquad \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T-v} X(t)\, X(t+v)\, dt = R(v)$$

where the limit holds either as a limit in quadratic mean or as a limit with probability one.

We now show that (9) holds as a limit with probability one. In the same way that Theorem 1 is proved, one may show that (9) holds, as a limit in probability one, if for each fixed v the sequence

$$\text{(10)} \qquad C_t(v) = \frac{1}{t} \sum_{s=1}^{t} \operatorname{cov}[X(s)\, X(s+v), X(t)\, X(t+v)]$$

satisfies, for some $q > 0$,

$$\text{(11)} \qquad \lim_{t\to\infty} t^q C_t(v) = 0$$

To obtain the covariance appearing in (10), we first compute the logarithm of the joint characteristic function of $X(s)$, $X(s+v)$, $X(t)$, and $X(t+v)$: for any real numbers u_1, u_2, u_3, u_4,

$$\log E[\exp i\,(u_1 X(s) + u_2 X(s+v) + u_3 X(t) + u_4 X(t+v))]$$

$$\text{(12)} \qquad = \sum_{\alpha=1}^{\infty} \log \varphi_\eta \{u_1 W(s-\alpha) + u_2 W(s+v-\alpha) + u_3 W(t-\alpha) + u_4 W(t+v-\alpha)\}$$

where $\phi_\eta(u) = E[\exp iu\eta(s)]$ and we define $W(\alpha) = 0$ for $\alpha < 0$. From (12) it follows that

$$(13)\quad \begin{aligned} \operatorname{cov}[X(s)X(s+v), X(t)X(t+v)] &= \operatorname{cov}[X(s),X(t)] \operatorname{cov}[X(s+v),X(t+v)] \\ &+ \operatorname{cov}[X(s), X(t+v)] \operatorname{cov}[X(s+v), X(t)] \\ &+ \lambda_4 \sum_{\alpha=1}^{\infty} W(s-\alpha)\, W(s+v-\alpha)\, W(t-\alpha)\, W(t+v-\alpha) \end{aligned}$$

where

$$(14)\quad \lambda_4 = E[\eta^4(s)] - 3\sigma^4$$

is the fourth cumulant of $\eta(s)$. Consequently, for $s \leq t$,

$$(15)\quad \operatorname{cov}[X(s)\, X(s+v), X(t)\, X(t+v)]$$

$$\begin{aligned} &= \sigma^4 \sum_{\alpha=-\infty}^{s-1} W(\alpha)\, W(t-s+\alpha) \sum_{\alpha=-\infty}^{s+v-1} W(\alpha)\, W(t-s+\alpha) \\ &+ \sigma^4 \sum_{\alpha=-\infty}^{s-1} W(\alpha)\, W(t+v-s+\alpha) \sum_{\alpha=-\infty}^{\min(s+v,t)-1} W(\alpha)\, W(|t-s-v|+\alpha) \\ &+ \lambda_4 \sum_{\alpha=-\infty}^{s-1} W(\alpha)\, W(v+\alpha)\, W(t-s+\alpha)\, W(t-s+v+\alpha). \end{aligned}$$

From (15) it may be shown that one may write

$$(16)\quad \operatorname{cov}[X(s)\, X(s+v), X(t)\, X(t+v)] = C_1(t-s) + C_2(t,s)$$

where

$$\begin{aligned} C_1(t-s) &= \sigma^4 \sum_{\alpha=-\infty}^{\infty} W(\alpha)\, W(t-s+\alpha) \sum_{\alpha=-\infty}^{\infty} W(\alpha)\, W(t-s+\alpha) \\ &+ \sigma^4 \sum_{\alpha=-\infty}^{\infty} W(\alpha)\, W(t-s+v+\alpha) \sum_{\alpha=-\infty}^{\infty} W(\alpha)\, W(t-s-v+\alpha) \\ &+ \lambda_4 \sum_{\alpha=-\infty}^{\infty} W(\alpha)\, W(v+\alpha)\, W(t-s+\alpha)\, W(t-s+v+\alpha) \\ &= R^2(t-s) + R(t-s+v)\, R(t-s-v) \\ &+ \lambda_4 \sum_{\alpha=-\infty}^{\infty} W(\alpha)\, W(v+\alpha)\, W(t-s+\alpha)\, W(t-s+v+\alpha) \end{aligned}$$

and for some constant M

$$|C_2(t,s)| \leq M\, e^{-\rho(t+s)}.$$

From (10) and (16) it follows that

$$(17) \qquad \lim_{t\to\infty} t\, C_t(v) = \sum_{u=0}^{\infty} \{R^2(u) + R^2(u+v)\, R(u-v)\}$$

$$+ \lambda_4 \sum_{u=0}^{\infty} \sum_{\alpha=-\infty}^{\infty} W(\alpha)\, W(v+\alpha)\, W(u+\alpha)\, W(u+v+\alpha)$$

which implies that (11) holds. Indeed it may be shown that

$$(18) \qquad \lim_{T\to\infty} T \operatorname{var}[R_T(v)] = \sum_{u=-\infty}^{\infty} \{R^2(u) + R(u+v)\, R(u-v)\} + \frac{\lambda_4}{\sigma^4} R^2(v).$$

6. Example III : *An oscillator with fluctuating frequency.*

A time series which arises in many contexts is given by

$$(1) \qquad X(t) = \cos(\omega_0 t + \varphi(t)) \qquad t \geq 0$$

where ω_0 is a given positive frequency and the phase $\varphi(t)$ is a stochastic process representing fluctuations in frequency. For discussions of the time series (1) together with references to its history in the physical literature, the reader is referred to Blachman [2] and Malakhov [8].

Since the frequency fluctuations may be considered to be the superposition of a large number of random and statistically independent disturbances, we may assume $\varphi(t)$ to be a normal process. Further we shall assume that $\varphi(t)$ has stationary independent increments, and $\varphi(0) = 0$, so that there are constants m and α such that for $0 \leq s < t$

$$(2) \qquad E[\varphi(t) - \varphi(s)] = m(t-s), \qquad \operatorname{Var}[\varphi(t) - \varphi(s)] = \alpha(t-s).$$

We shall assume here that $m = 0$, so that $\varphi(t)$ has zero means.

We first determine the product moment of $X(t)$:

$$(3) \qquad E[X(t)\, X(t+v)] = E[\cos\{\omega_0 t + \varphi(t)\} \cos\{\omega_0(t+v) + \varphi(t+v)\}]$$

$$= \frac{1}{2} E[\cos\{\omega_0 v + \varphi(t+v) - \varphi(t)\}]$$

$$+ \frac{1}{2} E[\cos\{\omega_0(2t+v) + \varphi(t) + \varphi(t+v)\}].$$

Now, for $v \geq 0$, $\omega_0 v + \varphi(t+v) - \varphi(t)$ is normal with mean $\omega_0 v$ and variance αv, while $\varphi(t) + \varphi(t+v)$ has variance $\alpha(v+4t)$. Since for a normal random variable Z with mean m and variance σ^2

$$(4) \qquad E[\cos uZ] = \cos um\; e^{-\sigma^2 u/2}$$

it follows that, for $v \geqslant 0$,

$$E[X(t)\,X(t+v)] = \frac{1}{2}\cos\omega_0 v\, e^{-\alpha v/2} + \frac{1}{2}\cos\omega_0(2t+v)\, e^{-\alpha(v+4t)/2}. \tag{5}$$

Consequently, the time series $[X(t),\ t \geqslant 0]$ is asymptotically weakly stationary in the sense that

$$R(v) = \lim_{t\to\infty} E[X(t)\,X(t+v)] \tag{6}$$

exists, and is given by (for $v \geqslant 0$)

$$R(v) = \frac{1}{2}\cos\omega_0 v\, e^{-\alpha v/2}. \tag{7}$$

It may be shown that, for $v \geqslant 0$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^{T-v} X(t)\,X(t+v)\,dt = R(v) \tag{8}$$

holds as a limit with probability one by showing that

$$t\,C_t(v) = \int \operatorname{cov}[X(s)\,X(s+v),\, X(t)\,X(t+v)]\,ds \tag{9}$$

is a bounded function of t.

7. *Other aspects of the spectrum.*

In this paper we have been concerned with defining the notion of spectrum for a wider class of time series than the class of ergodic stationary processes. We have not considered the problem of finding conditions under which the notion of spectrum defined in this paper possesses various properties of the spectrum of a stationary time series, such as the following :

(*i*) if $[X(t),\ t \geqslant 0]$ is a stationary time series with spectral distribution function $F(\omega)$ satisfying

$$\int_{-\infty}^{\infty} \omega^2\, dF(\omega) < \infty \tag{1}$$

then the derivative

$$X'(t) = \lim_{h\to 0} \frac{X(t+h) - X(t)}{h} \tag{2}$$

exists as a limit in quadratic mean for $t > 0$;

(*ii*) if $[X(t), t \geqslant 0]$ is a stationary time series with zero means and possessing a spectral density function $f(\omega)$, then

(3) $$\lim_{T \to \infty} T \operatorname{var}\left[\frac{1}{T}\int_0^T X(t)\, dt\right] = 2\pi f(0)$$

(4) $$\lim_{T \to \infty} T \operatorname{var}\left[\frac{1}{T}\int_0^T \cos \omega t\, X(t)\, dt\right] = \pi f(\omega) \qquad \text{if} \qquad \omega \neq 0.$$

One can readily give sufficient conditions under which (*i*) and (*ii*) hold for a time series possessing a generalized harmonic analysis (see, for example, Parzen [11]). It seems much more difficult to give, for each property of the spectrum of a stationary time series, necessary and sufficient conditions under which this property holds for the notion of spectrum defined in this paper. Another problem which remains to be investigated is that of obtaining expressions for the variance and covariance of the various estimates of the spectrum considered by Parzen [12].

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RÉSUMÉ

Dans la théorie de l'analyse des séries temporelles et la théorie statistique des communications, les notions de fonction de corrélation et de spectre jouent un rôle central. La question suivante se pose naturellement : dans quelles conditions peut-on dire qu'une série temporelle possède un spectre? Dans le présent article, il est montré comment construire une théorie de l'existence, de l'interprétation, et de l'estimation du spectre, qui me semble s'accorder mieux avec la manière dont les spécialistes des sciences physiques se servent de ces idées, que la définition largement acceptée du spectre, basée sur la notion d'un processus stationnaire.

On dit qu'une série temporelle $[X(t), t \geqslant 0]$. possède une analyse harmonique généralisée, avec fonction de covariance $R(v)$, si pour tout $v \geqslant 0$

$$R(v) = \lim_{T \to \infty} \frac{1}{T} \int_0^{T-v} X(t)\, X(t+v)\, dt,$$

prenant comme acquis que la limite existe selon un mode quelconque de convergence stochastique, et que $R(v)$ est une fonction non-aléatoire, continue, de v. Le but du présent article est de *(i)* déterminer les conditions dans lesquelles une série temporelle possède une analyse harmonique généralisée et *(ii)* de discuter l'interprétation et l'estimation du spectre d'une série temporelle possédant une analyse harmonique généralisée.

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