ON ASYMPTOTICALLY EFFICIENT CONSISTENT ESTIMATES OF THE SPECTRAL DENSITY FUNCTION OF A STATIONARY TIME SERIES

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On Asymptotically Efficient Consistent Estimates of the Spectral Density Function of a Stationary Time Series

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SUMMARY

Several general classes of estimates of the spectral density function of a stationary time series are introduced, which are shown to include most estimates previously suggested by various researchers. Asymptotic expressions are given for the mean square error of these estimates. We thus determine, for each estimate, the class of time series for whose spectral density it is a consistent estimate. Further, for a given time series, the minimum integrated mean square error with which its spectral density may be estimated is determined, and evaluated for large sample sizes. Efficient estimates are defined as those which achieve, asymptotically, the minimum mean integrated square error. The asymptotic formulae derived are used to make a numerical comparison of some estimates which have been suggested. However, in this paper no conclusions are drawn or principles enunciated as to how to proceed in practice to estimate the spectral density.

1. Introduction

The problem of estimating the spectral density function of a stationary time series has attracted much attention recently (see references) because of its increasing importance in scientific work. One of the difficulties one encounters in treating this problem is the fact that it is not clear what is meant by a solution to the problem. Indeed, there are many senses in which a solution may exist, and it is almost a question of taste (rather than of science) what constitutes a satisfactory solution from a practical point of view. This is especially true of the problem of estimating the spectral density function, since it is not completely clear for what purposes the estimate is desired. No wonder, then, that confusion exists as to what properties it is desirable that an estimate possess. In a situation such as this, it would appear that in first approaching the problem, one should obtain as many *theorems* as possible. One's criteria as to what constitutes a solution may change, but the theorems endure, as statements of incontrovertible facts which may or may not be relevant to the problem at hand.

Although it may be read by itself, this paper is a sequel to two previous papers (Parzen, 1957*a*, *b*) on statistical spectral analysis and, like them, deals simultaneously with discrete and continuous parameter time series.

On the one hand, as in (1957*a*), we consider in sections 4–6 the order of consistency of various general types of estimates of the spectral density function. The results obtained

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in the present paper constitute a more of less complete solution of the problem of estimating the spectral density from the point of view of obtaining estimates with as high an order of consistency as is possible. In particular, the notion of the *efficiency* of an estimate is introduced, and asymptotically most efficient estimates are exhibited. It is shown that the highest order of consistency which may be possessed by a family of estimates of the spectral density function $f(\omega)$ of a stationary time series with integrable covariance function $R(v)$ is determined by the smoothness (differentiability) of $f(\omega)$ as expressed by the manner in which $R(v)$ tends to 0 as the lag v tends to infinity. It turns out that the case where $R(v)$ tends to 0 exponentially is strikingly different from the case where $R(v)$ tends to 0 algebraically.

We also determine the *consistency* class of a family of estimates. Let S be one of the modes of consistency defined in section 4. Given a family of estimates $f_T^*(\omega)$, we define its *consistency class in the sense of S* to be the class of covariance functions $R(v)$ with respect to which the family of estimates is consistent in the sense S.

On the other hand, as in (1957*b*), we consider in section 7 the consequences of interpreting the asymptotic expressions which are obtained for the mean square error of the various estimates as being valid for finite (although large) sample sizes. We obtain certain formulae, which are used to compare estimates which have been introduced by various writers.

However, in the present paper we cannot give an unequivocal answer to the practical question of how to estimate the spectral density function. The difficulty is that there is no unequivocal answer to the question of what properties should the best estimate have; the reader is referred to Parzen (1957*b*), however, where one possible answer is discussed.

The contents of the paper are as follows. In section 2, we state the basic assumptions of the paper, namely that the stationary time series under observation have integrable covariance functions and fourth moment functions. In section 3, a class of estimates of the spectral density function is introduced, which is shown to include most estimates introduced previously. In section 4, various figures of merit of an estimate are stated. In section 5, we state various smoothness conditions which may be satisfied by the covariance function of the time series under observation. We then *summarize the main theorems of the paper*. However no proofs are given in the interest of economy; the methods of proof are essentially the same as in our paper (1957*a*). Complete proofs are to be found in a technical report, with the same title as this paper, issued as Technical Report No. 36 on Contract N6onr-25140, by the Applied Mathematics and Statistics Laboratory, Stanford University. A copy of this report has been deposited with the Library of the Royal Statistical Society. In section 6, we state the consistency class of an estimate. The contents of section 7 were described above.

2. Assumptions, Definitions, and Notation

Let $y(t)$ denote the stationary time series under observation. We seek to treat simultaneously both discrete and continuous parameter time series. Most equations that we write will hold for both cases, with the proper interpretation, which will be explained as we proceed. The domain of the variable t, in cases where it is not explicitly mentioned, is to be taken as the infinite real line $-\infty < t < \infty$, in the continuous parameter case, and as the set of integers $0, \pm 1, \pm 2, \ldots$ in the discrete parameter case.

It is assumed that the mean value function $m(t) = Ey(t)$ vanishes for all t. However, it is easy to extend the method of estimating the spectrum considered in this paper to the case where $m(t)$ does not vanish but may be written as a finite sum of known functions of t, with unknown coefficients to be estimated by a regression analysis.

It is assumed that $y(t)$ is wide sense stationary up to order four, in the sense that $E \mid y(t) \mid^4$ exists for all t, and the product moments

$$E[y(t)\, y(t+v)] = R(v) \tag{2.1}$$

$$E[y(t)\, y(t+v_1)\, y(t+v_2)\, y(t+v_3)] = P(v_1, v_2, v_3) \tag{2.2}$$

do not functionally depend on t. The domain of the variables v, v_1, v_2, v_3 is the same as that of t. The fourth cumulant function is defined by

$$Q(v_1, v_2, v_3) = P(v_1, v_2, v_3) - R(v_1)\, R(v_2 - v_3) \\ - R(v_2)\, R(v_3 - v_1) - R(v_3)\, R(v_1 - v_2). \tag{2.3}$$

If the stochastic process $y(t)$ is normally distributed, than $Q(v_1, v_2, v_3) = 0$. Thus $Q(v_1, v_2, v_3)$ may be thought of as the non-Gaussian part of the fourth-moment function. As an example, suppose that $y(t)$ is a discrete parameter linear process,

$$y(t) = \sum_v h(t - v)\, \xi(v), \tag{2.4}$$

where one assumes that the sequence $h(v)$ is absolutely convergent, and the random variables $\xi(v)$ are independent identically distributed random variables with finite fourth moment, and fourth cumulant k_4. Then

$$Q(v_1, v_2, v_3) = k_4 \sum_v h(v)\, h(v + v_1)\, h(v + v_2)\, h(v + v_3). \tag{2.5}$$

One calls $R(v)$ the covariance function of the process $y(t)$. It possesses a representation as a Fourier-Stieltjes integral (in the continuous parameter case, under the additional assumption that it is continuous)

$$R(v) = \int e^{iv\omega}\, dF(\omega), \tag{2.6}$$

where $F(\omega)$ is a bounded non-decreasing function, called the spectral distribution function of the process. Physically, the difference $F(\omega_2) - F(\omega_1)$ represents the fraction of the power (or mean square) of the time series $y(t)$ contained between the frequencies ω_1 and ω_2. The domain of the variable ω is $-\infty$ to ∞ in the continuous parameter case, and $-\pi$ to π in the discrete parameter case. The domain of integration of an integral involving ω is to be taken as the whole domain of ω, in cases where it is not otherwise specified.

It is assumed next that $R(v)$ and $Q(v_1, v_2, v_3)$ are absolutely summable over their domains.

Under the assumption that $R(v)$ is summable, it follows that the spectral distribution function $F(\omega)$ possesses everywhere a derivative $f(\omega)$, called the spectral density function of the time series $y(t)$. The following relations hold:

$$R(v) = \int e^{i\omega v} f(\omega)\, d\omega, \tag{2.7}$$

$$f(\omega) = \frac{1}{2\pi} \int e^{-i\omega v} R(v)\, dv, \tag{2.8c}$$

$$= \frac{1}{2\pi} \Sigma\, e^{-i\omega v} R(v). \tag{2.8d}$$

In cases where the limits of integration (or summation) of an integral (or sum) involving the variables u or v are omitted, they are to be assumed to be $-\infty$ to ∞. Henceforth, we write equations of the type of (2.8), involving the variables u or v, only once, for the continuous parameter case, with the understanding that for every such equation a corresponding equation may be written for the discrete parameter case by replacing the integral by a sum. For certain important equations we will write, without further explanation, two equations, with a suffix c for the continuous parameter case, and a suffix d for the discrete parameter case.

Various "smoothness" assumptions that will be made are introduced in section 5.

3. A Class of Estimates of the Spectral Density Function

Let $y(t)$, given for $0 \le t \le T$ in the continuous parameter case and for $t = 1, \ldots, T$ in the discrete parameter case, be a sample of length T of the time series under observation. Various authors (e.g., Grenander and Rosenblatt, 1957; Lomnicki and Zaremba, 1957; Parzen, 1956*a*) have considered estimates $f_T{}^*(\omega)$ of the spectral density function $f(\omega)$ of the following general form:

$$f_T^*(\omega) = \frac{1}{2\pi} \int_{-T}^{T} e^{-iv\omega}\, k_T(v)\, R_T(v)\, dv \tag{3.1c}$$

$$= \frac{1}{2\pi} \sum_{|v| \le T} e^{-iv\omega}\, k_T(v)\, R_T(v), \tag{3.1d}$$

where $R_T(v)$ is the sample covariance function, defined to be 0 for $|v| \ge T$, and for $|v| < T$,

$$R_T(v) = \frac{1}{T} \int_0^{T-|v|} y(t)\, y(t + |v|)\, dt \tag{3.2c}$$

$$= \frac{1}{T} \sum_{t=1}^{T-|v|} y(t)\, y(t + |v|) \tag{3.2d}$$

and where the constants $k_T(v)$ are to be chosen (as an *even* function of v) to satisfy some criterion of optimality.

Other authors have considered estimates which are either continuous averages over the periodogram (Grenander, 1951) or even only discrete averages over the periodogram (Whittle, 1957). We now show that estimates of these forms can be written in the form of (3.1).

The periodogram $f_T(\omega)$ is defined in terms of the sample values of $y(t)$ by

$$f_T(\omega) = \frac{1}{2\pi T} \left| \int_0^T e^{-it\omega}\, y(t)\, dt \right|^2 \tag{3.3c}$$

$$= \frac{1}{2\pi T} \left| \sum_{t=1}^{T} e^{-it\omega} y(t) \right|^2. \tag{3.3d}$$

In terms of the sample covariance function, it may be written

$$f_T(\omega) = \frac{1}{2\pi} \int_{-T}^{T} e^{-iv\omega} R_T(v)\, dv. \tag{3.4}$$

Indeed, the definition of the sample covariance function by (3.2) is motivated by the notion that $R_T(v)$ should be the Fourier transform of the periodogram:

$$R_T(v) = \int e^{iv\omega} f_T(\omega)\, d\omega. \tag{3.5}$$

Substituting (3.5) into (3.1) it is clear that one may write the estimated spectral density function as a continuous average of the periodogram:

$$f_T^*(\omega) = \int K_T(\omega - \lambda) f_T(\lambda)\, d\lambda, \tag{3.6}$$

where $K_T(\omega)$ is defined by

$$K_T(\omega) = \frac{1}{2\pi} \int_{-T}^{T} e^{iv\omega} k_T(v)\, dv \tag{3.7c}$$

$$K_T(\omega) = \frac{1}{2\pi} \sum_{|v| \leq T} e^{-iv\omega} k_T(v). \tag{3.7d}$$

We note without proof that $f_T^*(\omega)$ may be written as a discrete average over the values of the periodogram at the points

$$\left.\begin{aligned} \omega_m(T) &= \frac{\pi m}{T} \qquad &(3.8c) \\ &= \frac{2\pi m}{2T+1} \qquad &(3.8d) \end{aligned}\right\} \quad m = 0, \pm 1, \pm 2, \ldots$$

in virtue of the expressions

$$f_T^*(\omega) = \frac{\pi}{T} \sum_{m=-\infty}^{\infty} f_T(\omega_m(T))\, K_T(\omega - \omega_m(T)), \tag{3.9c}$$

$$= \frac{2\pi}{2T+1} \sum_{m=-T}^{T} f_T(\omega_m(T))\, K_T(\omega - \omega_m(T)). \tag{3.9d}$$

The class of estimates to be considered in this paper are formed in the following way. Let $h(u)$ be a bounded, even, square integrable function, defined for all real u, and satisfying additional smoothness properties to be specified in section 5. Some possible choices for $h(u)$ are

$$h(u) = \frac{1}{1 + |u|}, \tag{3.11}$$

$$h(u) = 1 - |u|, \quad |u| \leq 1 \tag{3.12}$$

$$= 0, \text{ otherwise}, \tag{3.12}$$

$$h(u) = \frac{\sin u}{u}. \tag{3.13}$$

Infinities of other possibilities are to be found in Parzen (1957*b*).

One class of estimates $f_T^*(\omega)$ of the spectral density function that we consider are defined by (3.1), with $k_T(v)$ given by

$$k_T(v) = h(A_T\, e^{\alpha|v|}) \tag{3.14}$$

where the A_T are positive constants tending to 0 as $T \to \infty$, and α is a positive constant. We call these the estimates of exponential type. An estimate similar to one of this type, with $h(u)$ given by (3.11), $A_T = a/T$, and $\alpha = \log \rho$, where a and ρ are positive constants, was considered by Lomnicki and Zaremba (1957).

Another class of estimates that we consider are of the form of (3.1), with

$$k_T(v) = h(B_T v) \tag{3.15}$$

where the B_T are positive constants tending to 0 as $T \to \infty$. We call these the estimates of algebraic type; they were considered in Parzen (1957*a*).

4. Figures of Merit of Estimates

In this section, we state (without comment in the interest of brevity) various figures of merit for estimates of the spectral density function. Let $f_T^*(\omega)$ be a sequence of estimates of the spectral density function $f(\omega)$, where T denotes the sample size; $T = 1, 2, \ldots$ in the discrete parameter case and $0 < T < \infty$ in the continuous parameter case. Let

$$\sigma^2[f_T^*(\omega)] = E\,|\,f_T^*(\omega) - Ef_T^*(\omega)\,|^2,\; b[f_T^*(\omega)] = Ef_T^*(\omega) - f(\omega) \tag{4.1}$$

$$\eta^2[f_T^*(\omega)] = E\,|\,f_T^*(\omega) - f(\omega)\,|^2 = \sigma^2[f_T^*(\omega)] + b^2[f_T^*(\omega)] \tag{4.2}$$

denote respectively the variance, bias, and mean square error of the estimate $f_T^*(\omega)$; we consider only estimates with finite mean and variance.

Let $N(T) \to \infty$ as $T \to \infty$. The family of estimates $f_T^*(\omega)$ is said to be (i) *consistent of order* $N(T)$ at ω if $N(T)\,\eta^2[f_T^*(\omega)]$ tends to a finite limit; (ii) *boundedly consistent of order* $N(T)$ at ω if $N(T)\,\eta^2[f_T^*(\omega)]$ is bounded in T; (iii) *uniformly consistent* of order $N(T)$ if $N(T)\,\eta^2[f_T^*(\omega)]$ converges uniformly in ω; (iv) *uniform-boundedly consistent* of order of $N(T)$ if $N(T)\,\eta^2[f_T^*(\omega)]$ is bounded in T and ω; (v) *functionally-uniformly consistent* of order $N(T)$ if $N(T)\,E[\sup_\omega |\,f_T^*(\omega) - f(\omega)\,|^2]$ is bounded in T.

Another criterion which regards the behaviour of the estimate $f_T^*(\omega)$ as a function of ω is the integrated square error of the estimate, defined by

$$I[f_T^*(\omega)] = \int |\,f_T^*(\omega) - f(\omega)\,|^2\, d\omega.$$

A family of estimates is said to be *integratedly consistent* of order $N(T)$ if $N(T)\;EI[f_T^*(\omega)$ tends to a finite non-zero limit axs $T \to \infty$. The integrated mean square error is particularly easy to treat mathematically. We are able to compute, for each sample size T, the *minimum* mean integrated square error, denoted by EI_T or $E[I_T \mid R(v)]$, of any estimate of the spectral density function which may be formed on the basis of a sample of size T. Further, under assumptions as to form of the covariance function $R(v)$ for v large, we are able to compute the order with which EI_T tends to 0 as $T \to \infty$.

Thus we are led to define the *asymptotic efficiency* of a family of estimates $f_T^*(\omega)$, given that $R(v)$ is the true covariance function of the time series being observed, by

$$e[f_T^*(\omega) \mid R(v)] = \lim_{T\to\infty} \frac{E[I_T \mid R(v)]}{EI[f_T^*(\omega)]}. \tag{4.3}$$

5. Smoothness Conditions and Summary of Results

Given a stationary time series with covariance function $R(v)$ and spectral density function $f(\omega)$, the highest order of consistency with which $f(\omega)$ can be estimated will depend on *the smoothness of $f(\omega)$ as expressed by the manner in which $R(v)$ behaves for v tending to infinity.* We distinguish two manners in which $R(v)$ may behave; it may decrease exponentially or algebraically.

Exponential decrease of coefficient ρ.—We say that the covariance function $R(v)$ decreases exponentially of coefficient ρ, where $\rho > 0$, if for large v it is of the form of $e^{-\rho|v|}$; that is, if for some constant R_0

$$|R(v)| \leq R_0\, e^{-\rho|v|} \text{ for all } v \tag{5.1}$$

and for almost all u in $0 < |u| < 1$,

$$\lim_{v\to\infty} e^{\rho v} |R(uv)| = \infty \tag{5.2c}$$

$$\lim_{v\to\infty} e^{\rho v} |R([uv])| = \infty, \tag{5.2d}$$

where $[x]$ denotes the largest integer less than or equal to x. Unfortunately, (5.2) is not general enough; it does not hold for the important special case $R(v) = \cos \omega v\, e^{-\rho|v|}$, where $\omega > 0$. We extend the above definition by requiring instead of (5.2) that, for almost all u in $0 < |u| < 1$

$$\limsup_{v\to\infty} e^{\rho v} |R(uv)| = \infty \tag{5.3}$$

and for any constant $c > 0$

$$\lim_{v\to\infty} \int_0^1 \frac{1}{1 + c\, e^{2\rho v} R^2(uv)}\, du = 0, \tag{5.4c}$$

$$\lim_{v\to\infty} \frac{1}{v} \sum_{|u|\leq v} \frac{1}{1 + c\, e^{2\rho v} R^2(u)} = 0. \tag{5.4d}$$

We remark that a stationary time series generated by an autoregressive scheme necessarily decreases exponentially (Doob, 1953, p. 503).

Algebraic decrease of degree r.—We say that the covariance function $R(v)$ decreases algebraically of degree $r > 0$ if for large v it is of the form of v^{-r}; that is, for some finite positive constant R_r

$$\lim_{v\to\infty} v^r |R(v)| = R_r. \tag{5.5}$$

Since $R(v)$ is assumed to be summable one has that necessarily $r > 1$.

Exponential decrease of degree r and coefficient ρ.—It should be pointed out that one may also consider covariance functions $R(v)$ which are, for v large, of the form of $e^{-\rho|v|r}$ where necessarily $0 < r \leq 2$. Using the methods of this paper, there is no difficulty in treating covariance functions of this form. However, we do not explicitly do so, except for one remark, in order not to overload the paper.

The asymptotic minimum mean integrated squared error.—It may be shown that for a covariance function $R(v)$ which decreases exponentially of coefficient ρ, the minimum mean integrated square error $E[I_T \mid R(v)]$ satisfies

$$\lim_{T\to\infty} \frac{T}{\log T} E[I_T \mid R(v)] = \frac{1}{2\pi\rho} S, \tag{5.6}$$

where

$$S = \int R^2(v)\, dv = 2\pi \int_{-\infty}^{\infty} f^2(\omega)\, d\omega \tag{5.7c}$$

$$= \Sigma\, R^2(v) = 2\pi \int_{-\pi}^{\pi} f^2(\omega)\, d\omega, \tag{5.7d}$$

It may also be shown that for a covariance function which decreases exponentially of degree $r \geqslant 1$ and coefficient ρ

$$\lim_{T\to\infty} \frac{T}{(\log T)^{1/r}} E[I_T \mid R(v)] = \frac{1}{\pi(2\rho)^{1/r}} S. \tag{5.8}$$

Throughout this paper we define, for any $p > 1$,

$$a(p) = 1 - 1/p. \tag{5.9}$$

It may be shown that for a covariance function which decreases algebraically of degree r,

$$\lim_{T\to\infty} T^{a(2r)} E[I_T \mid R(v)] = \frac{1}{2\pi} R_r{}^{1/r} S^{a(2r)} \int_{-\infty}^{\infty} (1 + u^{2r})^{-1}\, du. \tag{5.10}$$

Thus it is seen that the *highest order of consistency which may be possessed by a family of estimates of the spectral density function of a stationary time series with integrable covariance function R(v) is determined by the behaviour of R(v) for v large.* If $R(v)$ decreases exponentially (of degree 1), then the minimum mean integrated square error tends to 0 at the same rate as does $\log T/T$. However, if $R(v)$ decreases algebraically of degree r, then there is a highest power $a(2r)$ of T such that the mean integrated square error of any family of estimates cannot decrease to 0 at a rate faster than does $T^{-a(2r)}$.

We next consider the estimates introduced in section 3, and state how their properties depend on the smoothness of $R(v)$, as well as the choice of function $h(u)$ and constants α, A_T, B_T.

Mean integrated square error of the estimates of exponential type.—The estimates of exponential type are of the form

$$f_T^*(\omega) = \frac{1}{2\pi} \int_{-T}^{T} e^{-iv\omega}\, h(A_T\, e^{\alpha\,|\,v\,|})\, R_T(v)\, dv \tag{5.11c}$$

$$= \frac{1}{2\pi} \sum_{|\,v\,| \leq T} e^{-iv\omega}\, h(A_T\, e^{\alpha\,|\,v\,|})\, R_T(v). \tag{5.11d}$$

These estimates are of the form of (3.1), with $k_T(v)$ given by (3.14). The function $h(u)$ is even, defined for all real u, and assumed to satisfy the conditions for some positive constants H_0, H_1, and H,

$$|\,h(u)\,| \leq H_0, \text{ all } u \tag{5.12}$$

$$|\,1 - h(u)\,| \leq H_1\,|\,u\,|\,, \quad |\,u\,| \leq 1, \tag{5.13}$$

$$|\,uh(u)\,| \leq H, \quad |\,u\,| \geq 1. \tag{5.14}$$

The constants A_T will be assumed for convenience to be of the form

$$A_T = A\, T^{-b}. \tag{5.15}$$

One may prove the following theorem concerning the mean integrated square error of the estimates of exponential type.

Theorem.—Let $\rho > 0$, and suppose that $R(v)$ is dominated by $e^{-\rho\,|\,v\,|}$ in the sense that $R(v)$ satisfies (5.1). Let $h(u)$ satisfy (5.12)-(5.14). Choose $b > (1/2)$, and $A > 0$, and let A_T be given by (5.15). Choose $\alpha > 0$ so that $\alpha \leq 2b\rho$. Then the mean integrated square error of the estimates $f_T^*(\omega)$ defined by (5.11) satisfies

$$\lim_{T\to\infty} \frac{T}{\log T}\, E\, I[f_T^*(\omega)] = \frac{2b}{\alpha}\, \frac{1}{2\pi}\, S. \tag{5.16}$$

This theorem has several immediate consequences. First, it is seen that the asymptotic mean integrated square error of estimates of exponential type do not depend on the particular choice of A or of $h(u)$ so long as $h(u)$ satisfies (5.12)–(5.14).

Second, if it is known that $R(v)$ decreases exponentially with coefficient ρ, it is possible to choose a non-parametric estimate of the form of (5.11) which has asymptotic efficiency 1; chooce α and b so that $\rho = \alpha/(2b)$ (say, $b = 1$ and $\alpha = 2\rho$).

Third, we may characterize the consistency class (in the sense of integrated consistency) of a family of estimates of exponential type, corresponding to a given choice of $h(u)$, A, b, and α. Let $\rho(\alpha, b) = \alpha/(2b)$. Then the estimates $f_T^*(\omega)$ are integratedly consistent of order $\log T/T$ for any covariance function $R(v)$ decreasing exponentially with coefficient $\rho \geq \rho\ (\alpha, b)$. Further, the asymptotic efficiency of the estimates for such a covariance function is

$$e[f_T^*(\omega)\ |\ R(v)] = \rho(\alpha, b)/\rho. \tag{5.17}$$

Thus, for a covariance function which decreases exponentially, we have an unequivocal answer to the question of how to estimate its spectral density by means of an estimate of exponential type. For reasons to be explained later we take (3.11) for $h(u)$. The choice of b and A are not too important, and we take them both equal to 1. The resulting estimate is

$$f_T^*(\omega) = \frac{1}{2\pi} \int_{-T}^{T} e^{-iv\omega} \left(1 + \frac{1}{T} e^{\alpha |v|}\right)^{-1} R_T(v)\, dv. \tag{5.18}$$

If the coefficient ρ of exponential decrease of $R(v)$ is known, choose $\alpha = 2\rho$; the estimate (5.18) has then asymptotic efficiency 1. If the coefficient ρ is only known to satisfy the inequality $\rho \geq \rho_1$, one should choose $\alpha = 2\rho_1$; the estimate (5.17) has then asymptotic efficiency (ρ_1/ρ).

Mean integrated square error of the estimates of algebraic type.—The estimates of algebraic type are of the form

$$f_T^*(\omega) = \frac{1}{2\pi} \int_{-T}^{T} e^{-iv\omega}\, h(B_T\, v)\, R_T(v)\, dv, \tag{5.21c}$$

$$= \frac{1}{2\pi} \sum_{|v| \leq T} e^{-iv\omega}\, h(B_T\, v)\, R_T(v). \tag{5.21d}$$

These estimates are of the form of (3.1), with $k_T(v)$ given by (3.15). The function $h(u)$ is even, defined for all u, and assumed to satisfy the conditions, for some positive constants q, ϵ, H_0, H_q, and H,

$$|\, h(u) \,| \leq H_0, \text{ all } u, \tag{5.22}$$

$$|\, 1 - h(u) \,| \leq H_q \,|\, u \,|^{q}, \quad |\, u \,| \leq 1, \tag{5.23}$$

$$|\, u^{\frac{1}{2}+\epsilon} h(u) \,| \leq H, \quad |\, u \,| \geq 1. \tag{5.24}$$

A function $h(u)$ satisfying these conditions will be said to be of type q; the corresponding estimate (5.21) will be said to be of algebraic type q. We define

$$S(h) = \int_{-\infty}^{\infty} h^2(u)\, du, \tag{5.25}$$

$$S_p(h) = \int_{-\infty}^{\infty} \left(\frac{1 - h(u)}{|u|^p}\right)^2 du. \tag{5.26}$$

If $h(u)$ is of type q, $S_p(h)$ is finite for $p < q + 1/2$. It is to be stressed that these integrals are the only integrals in this paper involving the variables u or v which are *not* to be read as sums in the discrete parameter case.

The constants B_T will be assumed (for convenience) to be of the form

$$B_T = B\, T^{-b}. \tag{5.27}$$

One may prove the following theorem concerning the mean integrated square error of the estimates of algebraic type.

Theorem.—Let $r > 1/2$, and suppose that the covariance function $R(v)$ decreases algebraically of degree r, in the sense that (5.5) is satisfied. Choose p so that $1/2 \leq p \leq r$, and let $b = 1/(2p)$. Choose $B > 0$, and let B_T be given by (5.27). Choose $h(u)$ of type $q > p - 1/2$. Then the mean integrated square error of the estimate $f_T^*(\omega)$ defined by

(5.21) satisfies

$$\lim_{T\to\infty} T^{a(2p)}\, E\, I[f_T^*(\omega)] = \frac{S}{2\pi B}\, S(h) + \frac{R_p{}^2}{2\pi}\, B^{2p-1}\, S_p(h). \tag{5.28}$$

This theorem has several immediate consequences. First, it is seen that the asymptotic integrated mean square error of estimates of algebraic type depends very strongly on the choice of $h(u)$ and B_T. Indeed, the order of consistency of the estimate varies directly with the choice of B_T.

Second, if it is known that $R(v)$ decreases algebraically of degree r, and the *positive* quantities S and R_r are known, then it is possible to choose an estimate of algebraic type which will have asymptotic efficiency 1. Choose $h(u) = (1 + u^{2r})^{-1}$, which is of type $2r$. Choose $p = r$, so that $b = 1/(2r)$. Choose $B = (S/R_r{}^2)^{1/(2r)}$. The resulting estimate, given by (5.29) with $q = r$, has an asymptotic integrated mean square error which, in view of (5.30), may be verified to be equal to the right-hand side of (5.10).

Third, we may characterize the consistency class (in the sense of integrated consistency) of a family of non-parametric estimates of algebraic type q, corresponding to a given choice of $h(u)$ of type q. Choose $p < q + 1/2$. Let B_T be given by (5.27), with $B > 0$ and $b = 1/2(p)$. Then the corresponding estimates $f_T^*(\omega)$ are integratedly consistent of order $T^{a(2p)}$ for any covariance function $R(v)$ decreasing algebraically of degree $r \geq p$. The asymptotic efficiency of the estimate is non-zero only if $r = p$, in which case it is given by the ratio of the right-hand side of (5.10) to the right-hand side of (5.28).

Thus, a covariance function which decreases algebraically must be known fairly well, in terms of the parameters r, R_r, and S, in order to best estimate its spectral density by means of an estimate of algebraic type. An estimate of algebraic type which seems to have desirable properties is

$$f_T^*(\omega) = \frac{1}{2\pi} \int_{-T}^{T} e^{-iv\omega} \left(1 + \frac{1}{T}\,(B\,|\,v\,|\,)^{2q}\right)^{-1} R_T(v)dv. \tag{5.29}$$

This estimate is integratedly consistent of order $T^{a(2q)}$ for any covariance function decreasing algebraically of degree $r > q$. Its asymptotic integrated mean square error satisfies

$$\lim_{T\to\infty} T^{a(2q)}\, E\, I[f_T^*(\omega)] = \frac{S}{2\pi B}\, 2 \int_0^\infty \frac{1 + B^{2q}(R_q{}^2/S)\, u^{2q}}{(1 + u^{2q})^2}\, du. \tag{5.30}$$

As has been pointed out, for a covariance function $R(v)$ for which the parameters r, S and R_r are known, the estimate (5.29), for a suitable choice of q and B, yields an estimate of asymptotic efficiency 1.

Pointwise consistency.—The asymptotic mean square error of the estimates (5.11) and (5.21) may be evaluated.

It will be found that the integratedly-consistent estimates of exponential type are also pointwise consistent of the same order. Indeed, the following theorem concerning the asymptotic variance and bias of the estimates (5.11) may be proved.

Theorem.—Let $\rho > 0$ and suppose that $R(v)$ is dominated by $e^{-\rho|v|}$. Let $h(u)$ satisfy (5.12)-(5.14). Choose $b > (1/2)$, and $A > 0$, and let A_T be given by (5.15). Choose $\alpha > 0$ so that $\alpha \leq 2b\rho$.

Then the covariance of the estimate $f_T^*(\omega)$, defined by (5.11) satisfies, for any non-negative frequencies ω_1 and ω_2,

$$\lim_{T\to\infty} \frac{T}{\log T} \operatorname{cov}\,[f_T^*(\omega_1), f_T^*(\omega_2)] = \frac{2b}{\alpha} f^2(\omega_1)\{1 + \delta(0, \omega_1)\}\,\delta(\omega_1, \omega_2), \tag{5.31}$$

where $\delta(\omega_1, \omega_2) = 1$ or 0 according as $\omega_1 = \omega_2$ or $\omega_1 \neq \omega_2$. For every $\epsilon > 0$ the convergence in (5.31) is uniform in $\omega_1 \geq \epsilon$ and $\omega_2 \geq \epsilon$. Moreover, the quantities in (5.31) are uniformly bounded (in T and ω).

The supremum of the bias of the estimate $f_T^*(\omega)$ satisfies

$$\lim_{T\to\infty} \left(\frac{T}{\log T}\right)^{1/2} \sup_{\omega} \mid b[f_T^*(\omega)] \mid = 0. \tag{5.32}$$

Consequently, the mean square error of the estimate satisfies, for every ω,

$$\lim_{T\to\infty} \frac{T}{\log T} E \mid f_T^*(\omega) - f(\omega) \mid^2 = \frac{2b}{\alpha} f^2(\omega)\,\{1 + \delta(0, \omega)\}. \tag{5.33}$$

Thus the estimates $f_T^*(\omega)$ are uniformly-boundedly consistent (and, for any $\epsilon > 0$, uniformly consistent for $\omega \gtrsim \epsilon$) of order $T/\log T$.

For the estimates (5.11) of algebraic type, the situation is more complicated. It has been stated previously that for a covariance function decreasing algebraically of degree r, one can find estimates (5.21) which are integratedly consistent of order $T^{a(2r)}$. However, from the theorems in section 12, we are at best able to find estimates (5.21) which are boundedly consistent of order $T^{a(2r-1)}$. Next, if one desires estimates for which the limit in (4.6) exists, one must be content with estimates of order of consistency *less* than $T^{a(2r-1)}$. This follows from the fact that if $R(v)$ decreases algebraically of degree r, then the p^{th} absolute covariance moment

$$\int \mid v \mid^p \mid R(v) \mid dv < \infty \tag{5.34}$$

for $p < r - 1$.

One is able to evaluate the asymptotic variance of estimates of algebraic type without requiring any smoothness conditions on $R(v)$. The following theorem was proved in Parzen (1957*a*).

Theorem.—Let $h(u)$ satisfy (5.22) and (5.23). Let B_T be positive constants tending to 0 as $T \to \infty$ in such a way that $T B_T \to \infty$. The covariance of the estimate $f_T^*(\omega)$ defined by (5.21) satisfies, for any non-negative frequencies ω_1 and ω_2,

$$\lim_{T\to\infty} T B_T \operatorname{Cov}\,[f_T^*(\omega_1), f_T^*(\omega_2)] = f^2(\omega_1)\, S(h)\{1 + \delta(0, \omega_1)\}\,\delta(\omega_1, \omega_2). \tag{5.35}$$

For any $\epsilon > 0$, the convergence in (5.35) is uniform in $\omega_1 \gtrsim \epsilon$ and $\omega_2 \gtrsim \epsilon$. Moreover, the quantities in (5.35) are uniformly bounded (in T and ω).

With the aid of this theorem one may prove the following theorem.

Theorem.—Let $r > 1$, and let $R(v)$ decrease algebraically of degree r. Choose p so that $1 < p \leq r$, and let $b = 1/(2p - 1)$. Choose $B > 0$, and let B_T be given by (5.27). Choose $h(u)$ of type $q > p - 1$. Then the bias and mean square error of the estimates

$f_T^*(\omega)$ defined by (5.21) satisfy, for some finite constant C, for all T and ω,

$$B_T^{-p+1} \sup_{\omega} | b[f_T^*(\omega)] | < C, \tag{5.36}$$

$$T^{a(2p-1)} E | f_T^*(\omega) - f(\omega) |^2 < C. \tag{5.37}$$

One may prove also the following theorem.

Theorem.—Let $p > 0$. Suppose that $R(v)$ satisfies (5.34). Let $h(u)$ satisfy (5.22)-(5.24) and be such that

$$h^{(p)} = \lim_{u \to 0} \frac{1 - h(u)}{| u |^p} \tag{5.38}$$

exists and is finite. Let $b = 1/(2p + 1)$. Choose $B > 0$, and let B_T be given by (5.27). Let the generalized p^{th} derivative $f^{(p)}(\omega)$ of the spectral density $f(\omega)$ be defined by

$$f^{(p)}(\omega) = \frac{1}{2\pi} \int e^{-iv\omega} | v |^p R(v) \, dv \tag{5.39c}$$

$$= \frac{1}{2\pi} \Sigma \, e^{-iv\omega} | v |^p R(v). \tag{5.39d}$$

Then the bias and mean square error of the estimate $f_T^*(\omega)$ defined by (5.21) satisfies

$$\lim_{T \to \infty} B_T^{-p} | b[f_T^*(\omega)] | = | h^{(p)} f^{(p)}(\omega) | \tag{5.40}$$

$$\lim_{T \to \infty} T^{a(2p+1)} E | f_T^*(\omega) - f(\omega) |^2 = \frac{1}{B} f^2(\omega) \, S(h) \{1 + \delta(0, \omega)\} + B^{2p} | h^{(p)} f^{(p)}(\omega) |^2. \tag{5.41}$$

The convergence in (5.40) is uniform in all ω, and in (5.41) is uniform in $\omega \geq \epsilon$, for any $\epsilon > 0$. The integrated mean square error of the estimate satisfies

$$\lim_{T \to \infty} T^{a(2p+1)} E \, I[f_T^*(\omega)] = \frac{1}{2\pi B} S \, S(h) + \frac{B^{2p}}{2\pi} | h^{(p)} |^2 \int_{-\infty}^{\infty} v^{2p} R^2(v) \, dv. \tag{5.42}$$

Functionally-uniform consistency.—Concerning the functionally-uniform consistency of the estimates (5.11) and (5.21) one has the following theorems.

Theorem.—Let $\rho > 0$, and suppose that $R(v)$ is dominated by $e^{-\rho |v|}$. Let $h(u)$ satisfy (5.12)-(5.14). Choose $b > 1/2$, and $A > 0$, and let A_T be given by (5.15). Choose $\alpha > 0$ so that $\alpha \leq 2b\rho$. Then

$$\limsup_{T \to \infty} \frac{T}{(\log T)^2} E[\sup_{\omega} | f_T^*(\omega) - f(\omega) |]^2 < \infty,$$

so that the estimate $f_T^*(\omega)$ is functionally uniform consistent of order $T/(\log T)^2$.

Theorem.—Let $r > 1$, and let $R(v)$ decrease algebraically of degree r. Choose p so that $1 < p \leq r$, and let $b = 1/(2p)$. Choose $B > 0$, and let B_T be given by (5.27). Choose $h(u)$ of type $q > p - 1$. Then

$$\limsup_{T \to \infty} T^{1(-1/p)} E[\sup_{\omega} | f_T^*(\omega) - f(\omega) |]^2 < \infty,$$

so that the estimate $f_T^*(\omega)$ is functionally uniform consistent of order $T^{1-(1/p)}$.

Truncated estimates.—Consider an estimate $f_T^*(\omega)$ of the form of (5.11) or (5.21). By the truncated form of the estimate, denoted by $f_T^{*t}(\omega)$, we mean the estimate of the same form corresponding to the truncated function $h^t(u)$, defined by $h^t(u) = h(u)$ or 0 according as $|u| \leq 1$ or $|u| > 1$. As a rule, estimates of the form of (5.11) or (5.21) and their truncated forms will be consistent together, although not necessarily with the same asymptotic mean square error or integrated mean square error.

6. The Consistency Class of an Estimate

The theorems in the foregoing have all been concerned with the following problem: given a covariance function $R(v)$, to find the estimates of the form of (5.11) or (5.21) which are consistent estimates of the spectral density of $R(v)$. However, these theorems can all be interpreted to yield solutions to the following problem: given an estimate of the form of (5.11) or (5.21), to find the covariance functions for which it is a consistent estimate of the corresponding spectral density. In the theorems below we state the consistency classes of estimates of exponential type and algebraic type respectively.

Theorem.—Let $h(u)$ satisfy (5.12)–(5.14). Choose $\alpha > 0$, $b > 1/2$, and $A > 0$, and let A_T be given by (5.15). Let $\rho(\alpha, \beta) = \alpha/(2b)$. Then the estimate $f_T^*(\omega)$ defined by (5.11) is

(a) integratedly consistent and uniform-boundedly consistent (and uniformly consistent for $\omega \geq \epsilon$, for any $\epsilon > 0$) of order $T/\log T$ for any covariance function $R(v)$ dominated by $e^{-|v|\rho(\alpha,\beta)}$; the asymptotic integrated mean square error is then given by (5.12);

(b) functionally-uniformly consistent of order $T/(\log T)^2$ for any covariance function dominated by $e^{-|v|\rho(\alpha,\beta)}$.

Theorem.—Let $h(u)$ be of type $q > 0$, and such that $h^{(q)}$ exists. Choose b in the region specified below. Choose $B > 0$, and let B_T be given by (5.27). Then the estimate $f_T^*(\omega)$ defined by (5.21) is,

(a) if $1 > b > 1/(2q+1)$, integratedly consistent of order T^{1-b} for any covariance function $R(v)$ decreasing algebraically of degree $r \geq 1/(2b)$; the asymptotic integrated mean square error then satisfies (5.28) with $p = 1/(2b)$;

(b) if $1 > b > 1/(2q+1)$, uniform-boundedly consistent of order T^{1-b} for any covariance function $R(v)$ decreasing algebraically of degree $r \geq (1/(2b)) + (1/2)$;

(c) if $1 > b \geq 1/(2q+1)$, consistent (uniformly in $\omega \geq \epsilon$, for any $\epsilon > 0$) of order T^{1-b} for any covariance function $R(v)$ decreasing algebraically of degree $r > (1/(2b)) + 1/2$; the mean square error then satisfies (5.41) and the asymptotic integrated mean square error satisfies (5.42), with $p = ((1/b) - 1)/2$.

(d) if $1 > b \geq 1/(2q+1)$, functionally-uniformly consistent of order T^{1-2b} for any covariance function decreasing algebraically of degree $r \geq 1/(2b)$.

In applying the latter theorem note that the bias terms in (5.28), (5.41), and (5.42) will often vanish, since for a covariance function $R(v)$ decreasing algebraically of degree r, $R_p = 0$ for $p < r$, and for a function $h(u)$ for which $h^{(q)}$ exists, $h_p = 0$ for $p < q$.

7. A Comparison of Some Suggested Estimates

We now discuss in the light of these results a number of estimates of the spectral density which have been suggested by various authors.

Given an estimate of algebraic type q corresponding to a function $h(u)$ for which $h^{(q)}$ exists and is *positive*, we define B_T by (5.27), with $b = 1/(2q+1)$. Then the estimate $f(^*_T\omega)$ given by (5.21) is consistent (integratedly, uniform boundedly, and, for any $\epsilon > 0$, uniformly for $\omega \geq \epsilon$) of order $T^{2q/(2q+1)}$. If we regard the equations (5.41) and (5.42) as being approximately true for finite values of T, then the mean square error is approximately given by

$$T^{2q/2q+1}\,\eta^2[f_T^*(\omega)] = \frac{1}{B} f^2(\omega)\, S(h)\,\{1 + \delta(0,\omega)\} + B^{2q}\,|\,h^{(q)} f^{(q)}(\omega)\,|^2 \tag{7.1}$$

and the integrated mean square error is approximately given by

$$T^{2q/2q+1}\,EI[f_T^*(\omega)] = \frac{1}{B}\, S_0\, S(h) + B^{2q}\,|\,h^{(q)}|^2\, S_{2q}, \tag{7.2}$$

where we define, for $q \geq 0$,

$$S_{2q} = \frac{1}{2\pi}\int v^{2q}\, R^2(v)\, dv \tag{7.3c}$$

$$= \frac{1}{2\pi}\,\Sigma\, v^{2q}\, R^2(v). \tag{7.3d}$$

For fixed T, these expressions are functions of B. Let $\eta_T(\omega)$ be the minimum mean square error over all choices of B, and similarly let I_T be the minimum integrated mean square error over all choices of B. One obtains the following expressions for the *relative* minimum mean square error, and the *relative* minimum integrated mean square error:

$$\left(\frac{\eta_T{}^2(\omega)}{f^2(\omega)}\right)^{1+(1/(2q))} = \frac{1}{T}\, D(q)\, T(h)\,\left|\frac{f^{(q)}(\omega)}{f(\omega)}\right|^{1/q}\{1 + \delta(0,\omega)\}, \tag{7.4}$$

$$\left(\frac{I_T}{S_0}\right)^{1+(1/2q)} = \frac{1}{T}\, D(q)\, T(h)\left(\frac{S_{2q}}{S_0}\right)^{1/2q}, \tag{7.5}$$

where we define for $q > 0$

$$D(q) = (1+2q)^{1/2q}\left(1 + \frac{1}{2q}\right), \tag{7.6}$$

$$T(h) = |\,h^{(q)}\,|^{1/q}\, S(h). \tag{7.7}$$

The minimum value is attained at a value B which satisfies

$$\left(\frac{1}{B}\right)^{2q+1} = 2q\,\frac{S_{2q}}{S_0}\,\frac{|\,h^{(q)}\,|^2}{S(h)}. \tag{7.8}$$

Equivalently, the constants B_T in the estimate (5.21) satisfy

$$\left(\frac{1}{B_T}\right)^{2q+1} = T\left(\frac{1}{B}\right)^{2q+1} = 2qT\,\frac{S_{2q}}{S_0}\,\frac{|\,h^{(q)}\,|^{(2q+1)/q}}{T(h)}. \tag{7.8'}$$

In Parzen (1957*b*), we stated various general conclusions that could be drawn from the foregoing equations for the relative minimum mean square error. Here we obtain similar conclusions from the equations for the relative minimum integrated mean square error. We shall obtain these conclusions by means of the following example.

Consider a stationary time series which is Markov in the wide sense (Doob, 1953, p. 90). The covariance function of such a process is necessarily of the form (Doob, 1953, p. 234)

$$R(v) = R(0)\, e^{-\rho |v|} \tag{7.9c}$$

$$R(v) = R(0)\, r^{|v|} \tag{7.9d}$$

for some positive constants ρ or $r < 1$. We obtain

$$S_{2q} = \frac{R^2(0)}{\pi} \frac{\Gamma(2q+1)}{(2\rho)^{2q+1}}, \qquad q \geqq 0 \tag{7.10c}$$

$$S_0 = \frac{R^2(0)}{2\pi} \frac{1+r^2}{1-r^2} \tag{7.10d}$$

$$S_{2q} = \frac{R^2(0)}{2\pi} G^{(2q)}(0), \qquad q > 0,$$

where

$$G(\alpha) = 2 \sum_{v=1}^{\infty} (r^2 e^{\alpha})^v = 2 \frac{r^2 e^{\alpha}}{1 - r^2 e^{\alpha}}.$$

Consequently

$$\frac{S_{2q}}{S_0} = \frac{\Gamma(2q+1)}{(2\rho)^{2q}} \tag{7.11c}$$

$$\begin{aligned} \frac{S_{2q}}{S_0} &= 2 \frac{1-r^2}{1+r^2} G^{(2q)}(0), \qquad q > 0, \\ &= 2 \frac{r^2}{(1-r^2)^2}, \qquad q = 1, \\ &= 2 \frac{r^2(1+10r^2+r^4)}{(1-r^2)^4}, \qquad q = 2. \end{aligned} \tag{7.11d}$$

We next suppose that the discrete parameter process $x(n)$, $n = 0, \pm 1, \pm 2, \ldots$, whose covariance function is (7.9d) is in reality derived from the continuous parameter time series $x(t)$ whose covariance function is (7.9c) by means of *sampling at a rate of* $2W$ *samples a second.* It then follows that

$$r = e^{-\rho/(2W)}. \tag{7.12}$$

Next, if the continuous parameter time series is observed for a time of length T, one has $N = 2WT$ observations on the discrete parameter time series. In the equations (7.4) and (7.5) for the discrete parameter case, one should read N for T.

Under these assumptions we obtain the following expressions for the relative integrated mean square error. For $q = 1$, since $D(1) = \frac{1}{2}\, 3^{3/2} = 2{\cdot}598$,

$$\left(\frac{I_T}{S_0}\right)^{3/2} = \frac{1}{T} 2{\cdot}6\, T(h) \frac{\sqrt{2}}{2\rho} \tag{7.13c}$$

$$= \frac{1}{T} 2{\cdot}6\, T(h) \frac{\sqrt{2}}{2W} \frac{e^{-\rho/2W}}{1 - e^{-\rho/W}}. \tag{7.13d}$$

For $q = 2$, since $D(2) = \frac{1}{4}\, 5^{q/4} = 1\cdot 497$

$$\left(\frac{I_T}{S_0}\right)^{5/4} = \frac{1}{T} 1\cdot 5\, T(h)\, \frac{(24)^{1/4}}{2\rho} \tag{7.14c}$$

$$= \frac{1}{T} 1\cdot 5\, T(h)\, \frac{1}{2W} \frac{\{2e^{-\rho/W}(1 + 10\, e^{-\rho/W} + e^{-2\rho/W})\}^{1/4}}{1 - e^{-\rho/W}}. \tag{7.14d}$$

Several conclusions may be drawn from (7.13) and (7.14).

First, one sees that as the sampling rate W tends to infinity, the equations for the discrete parameter case tend to the equations for the continuous parameter case. Further, the discrete parameter covariance function tends to 1 as $W \to \infty$. Thus the analysis of discrete parameter time series is, in our opinion, without meaning, if one does not refer to the continuous parameter time series which is being sampled, for only the coefficient ρ has a physical significance independent of sampling rate.

Second, one sees that the mean square errors in (7.13) and (7.14) depends, in the continuous parameter case, inversely on the *product* of the sample size T and the correlation coefficient ρ. Among the implications of this fact is the following. Let us consider a discrete parameter time series with covariance function (7.9d), which has been obtained by sampling a continuous parameter time series with covariance function (7.9c).

From (7.12), $-\log_e r = \rho/2W$. Consequently

$$T\rho = 2WT\,(-\log_e r) = N\,(-\log_e r), \tag{7.15}$$

where T is the length of sample of the continuous parameter time series and N is the number of samples of the discrete parameter time series. As will be seen below, in view of (7.15), we are able to evaluate the effect of sampling on the variability of our estimates of the spectrum. It will be seen that, for a given choice of T (or N) and $h(u)$, the mean square errors in (7.13) and (7.14) are always in the continuous parameter case *greater* than in the discrete parameter case. This fact may be verified algebraically (as well as arithmetically, as below) for the case of stationary Markov time series. We have not been able to prove a theorem to this effect, but it appears plausible that the result holds in general. For the spectral density of the discrete parameter time series, which we may denote by $f^d(\omega)$, is not the same as the spectral density $f^c(\omega)$ of the continuous parameter time series, but is related to it by the formula, for $-\pi \leq \omega \leq \pi$,

$$f^d(\omega) = \sum_{m=-\infty}^{\infty} f^c(\omega - 4\pi\, Wm). \tag{7.16}$$

Thus $f^d(\omega)$ is a kind of smoothing of $f^c(\omega)$, which may imply that there is a smaller mean square error involved in estimating $f^d(\omega)$ than in estimating $f^c(\omega)$.

Third, it is seen from (7.13) and (7.14) that in order to compare the (minimum integrated) mean square errors of two estimates of the same type, one need only to compare the coefficients $T(h)$ of the corresponding functions $h(u)$. In Table 1 we compute $T(h)$ for various functions $h(u)$. As far as estimates of different types are concerned, it appears that the estimate of higher type number yields a lesser mean square error (at least, for large sample sizes).

We next study various estimates (of algebraic type) which have been introduced. In Table 1, we list eight functions $h(u)$. Bartlett's modified periodogram (Bartlett, 1950,

1955) corresponds to function 1. Daniell's suggestion that one average over the periodogram (Bartlett, 1946) corresponds to function 4. Tukey's estimate (see Tukey, 1949) corresponds to function 5. An estimate suggested by us in Parzen (1957*a*) corresponds to function 6. The functions 2, 3, 7, 8 correspond to estimates related to estimates suggested in this paper in section 5.

For purposes of numerical illustration, we give formulae for the relative minimum integrated mean square error I_T/S_0 in the case that $r^2 = 1/2$, which corresponds to an example treated by Lomnicki and Zaremba (1957). We have the following formulae (with $S_0 = 12/2\pi$) which follow immediately from (7.13d) and (7.14d):

$$\frac{I_T}{S_0} = 3\cdot 000 \left(\frac{T(h)}{N}\right)^{2/3}, \qquad q = 1, \tag{7.17}$$

$$= 3\cdot 607 \left(\frac{T(h)}{N}\right)^{4/5}, \qquad q = 2. \tag{7.17}$$

TABLE I

Estimates Listed by Corresponding Function h(u)

	$h(u)$	$h^{(q)}$	$T(h)$	$\lvert T(h) \rvert^{2q/(2q+1)}$	$\dfrac{\lvert h^{(q)} \rvert^{1/q}}{T(h)^{1/(1+2q)}}$
		Estimates of type $q = 1$			
1 .	$1 - \lvert u \rvert, \ \lvert u \rvert < 1$; $0, \ \lvert u \rvert > 1$	1	·666	·763	1·145
2 .	$\frac{1}{1 + \lvert u \rvert}, \ \lvert u \rvert < 1$; $0, \ \lvert u \rvert > 1$	1	1·000	1·000	1·000
3 .	$\frac{1}{1 + \lvert u \rvert}$	1	2·000	1·587	·794
		Estimates of type $q = 2$			
4 .	$\frac{\sin u}{u}$	0·167	1·283	1·220	·388
5 .	$1 - 2a + 2a \cos \pi u, \ \lvert u \rvert < 1$; $0, \ \lvert u \rvert > 1$	$\pi^2 a$	$2\pi\sqrt{a}$ $(1 - 4a + 6a^2)$		
5a .	$a = \cdot 23$ (Tukey's original choice)	2·270	1·197	1·155	1·453
5b .	$a = \cdot 25$	2·467	1·178	1·140	1·520
5c .	$a = \cdot 282 = \frac{6 + \sqrt{6}}{30}$ (minimizes $T(h)$)	2·780	1·165	1·130	1·617
5d .	$a = \cdot 333$	3·290	1·209	1·164	1·746
6 .	$1 - \lvert u \rvert^2, \ \lvert u \rvert < 1$; $0, \ \lvert u \rvert > 1$	1	1·067	1·053	·987
7 .	$\frac{1}{1 + \lvert u \rvert^2}, \ \lvert u \rvert < 1$; $0, \ \lvert u \rvert > 1$	1	1·285	1·222	·951
8 .	$\frac{1}{1 + \lvert u \rvert^2}$	1	1·571	1·435	·914

One may compare the mean square errors in the discrete and continuous parameter cases. In view of (7.15), $T\rho = N \log_e \sqrt{2}$. Thus from (7.13c) and (7.14c)

$$\frac{I_T}{S_0} = 3{\cdot}040 \left(\frac{T(h)}{N}\right)^{2/3}, \qquad q = 1,$$

$$= 4{\cdot}175 \left(\frac{T(h)}{N}\right)^{4/5}, \qquad q = 2. \tag{7.18}$$

A comparison of (7.17) and (7.18) shows numerically that the relative minimum integrated mean square error is larger in the continuous parameter case than in the discrete parameter case.

By means of (7.17) there is no difficulty in evaluating the minimum relative integrated mean square error I_T/S_0 for any given sample size N for each of the estimates corresponding to the functions $h(u)$ given in Table 1. In Table 2, we tabulate such values of I_T/S_0, for $N = 500$, 2,000 and 5,000.

TABLE 2

The Minimum Relative Integrated Mean Square Error

>

I_N/S_0, given by equation (7.17)

$h(u)$ \ N	500	2000	5000
1	·03634	·01442	·00783
2	·04762	·01890	·01026
3	·07560	·03000	·01629
4	·03051	·01006	·00484
5a	·02888	·00953	·00458
5b	·02850	·00940	·00452
5c	·02825	·00932	·00448
5d	·02911	·00960	·00461
6	·02633	·00869	·00417
7	·03056	·01008	·00484
8	·03588	·01184	·00569

For functions $h(u)$ which vanish for $|u| > 1$, the number of sample covariances $R_T(v)$ which must be computed in order to form the estimate (5.21) is less than the sample size. From (7.8′) it may be seen that the number of sample covariances M_T required to form the foregoing estimates which achieve the minimum relative integrated mean square error is given by

$$M_T = \frac{1}{B_T} = \left(2q\, T \frac{S_{2q}}{S_0}\right)^{1/(1+2q)} \frac{|\, h^{(q)} \,|^{1/q}}{T(h)^{1/(1+2q)}}. \tag{7.19}$$

For the Markov process with $r^2 = 1/2$, we obtain

$$M_N = 2\, N^{1/3} \frac{|\, h^{(1)} \,|}{T(h)^{1/3}}, \qquad q = 1,$$

$$= 3{\cdot}314\, N^{1/5} \frac{|\, h^{(2)} \,|^{1/2}}{T(h)^{1/5}}, \qquad q = 2. \tag{7.20}$$

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8. REPRESENTATIONS OF THE VARIANCE AND BIAS OF THE ESTIMATES

In this section, we obtain representations, in terms of the covariance function $R(v)$ and the fourth-order cumulant $Q(v_1,v_2,v_3)$, of the variance and bias of the estimates of the form of (3.1). These representations are very useful in the proofs of our results.

A basic role in our proofs is played by the following representation of the sample covariance function $R_T(v)$. Let, for $|v| < T$,

$$(8.1) \qquad R_T'(v) = \frac{1}{T}\int_0^{T-|v|} \left\{ y(t)\, y(t+|v|) - R(v) \right\} dt$$

$$= R_T(v) - E\, R_T(v).$$

Consequently, $\mathrm{Cov}\,[R_T(v_1), R_T(v_2)] = E[R_T'(v_1)\, R_T'(v_2)]$.

Now, it may be verified by a direct calculation (see [12]) that one may write, for non-negative v_1 and v_2,

$$(8.2) \qquad T\, E[R_T'(v_1)\, R_T'(v_2)] = \int_{-T}^{T} du\, U_T(u,v_1,v_2)$$

$$\left\{ Q(v_1,u,u+v_2) + R(u)\, R(u+v_1-v_2) + R(u+v_1)\, R(u-v_2) \right\}$$

where $U_T(u,v_1,v_2)$ is a function with values between 0 and 1 defined as follows:

$$
\begin{aligned}
U_T(u,v_1,v_2) &= 0 && u \leq -T + v_2 \\
&= 1 - \frac{v_2 + u}{T} && -T + v_2 \leq u \leq \min(0, v_2 - v_1) \\
(8.3) \qquad &= 1 - \frac{\max(v_1, v_2)}{T} && \min(0,v_2-v_1) \leq u \leq \max(0,v_2-v_1 \\
&= 1 - \frac{v_1 + u}{T} && \max(0,v_2-v_1) \leq u \leq T - v_1 \\
&= 0 && T - v_1 \leq u .
\end{aligned}
$$

The study of the properties of $f_T^*(\omega)$ defined by (3.1) may be reduced to a study of the properties of $R_T'(v)$. For example, the covariance between the estimates at two frequencies ω_1 and ω_2 may be written

$$
(8.4) \quad \mathrm{Cov}[f_T^*(\omega_1), f_T^*(\omega_2)] = \frac{4}{4\pi^2} \int_0^T \int_0^T dv_1\, dv_2 \cos \omega_1 v_1 \cos \omega_2 v_2\, k_T(v_1) k_T(v_2)\, E[R_T'(v_1) R_T'(v_2)].
$$

Next, by Parseval's formula

$$
(8.5) \qquad \int |f_T^*(\omega) - E\, f_T^*(\omega)|^2\, d\omega = \frac{1}{2\pi} \int_{-T}^{T} k_T^2(v)[R_T(v) - E\, R_T(v)]^2\, dv.
$$

Consequently, one may write the integrated variance by

$$(8.6) \qquad \int \sigma^2[f_T^*(\omega)]d\omega = \frac{1}{2\pi}\int_{-T}^{T} k_T^2(v)\ E[R_T'^2(v)]dv \ .$$

From the representation (8.2), we may obtain two important facts. First, there is a finite constant, which we denote by Q_o, which is a uniform bound for $T\ E[R_T'^2(v)]$; that is,

$$(8.7) \qquad T\ E[R_T'^2(v)] \le Q_o, \quad \text{for all } T \text{ and } v.$$

Second, one may infer that (where S is defined by (5.7))

$$(8.8) \qquad \lim_{T\to\infty} T\ E[R_T'^2(n(T))] = S$$

for any function $n(T) \to \infty$ as $T \to \infty$ in such a way that $n(T)/T \to 0$.

The expectation of the estimate $f_T^*(\omega)$ is given by

$$(8.9) \qquad E[f_T^*(\omega)] = \frac{1}{2\pi}\int_{-T}^{T} e^{-iv\omega}\, k_T(v)\ R(v)\ \left(1 - \frac{|v|}{T}\right)dv.$$

Consequently the bias is given by

$$(8.10) \qquad \begin{aligned} b[f_T^*(\omega)] &= E[f_T^*(\omega)] - f(\omega) \\ &= \frac{1}{2\pi}\int_{-T}^{T} e^{-iv\omega}\ (k_T(v) - 1)\ R(v)dv \\ &- \frac{1}{2\pi T}\int_{-T}^{T} e^{-iv\omega}\, k_T(v)\ |v|\ R(v)dv \\ &- \frac{1}{2\pi}\int_{|v|\ge T} e^{-iv\omega}\ R(v)dv. \end{aligned}$$

By Parseval's formula the squared bias is given by

$$(8.11) \qquad \int b^2[f_T^*(\omega)]d\omega = \frac{1}{2\pi}\int_{-T}^{T} [1 - k_T(v)\ (1 - \frac{|v|}{T})]^2\ R^2(v)dv$$

$$+ \frac{1}{2\pi}\int_{|v| \geq T} R^2(v)dv.$$

The mean integrated square error $E\ I[f_T^*(\omega)]$ is given by the sum of (8.6) and (8.11).

The following representations will be useful. Let H_0 be an upper bound for $k_T(v)$ for all T and v. Then we have the following upper bound for the supremum of the bias:

$$(8.12) \qquad 2\pi \sup_{\omega} |b[f_T^*(\omega)]| \leq I_1 + H_0\ I_2 + I_3$$

where

$$I_1 = \int_{-T}^{T} |1 - k_T(v)|\ |R(v)|\,dv$$

$$I_2 = \frac{1}{T}\int_{-T}^{T} |v|\,|R(v)|\,dv$$

$$I_3 = \int_{|v| \geq T} |R(v)|\,dv.$$

We have the following upper bound for the integrated squared bias;

$$2\pi \int b^2[f_T^*(\omega)]d\omega \leq J_1 + (H_0+1)H_0 \; J_2 + H_0^2 \; J_3 + J_4 \tag{8.13}$$

where

$$J_1 = \int_{-T}^{T} (1 - k_T(v))^2 \; R^2(v)dv$$

$$J_2 = \frac{1}{T} \int_{-T}^{T} |v| \; R^2(v)dv$$

$$J_3 = \frac{1}{T^2} \int_{-T}^{T} |v|^2 \; R^2(v)dv$$

$$J_4 = \int_{|v| \geq T} R^2(v)dv.$$

9. THE ASYMPTOTIC MINIMUM MEAN INTEGRATED SQUARE ERROR

In order to evaluate the asymptotic minimum mean integrated square error, we first determine the weights, which we shall denote by $K_T(v)$, such that the corresponding estimate $f_T^*(\omega)$ minimizes the mean integrated square error. Define

$$M[k_T] = 2\pi \; E \; I[f_T^*(\omega)] - 2\pi \int_{|v| \geq T} R^2(v)dv \tag{9.1}$$

if $k_T(v)$ are the weights employed in the definition of $f_T^*(\omega)$. In view of (8.6) and (8.11) one may write $M[k_T]$ as a quadratic functional in k_T:

$$(9.2) \qquad M[k_T] = \int k_T^2(v)\ c_2(v)dv - 2\int k_T(v)\ c_1(v)dv + \int c_o(v)dv\ ,$$

where all the integrations are over the interval $|v| \leq T$, and for $|v| \leq T$,

$$(9.3) \qquad c_2(v) = E\ R_T{}'^2(v) + R^2(v)\ (1 - \frac{|v|}{T})^2$$

$$(9.4) \qquad c_1(v) = R^2(v)\ (1 - \frac{|v|}{T})$$

$$(9.5) \qquad c_o(v) = R^2(v).$$

It is then clear that if one defines

$$(9.6) \qquad K_T(v) = \frac{c_1(v)}{c_2(v)} = \frac{(1 - \frac{|v|}{T})R^2(v)}{E\ R'^2(v) + (1 - \frac{|v|}{T})^2\ R^2(v)}$$

that

$$(9.7) \qquad M[k_T] - M[K_T] = \int [k_T(v) - K_T(v)]^2\ c_2(v)dv$$

and

$$(9.8) \qquad M[K_T] = \int \left\{c_o(v) - \frac{c_1^2(v)}{c_2(v)}\right\}dv = \int_{-T}^{T} \frac{E\ R_T{}'^2(v)\ R^2(v)}{ER_T{}'^2(v) + R^2(v)\left(1 - \frac{|v|}{T}\right)^2}\ dv$$

It is clear that the choice of weights $K_T(v)$ which minimize the mean integrated square error is given by (9.6). The corresponding minimum mean integrated square error, denoted by $E\ I_T$, is given by

$$(9.9) \qquad 2\pi\ E\ I_T = M[K_T] + \int_{|v| \geq T} R^2(v)dv.$$

We now prove (5.6) and (5.10).

THEOREM 9A: Let the covariance function $R(v)$ decrease exponentially of coefficient ρ; that is $R(v)$ satisfies (5.1), (5.3), and (5.4). Then (5.6) holds.

PROOF: From (9.9) it follows that

$$L \equiv \lim_{T \to \infty} \frac{T}{\log T} E[I_T | R(v)] = \lim_{T \to \infty} \frac{T}{\log T} \frac{1}{2\pi} M[K_T].$$

Let $n(T) = (\log T)/2\rho$. Define

$$I_o(T) = \int_{|v| \leq n(T)} E\ R_T'^2(v)\ K_T(v)(1 - \frac{|v|}{T})^{-1}\ dv$$

and $I_1(T) = M[K_T] - I_o(T)$. Now

$$|I_1(T)| \leq \int_{|v| \geq n(T)} R^2(v)dv \leq R_o\ e^{-2\rho n(T)} = R_o\ T^{-1}.$$

Consequently,

$$L = \lim_{T \to \infty} \frac{T}{\log T} \frac{1}{2\pi} I_o(T) = \lim_{T \to \infty} \frac{1}{4\pi} \frac{1}{n(T)} \int_{|v| \leq n(T)} a_T(v)dv$$

where we define, for $|v| \leq T$,

$$a_T(v) = T\, E[R_T{'}^2(v)] K_T(v) \left(1 - \frac{|v|}{T}\right)^{-1}. \tag{9.10}$$

Let S be defined by (5.5). Write $a_T(v) - S = a_{T,1}(v) + a_{T,2}(v) + a_{T,3}(v)$, where one defines

$$a_{T,1}(v) = \frac{(T\, E[R_T{'}^2(v)] - S) T\, R^2(v)}{T\, E[R_T{'}^2(v)] + T\, R^2(v)(1 - \frac{|v|}{T})^2}$$

$$a_{T,2}(v) = \frac{-S\left\{1 - \left(1 - \frac{|v|}{T}\right)^2\right\} T\, R^2(v)}{T\, E[R_T{'}^2(v)] + T\, R^2(v)(1 - \frac{|v|}{T})^2}$$

$$a_{T,3}(v) = \frac{T\, E[R_T{'}^2(v)]}{T\, E[R_T{'}^2(v)] + T\, R^2(v)(1 - \frac{|v|}{T})^2} .$$

The desired conclusion will be at hand if one shows that for $j=1,2,3$.

$$\lim_{T \to \infty} \int_{-1}^{1} |a_{T,j}(un(T))| \, du = 0. \tag{9.11}$$

Now from (8.8) it follows that $T\, E[R_T{'}^2(u\, n(T))]$ tends boundedly to S as $T \to \infty$, for each $u \neq 0$. Next, from (5.3) it follows that, for each $u \neq 0$, $\limsup T\, R^2(u\, n(T)) = \infty$. Consequently, $\limsup |a_{T,j}(u\, n(T))| = 0$ for $j=1,2$. Since $a_{T,j}(u)$ is bounded in u and T, (9.11) follows by Fatou's lemma. Next, (9.11) for $j=3$ follows from (5.4). The theorem is proved.

REMARK: To adapt the foregoing proof to the discrete parameter case, one requires the following lemma.

LEMMA 8: Let the constants $a_T(v)$, defined for $T=1,2,\ldots$ and $|v| \leq T$ be uniformly bounded. Let $0 < n(T) \leq T$. Suppose that, for $0 < |u| < 1$,

$$\limsup_{T \to \infty} a_T([u\, n(T)]) = 0.$$

Then

$$\limsup_{T \to \infty} \frac{1}{n(T)} \sum_{-n(T) \leq v < n(T)} a_T(v) = 0.$$

The lemma follows immediately from the observation that if one defines a function $f_T(u)$ by $f_T(u) = a_T([u\, n(T)])$, then the above sum is equal to

$$\int_{-1}^{1} f_T(u) du.$$

REMARK: To adapt the foregoing proof to the case of covariance functions $R(v)$ which decrease exponentially of degree r and coefficient ρ, where $r \geq 1$, define $n(T) = (\log T/2\rho)^{1/r}$, and use the inequality

$$\int_{|v| > M} e^{-2\rho v^r} dv \leq \frac{1}{2r\rho} M^{1-r} e^{-2\rho M^r}$$

THEOREM 9B: Let the covariance function $R(v)$ decrease algebraically of degree $r > 1/2$; that is, $R(v)$ satisfies (5.5). Then (5.10) holds.

PROOF: Let $a_T(v)$ be defined by (9.10), and let $m(T) = T^{1/2r}$. Then one may write

$$(9.12) \quad 2\pi\ T^{a(2r)}\ E[I_T|R(v)] = \int_{u \le T/m(T)} a_T(um(T))du + \int_{|u| \ge T/m(T)} TR^2(um(T))du.$$

Define $g(u) = R_r^2\ u^{-2r}$. From (5.5), it follows that, as $T \to \infty$, $TR^2(um(T)) \to g(u)$. Consequently, from (8.8), $a_T(um(T))$ converges boundedly and dominatedly to

$$S[1 + \frac{S}{g(u)}]^{-1}.$$

That the convergence is dominated follows from the fact that $a_T(um(T)) \le TR^2(um(T)) \le Cu^{-2r}$ for some constant C. The desired conclusion may now be inferred by letting $T \to \infty$ in (9.12).

10. THE ASYMPTOTIC MEAN INTEGRATED SQUARE ERROR OF THE ESTIMATES

In this section we prove Theorems 10A and 10B on the asymptotic mean integrated square error of the families of estimates (5.11) and (5.21). The proofs are written out only for the case of continuous parameter time series, but they are easily adapted to the case of discrete parameter time series.

PROOF OF THEOREM 10A: Let $n(T)$ be defined by (6.3). Let Q_o be defined by (8.7). We evaluate first the asymptotic integrated variance, given by (8.6). We have

$$\frac{T}{\log T} \int_{|v| \ge n(T)} h^2(A_T\ e^{\alpha|v|})E\ R_T'^2(v)dv \lesssim \frac{Q_o}{\log T} \int_{|v| \ge n(T)} h^2(A_T\ e^{\alpha|v|})dv \to 0$$

$$\frac{T}{\log T}\int_{|v|\leq n(T)}(1-h^2(A_T\,e^{\alpha|v|}))E\,R_T{'}^2(v)dv \leq \frac{Q_o}{\log T}\int_{|v|\leq n(T)}(1-h^2(A_T e^{\alpha|v|}))dv \to 0$$

$$\frac{T}{\log T}\int_{|v|\leq n(T)} E\,R_T{'}^2(v)dv \to 2S\lim_{T\to\infty}\frac{n(T)}{\log T} = \frac{2b}{\alpha}\,S.$$

From these relations, it follows that

$$\frac{T}{\log T}\int\sigma^2[f_T*(\omega) - f_T^t(\omega)]d\omega \to 0$$

$$\frac{T}{\log T}\int\sigma^2[f_T*(\omega)]d\omega \to \frac{2b}{\alpha}\,S.$$

Next, we evaluate the asymptotic integrated bias, given by (8.11). We use (8.13). Clearly, for $i=2,3,4$, $(T/\log T)J_i \to 0$. We next show that $(T/\log T)J_1 \to 0$. We have

$$\frac{T}{\log T}\int_{|v|\geq n(T)}[1-k_T(v)]^2\,R^2(v)dv \leq \frac{T}{\log T}\,(1+H_o)^2\,R_o^2\,e^{-2\rho n(T)}$$

$$= (1+H_o)^2\,R_o^2\,\frac{T}{\log T}\,(A_T)^{2\rho/\alpha} \to 0,$$

$$\frac{T}{\log T}\int_{|v|\leq n(T)}[1-k_T(v)]^2\,R^2(v)dv \leq \frac{T}{\log T}\,2H_1^2\,R_o^2\int_0^{n(T)} A_T^2\,e^{2v(\alpha-\rho)}dv$$

$$= \frac{H_1^2\,R_o^2}{\alpha-\rho}\,\frac{T}{\log T}\,(A_T^{2\rho/\alpha} - A_T^2) \to 0, \text{ if } \alpha > \rho,$$

$$\leq 2H_1^2\,R_o^2\,\frac{n(T)}{\log T}\,TA_T^2 \to 0, \text{ if } \alpha \leq \rho.$$

PROOF OF THEOREM 10B: We first evaluate the asymptotic integrated variance. We have

$$T\ B_T \int_{|v| \leq T} h^2(B_T\ v) E\ R_T{}'^2(v) dv = \int_{|u| \leq T B_T} h^2(u) T\, E\, R_T{}'^2\left(\frac{u}{B_T}\right) du \to S\ S(h).$$

Thus

$$T^{1-b} \int \sigma^2[f_T{}^*(\omega)] d\omega \to \frac{1}{2\pi B}\ S\ S(h).$$

We next evaluate the asymptotic integrated bias. We have

$$B_T^{-2p+1} \int_{|v| \leq T} (1-h(B_T\ v))^2\ R^2(v) = \int_{|u| \leq T B_T} \left(\frac{1-h(u)}{u^p}\right)^2 \left(\frac{u}{B_T}\right)^{2p} R^2\left(\frac{u}{B_T}\right) du$$

$$\to R_p^2\ S_p(h).$$

Next, since $v^{1/b}\ R^2(v)$ is bounded, it follows that

$$\int |v|^{1-b}\ R^2(v) < \infty .$$

From this it may be inferred that $T^{1-b}\ J_i \to 0$ for $i=2,3,4$. Consequently

$$T^{1-b} \int b^2[f_T{}^*(\omega)] d\omega \to B^{2p-1}\ R_p^2\ S_p(h).$$

The proof of Theorem 10B is completed.

11. THE ASYMPTOTIC BIAS, COVARIANCE, AND MEAN SQUARE ERROR OF ESTIMATES OF EXPONENTIAL TYPE.

In this section we prove Theorem 11. The proof is again written out only for the continuous parameter case. To prove the theorem, we need to prove (5.31) and (5.32).

PROOF OF (5.31): The desired limit is given by the limit, as $T \to \infty$, of $(T/\log T)$ multiplied by (8.4). In view of (8.2), one may write (8.4) as a sum of three 3-fold integrals. Because of the absolute summability of $Q(v_1,u,u+v_2)$, the term involving that quantity vanishes in the limit, uniformly in ω_1 and ω_2.

Next, we show that the term involving $R(u+v_1)\,R(u-v_1)$ also vanishes in the limit uniformly in ω_1 and ω_2. For this term is less than

$$(11.1) \qquad H_o^2 \frac{1}{\log T} \int_0^T dv_2 \int_0^T dv_1 \int_{-T}^T du\,|R(u+v_1)\,R(u-v_2)|\,.$$

Making the change of variables, for fixed v_2, $v_1 = z_1 - v_2$, and $u = z+v_2$, and using (5.1), one obtains that (11.1) is less than

$$(11.2) \qquad H_o^2\,R_o^2\,\frac{1}{\log T} \int_0^T dv_2 \int_{v_2}^{T+v_2} dz_1 \int_{-T-v_2}^{T-v_2} dz\; e^{-\rho|z|}\,e^{-\rho|z+z_1|}\,.$$

It may be verified that the triple integral in (11.2) is bounded for all T. Consequently, as $T \to \infty$, (11.1) tends to 0.

The value of (5.31) is then given by the limit of

$$\frac{1}{\pi^2} \frac{1}{\log T} \int_0^T \int_0^T dv_1\, dv_2 \cos \omega_1 v_1 \cos \omega_2 v_2\, k_T(v_1)\, k_T(v_2)$$

$$\int_{-T}^{T} du\, U_T(u,v_1,v_2)\, R(u)\, R(u+v_1-v_2). \tag{11.3}$$

To evaluate the limit of (11.3), we show that it is equal, uniformly in ω_1 and ω_2, to the limit of

$$\frac{1}{\pi^2} \frac{1}{\log T} \int_0^{n(T)} \int_0^{n(T)} dv_1\, dv_2 \cos \omega_1 v_1 \cos \omega_2 v_2$$

$$\int_{-T}^{T} du\, U_T(u,v_1,v_2)\, R(u)\, R(u+v_1-v_2) \tag{11.4}$$

where $n(T)$ is defined by (6.3).

From (3.12) and (3.13), one has the inequalities, for constants H_3 and H_4,

$$|1-k_T(v_1)\, k_T(v_2)| \leq H_3\, A_T[e^{\alpha v_1} + e^{\alpha v_2}] \quad \text{for} \quad 0 \leq v_1,\ v_2 \leq n(T)$$

$$|k_T(v_1)\, k_T(v_2)| \leq H_4 \frac{1}{A_T} e^{-\alpha v_1} \qquad \text{if} \quad v_1 > n(T)$$

$$\leq H_4 \frac{1}{A_T} e^{-\alpha v_2} \qquad \text{if} \quad v_2 > n(T).$$

Consequently, the difference between (11.3) and (11.4) is in absolute value less than, letting $W(v_1-v_2) = \int du|R(u)\ R(u+v_1-v_2)|$,

$$\frac{1}{\log T}\left\{H_3 \int_0^{n(T)} dv_1 \int_0^{n(T)} dv_2\ A_T(e^{\alpha v_1} + e^{\alpha v_2})\ W(v_1-v_2)\right.$$

$$+ H_4 \int_{n(T)}^{\infty} dv_1 \int_0^{\infty} dv_2\ \frac{1}{A_T} e^{-\alpha v_1}\ W(v_1-v_2)$$

$$\left. + H_4 \int_0^{\infty} dv_1 \int_{n(T)}^{\infty} dv_2\ \frac{1}{A_T} e^{-\alpha v_2}\ W(v_1-v_2)\right\}$$

which tends to 0 as $T \to \infty$ since the foregoing integrals are all bounded in T.

We next evaluate the limit of (11.4). By the change of variables $u_1 = v_1-v_2$, $u_2 = v_2$, (11.4) may be written

$$\frac{1}{\pi^2}\ \frac{1}{\log T} \int_0^{n(T)} du_2 \int_{-u_2}^{n(T)-u_2} du_1\ \cos\ \omega_1(u_1+u_2)\ \cos\ \omega_2 u_2$$

$$\int_{-T}^{T} du\ U_T(u,u_1+u_2,u_2)\ R(u)\ R(u+u_1)\ . \tag{11.5}$$

By the change of variable $z = u_2/n(T)$, one obtains that (11.5) is equal to

$$\frac{1}{\pi^2}\ \frac{n(T)}{\log T} \int_0^1 dz \int_{-zn(T)}^{n(T)(1-z)} du_1$$

$$\left\{\cos[zn(T)\ (\omega_1-\omega_2) + u_1\omega_1] + \cos[zn(T)(\omega_1+\omega_2) + u_1\omega_1]\right\} \tag{11.6}$$

$$\int_{-T}^{T} du\ U_T(u,u_1+zn(T),\ zn(T))\ R(u)\ R(u+u_1).$$

By referring to (8.3), it may be verified that, as $T \to \infty$, $U_T(u,u_1+zn(T), zn(T)) \to 1$ for fixed u, z, and u_1.

Now, to evaluate (11.6), one may distinguish three cases: case I, $\omega_1 \neq \omega_2$; case II, $\omega_1 = \omega_2 = \omega \neq 0$; case III, $\omega_1 = \omega_2 = 0$. In view of the Riemann-Lebesgue lemma, the first term in (11.6) vanishes in the limit if $\omega_1-\omega_2 \neq 0$, and the second term vanishes in the limit if $\omega_1 + \omega_2 \neq 0$. Further, for any $\epsilon > 0$, the convergence to 0 is uniform in ω_1 and ω_2 such that $\omega_1 \geq \epsilon$ and $\omega_2 \geq \epsilon$. Thus, one obtains that, in the limit as $T \to \infty$, the value of (11.6) is 0 in case I; in case II, it is equal to

$$\frac{1}{2\pi^2}\left(\lim_{T\to\infty} \frac{n(T)}{\log T}\right) \int_{-\infty}^{\infty} du_1\ \cos \omega u_1 \int_{-\infty}^{\infty} du\ R(u)\ R(u+u_1) \tag{11.7}$$

and in case III, it is equal to twice (11.7). One may verify that (5.31) and (11.7) are equal.

PROOF OF (5.32): In view of the upper bound for the supremum of the bias given by (8.12), if we define

$$I_0 = \int_{|v| \leq n(T)} |1-k_T(v)| \, |R(v)| dv$$

$$I_1 = \int_{|v| \geq n(T)} |1-k_T(v)| \, |R(v)| dv$$

the desired conclusion may be inferred from the following inequalities:

$$|I_3| \leq R_o \, e^{-\rho T}, \quad |I_2| \leq R_o \, T^{-1}$$

$$|I_1| \leq (H_o+1) \frac{1}{\rho} R_o \, e^{-\rho n(T)} = \frac{1}{\rho} (H_o+1) \, (A_T)^{\frac{\rho}{\alpha}}$$

$$|I_o| \leq 2H_1 \, R_o \int_0^{n(T)} A_T \, e^{(\alpha-\rho)v} \, dv$$

$$= \frac{1}{\alpha-\rho} H_1 \, R_o \, (A_T^{\rho/\alpha} - A_T) \quad \text{if} \quad \alpha > \rho$$

$$\leq H_1 \, R_o \, A_T n(T) \qquad \text{if} \quad \alpha \leq \rho .$$

12. THE ASYMPTOTIC BIAS, COVARIANCE, AND MEAN SQUARE ERROR OF ESTIMATES OF ALGEBRAIC TYPE

The proofs of Theorems 12A and 12C are essentially given in [12]. In this section we prove Theorem 12B. It suffices to prove (5.36). Let $n(T) = 1/B_T$, and let I_1, I_2, and I_3 be defined by (8.12). Then

$$B_T^{-p+1} \, I_1 = \int_{|u| \leq B_T T} \left| \frac{1-h(u)}{u^p} \right| \left(\frac{u}{B_T}\right)^p R\left(\frac{u}{B_T}\right) du,$$

is uniformly bounded in T. Further $B_T^{-p+1} I_i \to 0$ for $i=1,2$, since the covariance moment

$$\int |v|^{\frac{p-1}{2p-1}} |R(v)| \ dv < \infty$$

from which it follows, letting $s = (p-1)/2p-1$ for the moment, that $T^{s-1} I_2 \to 0$, $T^s I_3 \to 0$. From these facts, one immediately obtains (5.36), and the proof of Theorem 12B is completed.

13. FUNCTIONALLY-UNIFORM CONSISTENCY

In this section conditions are obtained for the estimates (5.11) and (5.21) to be functionally-uniform consistent.

We first note that for an estimate of the form of (3.1),

$$\sup_{\omega} | f_T^*(\omega) - E f_T^*(\omega) | \leq \frac{1}{2\pi} \int_{-T}^{T} |k_T(v)| |R_T'(v)| dv. \tag{13.1}$$

Let Q_o be an upper bound for $T E R_T'^2(v)$. From (13.1) we obtain by Minkowski's inequality

$$E^{1/2} [\sup_{\omega} | f_T^*(\omega) - f_T^*(\omega) |]^2 \leq Q_o \ T^{-1/2} \int_0^T |k_T(v)| dv. \tag{13.2}$$

Now for $k_T(v)$ given by (3.14), and A_T given by (5.15)

$$\int_0^T h(A_T \ e^{\alpha v}) dv = \int_{A_T}^{A_T \ e^{\alpha T}} \frac{h(u)}{\alpha u} du \sim \frac{b}{\alpha} \log T \tag{11.3}$$

while for $k_T(v)$ given by (3.15), and B_T given by (5.27)

$$(13.4) \qquad \int_0^T h(B_T\, v)dv = \frac{1}{B_T} \int_0^{TB_T} h(u)du \sim \frac{1}{B}\, T^b$$

if $h(u)$ is integrable. In view of these facts, and the equations (5.32) and (5.36) on the asymptotic bias, we obtain immediately the following theorems.

THEOREM 13A: Let $\rho > 0$, and suppose that $R(v)$ is dominated by $e^{-\rho|v|}$. Let $h(u)$ satisfy (5.12)-(5.14). Choose $\underline{b} > 1/2$, and $\underline{A} > 0$, and let A_T be given by (5.15). Choose $\alpha > 0$ so that $\alpha \leq 2b\rho$. Then

$$\limsup_{T \to \infty} \frac{T}{(\log T)^2}\, E[\sup_{\omega} |f_T^*(\omega) - f(\omega)|\,]^2 < \infty$$

so that the estimate $f_T^*(\omega)$ is functionally-uniform consistent of order $T/(\log T)^2$.

THEOREM 13B: Let $r > 1$, and let $R(v)$ decrease algebraically of degree r. Choose p so that $1 < p \leq r$, and let $b = 1/2p$. Choose $B > 0$, and let B_T be given by (5.27). Choose $h(u)$ of type $q > p-1$. Then

$$\limsup_{T \to \infty} T^{1-(1/p)}\, E[\sup_{\omega} |f_T^*(\omega) - f(\omega)|\,]^2 < \infty$$

so that the estimate $f_T^*(\omega)$ is functionally-uniform consistent of order $T^{1-(1/p)}$.

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