Edgeworth expansions for spectral density estimators

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Abstract

In this paper we obtain valid Edgeworth expansions for a class of spectral density estimators for a stationary time series. The spectral estimators are based on tapered periodograms of overlapping blocks of observations. We give conditions for the validity of a general order Edgeworth expansion under an approximate strong mixing condition on the random variables, and also establish a moderate deviation inequality. We also verify the conditions explicitly for linear time series, which are satisfied under mild and easy-to-check conditions on the innovation variables and on their nonrandom co-efficients. We also provide two term Edgeworth expansions for the studentized version of the spectral density estimate. Corresponding two terms expansions for quantiles are also obtained.

1 Introduction

Spectral densities play an important role in the frequency domain analysis of time series data. Accurate estimation of the spectral density is therefore a central issue for eliciting

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second order characteristics of a time series from the observed data. This has prompted a large amount of work on consistency and asymptotic normality of spectral estimators. But, much less is known about higher order asymptotic properties of spectral density estimators, which are most important for investigating accuracy of different interval estimation methods for the spectral density. In this paper, we construct valid Edgeworth expansions (EEs) for a class of lag-window spectral density estimators for a stationary time series. The EE results of this paper can be used to study coverage accuracy of confidence intervals (CIs) resulting from the standard Studentization approach and from R.A. Fisher's Variance Stabilizing Transformation (VST) approach, and also to investigate higher order properties of various block bootstrap CIs for the spectral density.

Let $\{X_t : t \in \mathbb{Z}\}$ be a real valued time series with $\mathbf{E}(X_1) = 0$ and spectral density $f(\lambda)$. Let $\mathcal{X}_n = \{X_1, \ldots, X_n\}$ denote the observations from this time series. We construct an estimator of $\widehat{f}_n(\lambda)$ based on blocks of observations from $\mathcal{X}_n$. Let $\mathcal{X}_{i,l} = (X_i, \ldots, X_{i+l-1})$, $i = 1, \ldots, N = (n - l + 1)$, denote a block of length l, where $l \equiv l_n \in [1, n]$ is some integer sequence with $l_n \uparrow \infty$ as $n \to \infty$. Let $\{h_r, r = 1, \ldots, l\}$ be a data-taper. Based on the blocks $\mathcal{X}_{i,l}$ and the taper, the estimator $\widehat{f}_n(\lambda)$ is defined as

$$\widehat{f}_n(\lambda) = \frac{1}{N} \sum_{j=1}^{N} Y_{j,n}, \quad \lambda \in (-\pi, \pi), \tag{1.1}$$

where, with $\iota = \sqrt{-1}$,

$$Y_{j,n} = \frac{\left| \sum_{r=1}^{l} h_r X_{r+j-1} \exp(\iota \lambda r) \right|^2}{2\pi \sum_{r=1}^{l} h_r^2} \tag{1.2}$$

denotes the tapered periodogram for the block $\mathcal{X}_{j,l}$ at frequency λ, $j = 1, \ldots, N$. Thus, the estimator $\widehat{f}_n(\lambda)$ is the average of the tapered periodograms over all blocks contained in $\{X_1, \ldots, X_n\}$. Estimators of this type of are called lag-window spectral density estimators and were previously considered by Bartlett (1946, 1950), Welch (1967), Brillinger (1975) and Zhurbenko (1979, 1980). These lag-window estimators have a close connection with kernel based estimators pioneered by Grenander and Rosenblatt (1957), Blackman and Tukey (1959) and Parzen (1961). It can be shown (see Priestley (1981)) by computing a tapered periodogram of a block of observations with an appropriate choice of taper (lag-window), we can obtain correspondence with kernel estimators.

The main results of the paper give valid EEs versions of the tapered spectral density estimator $\widehat{f}_n(\lambda)$ under a set of regularity conditions on the process $\{X_t\}$. We also derive *simple* sufficient conditions for the validity of the expansions when the time series is a linear process driven by a sequence of independent and identically distributed (iid) random variables. These sufficient conditions only involve the co-efficients of the linear process and the marginal distribution of the innovations and are easy to verify. Unlike the case of the sample mean of independent or weakly dependent random variables, a striking feature of the EEs here is that these are given by a series of terms in powers of $b^{-1/2}$, where $b = o(n)$ (more precisely, $b \sim n/l$ as $n \to \infty$). This is a consequence of the fact that the dependence among neighboring overlapping blocks $\mathcal{X}_{i,l}$'s is very strong, which leads to a behavior that is different from sums of weakly dependent random variables. Indeed, the local strong dependence among the $Y_{j,n}$ results in a slower rate of normal approximation for the spectral density estimator $\widehat{f}_n(\lambda)$. Therefore, in this context, statements about second, third, ... order properties refer to those of the terms of order $b^{-1/2}$, b^{-1}, etc. With this qualification, we also give explicit expressions for the polynomials in the second- and third-order expansions, which are useful for constructing higher-order-accurate large sample tests and CIs for $f(\cdot)$. We then use the derived EEs to construct two-term Edgeworth expansions for the studentized version of $\widehat{f}_n(\lambda)$. As a consequence we also provide corresponding two term Cornish-Fisher expansions for the quantiles of the studentized statistic.

In an important work, Götze and Hipp (1983) derived valid Edgeworth expansions for sample mean of weakly dependent random variables. Extensions and refinements of their results for the sample mean are given by Lahiri (1993) and Lahiri (1996). Janas (1994) derived valid EEs for a class of estimators, known as the spectral mean estimators, for weighted integrals of the spectral density. Unlike the case of the spectral density estimators, the spectral mean estimators are $n^{1/2}$ consistent and are amenable to the EE theory of Götze and Hipp (1983), that gives expansions in powers of $n^{-1/2}$. For the spectral density itself, Velasco and Robinson (2001) recently constructed valid EEs for kernel based nonparametric estimators of the spectral density (and also for studentized sample mean) for a *Gaussian* stationary time series. In this paper, we drop the Gaussianity assumption and derive the EEs under the general set-up introduced in Lahiri (2007) for EEs of estimators based on 'block' variables.

The rest of the paper is organised as follows. Section 2 describes the condition used for deriving the EE for the spectral density and provides simple sufficient conditions for linear processes. Section 3 gives the main results on the EE and moderate deviation bounds for the spectral density estimators. In section 4, the results for EEs of studentized version of $\widehat{f}_n(\lambda)$ are described. Proofs of the results are given in Section 5.

2 Conditions for the Edgeworth expansion

2.1 Theoretical framework and general conditions

To derive the EEs for the spectral density estimator for non-Gaussian processes, we adopt a framework similar to Lahiri (2007) for sums of block variables, which is an extension of Götze and Hipp (1983)'s framework for sums of weakly dependent random variables. Suppose the $\{X_t : t \in \mathbb{Z}\}$ are defined on a probability space $(\Omega, \mathcal{F}, P)$. Also suppose that $\{\mathcal{D}_j : j \in \mathbb{Z}\}$ be a collection of sub σ-fields of $\mathcal{F}$. Let $\mathcal{D}_p^q = \sigma\langle\{\mathcal{D}_j : j \in \mathbb{Z},\ p \le j \le q\}\rangle$, $-\infty \le p < q \le \infty$. Let $Z_{j,n} = Y_{j,n} - \mathbf{E}Y_{j,n}$, $1 \le j \le N$ and let $W_{k,n} = \frac{1}{l}\sum_{j=(k-1)l+1}^{(kl\wedge N)} Z_{j,n}$, $1 \le k \le b_0$, where $b_0 \equiv b_{0n} = \lceil N/l \rceil$, the smallest integer not less than N/l. The $W_{k,n}$ is a function (sample mean) of the block of variables $\{Z_{(k-1)l+1,n}, \ldots, Z_{kl,n}\}$. Let $b \equiv b_n = N/l$. With this, the centered and scaled version of the spectral density estimator $\widehat{f}_n(\lambda)$ can be written as

$$\begin{aligned} T_n(\lambda) &= \sqrt{b}\left(\frac{1}{N}\sum_{j=1}^{N}(Y_{j,n} - \mathbf{E}(Y_{j,n}))\right) \\ &= \frac{1}{\sqrt{b}}\sum_{k=1}^{b_0} W_{k,n}. \end{aligned} \tag{2.1}$$

We will use the following conditions.

(C.1) We assume that there exists a constant $\kappa \in (0,1)$ such that for all $n > \kappa^{-1}$,

$$\kappa \log n < l < \kappa^{-1} n^{1-\kappa} \quad \text{and} \quad \max\{|h_r| : r = 1, \ldots, l\} < \kappa^{-1}. \tag{2.2}$$

(C.2) There exists a constant $\kappa \in (0,1]$ and an integer $s \ge 3$ such that for all $n \ge \kappa^{-1}$,

$$\max\{\mathbf{E}|Y_{j,n}|^{(s+\kappa)} : j = 1, \ldots, N\} < \kappa^{-1}. \tag{2.3}$$

Further, $\lambda \in [0, \pi)$, and $\lim_{n\to\infty} \mathbf{V}(T_n(\lambda)) = \sigma_\infty^2$ exists and is non-zero.

(C.3) We assume that there exists a constant $\kappa \in (0,1)$ such that for all $n, m > \kappa^{-1}$ and for all $j \geq 1$, $1 \leq i \leq n$, there exists a $\mathcal{D}_{j-m}^{j+m}$-measurable $X_{j,m}^\dagger$ such that

$$\mathbf{E}|X_j - X_{j,m}^\dagger|^2 \;\leq\; \kappa^{-1} \exp(-\kappa m). \tag{2.4}$$

(C.4) There exists a constant $\kappa \in (0,1)$, such that for all $n, m = 1, 2, \ldots$, and $A \in \mathcal{D}_{-\infty}^n$ and $B \in \mathcal{D}_{n+m}^\infty$,

$$|\mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B)| \leq \kappa^{-1} \exp(-\kappa m). \tag{2.5}$$

(C.5) There exists a constant $\kappa \in (0,1)$, such that for all $i, j, k, r, m = 1, 2, \ldots$, and $A \in \mathcal{D}_j^i$ with $i < k < r < j$ and $m > \kappa^{-1}$,

$$\mathbf{E}\left|\mathbf{P}(A|\mathcal{D}_j : j \notin [k,r]) - \mathbf{P}(A|\mathcal{D}_j : j \in [i-m, k) \cup (r, j+m])\right| \leq \kappa^{-1} \exp(-\kappa m). \tag{2.6}$$

(C.6) There exist constants $a \in (0, \infty)$, $\kappa \in (0,1)$ and sequences $\{m_n\} \subset \mathbb{N}$ and $\{d_n\} \subset [1, \infty)$ with $m_n^{-1} + m_n b^{-1/2} = o(1)$, $d_n = O(l + b^a)$ and $d_n^2 m_n = O(b^{(1-\kappa)})$ such that for all $n \geq \kappa^{-1}$,

$$\max_{j_0 \in J_n} \sup_{t \in A_n} \mathbf{E}\left|\mathbf{E}\left\{\exp\left(it \sum_{j=j_0-m_n}^{j_0+m_n} W_{j,n}\right) \middle| \tilde{\mathcal{D}}_{j_0}\right\}\right| \leq (1-\kappa), \tag{2.7}$$

where $J_n = \{m_n + 1, \ldots, b - m_n - 1\}$, $A_n = \{t \in \mathbb{R} : \kappa d_n \leq |t| \leq [b^a + l]^{(1+\kappa)}\}$, and $\tilde{\mathcal{D}}_{j_0} = \sigma\langle\{\mathcal{D}_j : j \notin [(j_0 - \lfloor \frac{m_n}{2} \rfloor)l + 1, (j_0 + \lfloor \frac{m_n}{2} \rfloor + 1)l]\}$.

Condition (C.1) states the growth rate of the block size l and allows l to grow at a rate of $O(n^{1-\kappa})$ for arbitrarily small $\kappa > 0$. It also requires the taper-weights to be bounded, which is satisfied in most applications. The first part of Condition (C.2) gives a sufficient condition for the existence of $(s+\kappa)$-order absolute moment of the block variables $W_{k,n}$. Part (ii) of Condition (C.2) ensures that asymptotic variance of $T_n(\lambda)$ exists and is nonzero. By symmetry, this covers all $\lambda \in (-\pi, \pi)$. Note that the problem of existence of the asymptotic variance of $T_n(\lambda)$, when the taper weights h_r's derive from a taper function $h : [0,1] \to \mathbb{R}$ as

$$h_r = h(r/l), \;\; r = 1, \ldots, l. \quad n \geq 1, \tag{2.8}$$

is well-studied and different sufficient conditions for (C.2) are available in the literature. A set of sufficient conditions for this are given by (cf. Dahlhaus (1985)):

$$\left.\begin{array}{l}(i)\ h\ \text{ is continuously differentiable on } [0,1] \text{ with } \int_0^1 h^2(x)dx \in (0,\infty) \\ (ii)\ l = o(n) \quad \text{and} \\ (iii)\ f, f_4 \text{ are bounded and } f \text{ is continuous at } \ \lambda, \end{array}\right\} \tag{2.9}$$

where f_4 denotes the fourth order cumulant density of $\{X_t\}$. Note that in our set up, existence of a bounded f_4 is guaranteed if $E|X_1|^{4+\kappa} < \infty$ for some $\kappa > 0$ and (C.3)-(C.4) hold. Under (2.9), the asymptotic variance of $T_n(\lambda)$ is given by

$$\sigma_\infty^2 = c_2(h) \cdot f^2(\lambda)(1 + \eta(2\lambda)), \tag{2.10}$$

where $c_2(h) = 2\Big[\int_0^1 h(x)^2dx\Big]^{-2} \int_0^1 \big[\int_0^{1-x} h(y)h(x+y)dy\big]^2 dx$ and $\eta(\omega) = 1$ or 0 according as $\omega = 0 \pmod{2\pi}$ or not.

Next consider the regularity conditions (namely (C.3)-(C.6)) that are exclusively used for deriving the EEs for $T_n(\lambda)$ here. Condition (C.3) is an approximation condition that connects the variables X_j to the strong mixing property (C.4) of the auxiliary σ-fields $\mathcal{D}_j$'s. Thus, in our formulation, the variables $\{X_t\}$ are allowed to be only approximately strongly mixing. Condition (C.5) is an approximate Markovian condition and is a variant of a similar condition used in Götze and Hipp (1983). In particular, this condition holds if the σ-fields $\mathcal{D}_j$'s have the Markov property. Finally, consider Condition (C.6), which is a Cramer-type condition on the block variables $W_{j,n}$. In many applications, the natural choice $\mathcal{D}_j = \sigma\langle X_j\rangle$ is *not* the most convenient for verifying (C.6), although this renders verification of (C.3) trivial (e.g., with $X_{j,m}^\dagger = X_j$ for all j, m). A judicious choice of the auxiliary σ-fields $\mathcal{D}_j$'s often facilitates the verification of (C.6). For examples of choices of $\mathcal{D}_j$ in different time series models, see Götze and Hipp (1983) and Lahiri (2003).

2.2 Sufficient conditions for linear processes

To give a specific example, we now consider an important case where $\{X_t\}$ is a linear process and derive simple sufficient conditions for (C.3)-(C.6). Thus, for the rest of this subsection, suppose that $\{X_t : t \in \mathbb{Z}\}$ is a linear process, *i.e.*

$$X_t = \sum_{k \in \mathbb{Z}} \epsilon_{t-k} a_k \tag{2.11}$$

where $\{\epsilon_k\}_{k\in\mathbb{Z}}$ is a collection of iid random variables with $E\epsilon_1 = 0$ and $E\epsilon_1^2 = \sigma^2 \in (0, \infty)$ and where $\{a_k : k \in \mathbb{Z}\}$ are real numbers satisfying

$$|a_k| = O(|c_1|^{|k|}) \quad \text{as} \quad n \to \infty, \tag{2.12}$$

for some constant $0 < c_1 < 1$. For the process $\{X_t\}$ in (2.11), we take $\mathcal{D}_j = \sigma\langle \epsilon_j \rangle$, $j \in \mathbb{Z}$. Then, the σ-fields $\mathcal{D}_j$'s are independent and consequently, the strong-mixing condition (C.4) and the approximate Markovian condition (C.5) on the $\mathcal{D}_j$'s hold trivially. To verify (C.3), we set

$$X_{j,m} = \sum_{k=-m}^{m} \epsilon_{j-k} a_k, \ j \in \mathbb{Z}, m \geq 1.$$

Then, it is evident from (2.11), (2.12) and the definition of $X_{j,m}$ that Condition (C.3) holds. Next consider condition (C.6). It turns out that by choosing the sequences $\{d_n\}$ and $\{m_n\}$ appropriately, the conditional Cramer's condition on the block-variables $W_{k,n}$ can be ensured under some mild conditions on the taper and under the following condition on the joint distribution of $(\epsilon_1, \epsilon_1^2)$:

$$\limsup_{\max\{|t|,|s|\}\to\infty} \left| \mathbf{E} \exp(\iota[s\epsilon_1 + t\epsilon_1^2]) \right| < 1. \tag{2.13}$$

In particular, if the marginal distribution of ϵ_1 has an absolutely continuous component (w.r.t. the Lebesgue measure on the real line), then (2.13) holds. Thus, for the linear process in (2.11), the EE results of this paper remain valid under some simple sufficient conditions on $\{h_r\}$, $\{a_k\}$ and the marginal distribution of ϵ_1. The result is formally stated as Proposition 2.1 below and the proof is given in Section 5.

Proposition 2.1. *Suppose that $\{X_t\}$ is the linear process given by (2.11) and the taper weights $\{h_r : r = 1, \ldots, l\}$ are given by (2.8) for some function h satisfying (i) of (2.9). Also, suppose that for some $\kappa \in (0, 1)$ and an integer $s \geq 3$, $\mathbf{E}|\epsilon_1|^{2(s+1)+\kappa} < \infty$, $\kappa^{-1} \log n < l < \kappa^{-1} n^{(1-\kappa)/3}$, and ϵ_1 satisfies (2.13). If, in addition, $f(\lambda)c_2(h) > 0$, then condition (C.1)-(C.6) hold for every $a \in (0, \infty)$ in (C.6), and $\mathbf{V}(T_n(\lambda))$ in (C.2) is given by $\mathbf{V}(T_n(\lambda)) = c_2(h) \cdot f^2(\lambda)(1 + \eta(2\lambda))$ (cf. (2.10)).*

3 Edgeworth expansion for the spectral density function

3.1 Density of the Edgeworth expansion

We define the Edgeworth polynomials $\tilde{p}_{r,n}(t)$ for $t \in \mathbb{R}$ by the identity (in $u \in \mathbb{R}$)

$$\exp\left(\sum_{r=3}^{s}(r!)^{-1}u^{r-2}b^{(r-2)/2}\chi_{r,n}(t)\right) = 1 + \sum_{r=1}^{\infty} u^r \tilde{p}_{r,n}(t), \tag{3.1}$$

where $\chi_{r,n}(t) = t^r \chi_{r,n}$ and $\chi_{r,n}$ is the r-th cumulant of $T_n(\lambda)$ defined by the equation

$$\iota^r \chi_{r,n} = \frac{d^r}{du^r} \log \mathbf{E} \exp(\iota u T_n)\bigg|_{u=0}. \tag{3.2}$$

The density of the $(s-2)$-th order Edgeworth expansion $\psi_{s,n}(x)$ of $T_n(\lambda)$ is defined through its Fourier transform $\hat{\psi}_{s,n}(t) \equiv \int e^{\iota t x}\psi_{s,n}(x)dx$, $t \in \mathbb{R}$ by

$$\hat{\psi}_{s,n}(t) = \exp(-\chi_{2,n}(t)/2)\left[1 + \sum_{r=1}^{s-2} b^{-r/2}\tilde{p}_{r,n}(\iota t)\right], \quad t \in \mathbb{R}. \tag{3.3}$$

It can be shown (cf. Lemma 4.1, Lahiri (2007)) that under conditions (C.1)-(C.5), for any fixed $t \in \mathbb{R}$, the rth order cumulant $\chi_{r,n}(t)$ is $O(b^{-(r-2)/2})$ for each $2 \le r \le s$ and hence, the co-efficients of the polynomials $\tilde{p}_{r,n}(t)$, $2 \le r \le s$ are $O(1)$ as $n \to \infty$. This shows that the density of the $(s-2)$th order EE for $T_n = T_n(\lambda)$ is given by adding terms order $O(b^{-r/2})$ for $r = 1, \ldots, s-2$ to a normal density function.

It is worth noting that although the choice of the multiplier $b^{(r-2)/2}$ in (3.1) is a natural one, by no means it is unique. Indeed, b in (3.1) can be replaced by another factor b^*, say, such that $b/b_* = 1 + o(1)$ as $n \to \infty$, without altering the *functional form* of the EE. This is because the terms $b^{-r/2}\tilde{p}_{r,n}(t)$ in (3.3) are invariant w.r.t. b, as implied by (3.1). To illustrate this fact and for future reference, we now consider the special case $s = 4$ and write down the EE explicitly. Let $\mu_{r,n} = \mathbf{E}(T_n^r(\lambda))$. Since $\mathbf{E}(T_n(\lambda)) = \mu_{1,n} = 0$, $\mu_{r,n} = \chi_{r,n}$ for all $r = 1, 2, 3$ and $\chi_{4,n} = \left(\mu_{4,n} - \mu_{2,n}^2\right)$. Using the identity (3.1), we find that

$$\begin{aligned} \tilde{p}_{1,n}(t) &= \frac{b^{1/2}\chi_{3,n}}{6}t^3, \quad \text{and} \\ \tilde{p}_{2,n}(t) &= \frac{b\chi_{4,n}}{24}t^4 + \frac{b\chi_{3,n}^2}{72}t^6, \end{aligned} \tag{3.4}$$

which yields

$$\hat{\psi}_{4,n}(t) = \exp\left(\frac{-t^2\mu_{2,n}}{2}\right)\left[1 + \frac{(\iota t)^3}{6}\mu_{3,n} + \frac{(\iota t)^4}{24}(\mu_{4,n} - \mu_{2,n}^2) + \frac{(\iota t)^6}{72}\mu_{3,n}^2\right]. \tag{3.5}$$

Note that the final expression only involves the cumulants of T_n and does not depend on the multiplicative factors $b^{-r/2}$ explicitly. The density $\psi_{s,n}(x)$ can now be found using the inversion formula $\psi_{s,n}(x) = (2\pi)^{-1}\int_{\mathbb{R}} e^{-\iota tx}\hat{\psi}_{s,n}(t)dt$ and the identity:

$$\left[\sigma^{-k}H_k(\sigma^{-1}x)\right]\phi_\sigma(x) = (2\pi)^{-1}\int_{\mathbb{R}} \exp(-\iota tx)(\iota t)^k\hat{\phi}_\sigma(t)dt \tag{3.6}$$

for all $x \in \mathbb{R}$ and $k = 1, 2, \ldots$, where $\phi_\sigma(x) = (2\pi)\sigma^{2^{-1/2}}\exp(-x^2/[2\sigma^2])$, $x \in \mathbb{R}$ and $\hat{\phi}_\sigma(t) = \exp(-t^2\sigma^2/2)$, $t \in \mathbb{R}$ are the probability density function and the characteristic function of the $N(0, \sigma^2)$ distribution, $\sigma \in (0, \infty)$, and where $H_k(x)$ is the kth order Hermite polynomial. Thus,

$$\psi_{4,n}(x) = \phi_{\sigma_n}(x)\left[1 + \frac{\mu_{3,n}}{6\sigma_n^3}H_3(\sigma_n^{-1}x) + \frac{(\mu_{4,n} - \mu_{2,n}^2)}{24\sigma_n^4}H_4(\sigma_n^{-1}x) + \frac{\mu_{3,n}^2}{72\sigma_n^6}H_6(\sigma_n^{-1}x)\right], \tag{3.7}$$

where $\sigma_n^2 = \mu_{2,n} = \mathbf{V}(T_n)$.

3.2 Main Results

We now state the EE results and a moderate deviation result for T_n under the conditions of Section 2.1. To that end, let $s_0 = 2\lfloor s/2 \rfloor$. For a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ and $\epsilon > 0$, let

$$\omega(f : \epsilon) = \int \sup\Big\{ |f(x+y) - f(x)| : |y| \le \epsilon \Big\}\phi_{\sigma_\infty^2}(x)dx,$$

where σ_∞^2 is as in condition (C.2). Let $\mathbf{1}(\cdot)$ denote the indicator function. Then, we have the following EE result for $T_n(\lambda)$:

Theorem 3.1. *Suppose that conditions (C.1)-(C.6) hold for some $a \in ((s-2)/2, \infty)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function with $M_f \equiv \sup\{(1 + |x|^{s_0})^{-1}|f(x)| : x \in \mathbb{R}\} < \infty$. Then, there exist constants $C_1 = C_1(a)$, $C_2 \in (0, \infty)$ (neither depending on f) such that*

$$\left|\mathbf{E}f\Big(T_n(\lambda)\Big) - \int f(x)\psi_{s,n}(x)dx\right| \le C_1\omega(\tilde{f} : b^{-a}) + C_2M_fb^{-(s-2)/2}(\log n)^{-2} \tag{3.8}$$

for all $n > C_2$, where $\tilde{f}(x) = f(x)(1 + |x|^{s_0})^{-1}$, $x \in \mathbb{R}$.

As a corollary to the above theorem, we have the following result for the distribution function of $T_n(\lambda)$:

Corollary 3.2. *Under the conditions of Theorem 3.1,*

$$\sup_{u\in\mathbb{R}} \left|\mathbf{P}\Big(T_n(\lambda) \leq u\Big) - \int_{-\infty}^{u} \psi_{s,n}(x)dx\right| = O\left(b^{-(s-2)/2}(\log n)^{-2}\right). \tag{3.9}$$

In some applications, e.g., for deriving stochastic approximations to the studentized sample mean, where the studentization is ensured using the spectral density estimator $\widehat{f}_n(\lambda)$ (with $\lambda = 0$), moderate deviation inequalities for the estimator $\widehat{f}_n(\lambda)$ are often very useful. Here we state a moderate deviation bound for $\widehat{f}_n(\lambda)$ under the present framework that is valid without the conditional Cramer condition (C.6).

Theorem 3.3. *Suppose that conditions (C.1)-(C.5) hold. Then, for any $\gamma \in (\sigma_\infty^2, \infty)$, there exists a constant $C_3 \in (0,\infty)$ (depending only on γ, s, κ, $E|X_1|^{s+\kappa}$) such that for all $n \geq 2$,*

$$\mathbf{E}\Big(\big[1 + |T_n(\lambda)|^{s_0}\big]\mathbf{1}\left(|T_n(\lambda)| > [(s-2)\gamma\log n]^{1/2}\right)\Big) \leq C_3 b^{-(s-2)/2}(\log n)^{-2}. \tag{3.10}$$

4 Edgeworth expansions for the studentized estimate

It is of separate interest to construct valid EE's for studentized statistics. We define the studentized version of $\widehat{f}_n(\lambda)$ as

$$T_{1,n}(\lambda) = \frac{\sqrt{b}\left(\widehat{f}_n(\lambda) - f(\lambda)\right)}{\widehat{\sigma}_n}, \quad n \geq 1, \tag{4.1}$$

where, $\widehat{\sigma}_n^2$ is a suitably chosen estimate of the asymptotic variance of $T_n(\lambda)$. The studentized version of $\widehat{f}_n(\lambda)$ is useful for constructing valid asymptotic confidence intervals for $f(\lambda)$, specially when the asymptotic variance is unknown. Define

$$Z_n(\lambda) = \frac{\sqrt{b}\left(\widehat{f}_n(\lambda) - \mathbf{E}\widehat{f}_n(\lambda)\right)}{\sigma_n}, \quad n \geq 1,$$

which is the normalized form of $\widehat{f}_n(\lambda)$. Using Corollary 3.2, we can derive a similar two-term EE for the distribution function of $Z_n(\lambda)$. For convenience we state this as a separate corollary,

Corollary 4.1. *Under the conditions of Theorem 3.1 and using the relation* $\lim_{n\to\infty}\sigma_n^2 = \sigma_\infty^2$, *the following expansion holds:*

$$\mathbf{P}\Big(Z_n(\lambda) \le u\Big) = \Phi(u) + b^{-1/2}p_{1,n}(u)\,\phi(u) + b^{-1}p_{2,n}(u)\,\phi(u) + o\left(b^{-1}\right), \quad \text{for all } u \in \mathbb{R}, \tag{4.2}$$

where,

$$p_{1,n}(u) = -\frac{\sqrt{b}\kappa_{3,n}}{6}H_2(u), \quad \text{and,}$$
$$p_{2,n}(u) = -\left(\frac{b\kappa_{4,n}}{24}H_3(u) + \frac{b\chi_{3,n}^2}{72}H_5(u)\right),$$

$\kappa_{r,n} = r^{th}$ *cumulant of* $Z_n(\lambda)$ *and* $H_r(u)$ *is the* r^{th} *Hermite polynomial.*

For any tapering sequence $\{h_r : 1 \le r \le l\}$ of the form (2.8), define

$$\phi_l(x) = \left(\sum_{r=1}^{l} h_r^2\right)^{-1}\sum_{r=1}^{l} h_r \exp\left(-\iota r x\right), \quad \text{for all } x \in \Pi = [-\pi, \pi]. \tag{4.3}$$

The studentizing sequence is then defined as

$$\left.\begin{aligned} \widehat{\sigma}_n^2 = c_n^2\widehat{f}_n(\lambda)^2, \quad \text{with} \quad c_n^2 = \frac{b}{N^2}\int_{|x|\le l^{-1},|y|\le l^{-1}} \tau_n(x,y)\,dxdy, \\ \text{and,} \quad \tau_n(x,y) = |\phi_l(x)|^2|\phi_l(y)|^2\,\frac{\sin\Big(N(x+y)/2\Big)}{\sin\Big((x+y)/2\Big)}. \end{aligned}\right\} \tag{4.4}$$

Now we can state the result on EE's for $T_{1,n}(\lambda)$.

Theorem 4.2. *Define the following sequence of constants:*

$$\left.\begin{aligned} B_{1,n} = b^{1/2}\sigma_n^{-1}\left(\mathbf{E}\widehat{f}_n(\lambda) - f(\lambda)\right), \qquad B_{2,n} = \left(c_n\sigma_n^{-1}\mathbf{E}\widehat{f}_n(\lambda) - 1\right),\ n \ge 1, \\ \text{with,} \quad a_{0,n} = B_{1,n}(1 - B_{2,n}), \qquad a_{1,n} = 1 - B_{2,n} + B_{2,n}^2 - n^{-1}c_nB_{1,n}, \\ a_{2,n} = b^{-1/2}c_n(2B_{2,n} - 1), \qquad a_{3,n} = b^{-1}c_n^2, \quad \text{and} \\ \tilde{a}_{j,n} = a_{j,n}/a_{1,n}, \quad \text{for } j = 2, 3. \end{aligned}\right\} \tag{4.5}$$

Under the conditions of Theorem 3.1, the following two term EE is valid for $T_{1,n}(\lambda)$,

$$\mathbf{P}\Big(T_{1,n}\left(\lambda\right)\leq u\Big)=\Phi\left(u_n\right)+q_{1,n}\left(u_n\right)\phi\left(u_n\right)+q_{2,n}\left(u_n\right)\phi\left(u_n\right)+o\left(b^{-1}\right),\quad \textit{for all } u\in\mathbb{R}, \tag{4.6}$$

where $u_n = a_{1,n}^{-1}\left(u-a_{0,n}\right)$, *with*

$$\left.\begin{aligned} q_{1,n}(u) &= \frac{\kappa_{3,n}}{6}H_2(u)-\tilde{a}_{2,n}u^2, \quad \textit{and} \\ q_{2,n}(u) &= -\Big(b_{2,n}H_1(u)+b_{4,n}H_3(u)+b_{6,n}H_5(u)\Big), \end{aligned}\right\} \tag{4.7}$$

with $H_r(u)$ *being the* r^{th} *Hermite polynomial, and the constants* $b_{j,n}$*'s are defined as*

$$\begin{aligned} b_{2,n} &= 3\tilde{a}_{3,n}+\frac{3}{2}\tilde{a}_{2,n}^2+\tilde{a}_{2,n}\kappa_{3,n}, \\ b_{4,n} &= \frac{\kappa_{4,n}}{24}+\tilde{a}_{3,n}+3\tilde{a}_{2,n}^2+\frac{7}{6}\tilde{a}_{2,n}\kappa_{3,n}, \quad \textit{and}, \\ b_{6,n} &= \frac{1}{72}\kappa_{3,n}^2+\frac{1}{2}\tilde{a}_{2,n}^2+\frac{1}{6}\tilde{a}_{2,n}\kappa_{3,n}, \end{aligned}$$

with $\kappa_{r,n} = r^{th}$ *cumulant of* $Z_n(\lambda)$.

As a corollary to the above result we can derive corresponding expansions for quantiles of $T_{1,n}(\lambda)$. Let $\xi_{\alpha,n} = \alpha$-quantile of $T_{1,n}(\lambda)$ and $\tau_\alpha = \Phi^{-1}(\alpha)$, $\alpha\in(0,1)$.

Corollary 4.3. *The following two term expansion holds*

$$\xi_{\alpha,n}=a_{0,n}+a_{1,n}\Big[\tau_\alpha+h_{1,n}(\tau_\alpha)+h_{2,n}(\tau_\alpha)\Big]+o\left(b^{-1}\right),\quad \textit{for all}\ \ \alpha\in(0,1), \tag{4.8}$$

where the $h_{j,n}$*'s can be expressed in terms of* $q_{j,n}$*'s (cf. (4.7)) as follows,*

$$\begin{aligned} h_{1,n}\left(u\right) &= -q_{1,n}\left(u\right), \quad \textit{and} \\ h_{2,n}\left(u\right) &= q_{1,n}\left(u\right)q_{1,n}'\left(u\right)-\frac{1}{2}uq_{1,n}^2\left(u\right)-q_{2,n}\left(u\right). \end{aligned}$$

5 Proofs

Proof of Proposition 2.1. From the discussion in Section 2, it is evident that with the choice $\mathcal{D}_j = \sigma\langle\epsilon_j\rangle$, $j\in\mathbb{Z}$, conditions (C.1)-(C.5) holds. Hence, we concentrate on verification of

(C.6). The tapered DFT $d_j(\lambda)$ for the set of observations in the j-th block $\mathcal{X}_{j,l}$ is

$$\begin{aligned} d_j(\lambda) &= \sum_{r=1}^{l} h_r X_{r+j-1} \exp(\iota\lambda r) \\ &= \sum_{r=1}^{l} h_r \exp(\iota\lambda r) \sum_{m\in\mathbb{Z}} \epsilon_m a_{r+j-m-1} \\ &= \sum_{m\in\mathbb{Z}} \epsilon_m \left(\sum_{r=1}^{l} h_r \exp(\iota\lambda r) a_{r+j-m-1} \right). \end{aligned} \tag{5.1}$$

For $j, m \in \mathbb{Z}$, let

$$c_{jm} = \frac{\sum_{r=1}^{l} h_r (\cos\lambda r) a_{j-m+r-1}}{[2\pi \sum_{r=1}^{l} h_r^2]^{1/2}} \quad \text{and} \quad s_{jm} = \frac{\sum_{r=1}^{l} h_r (\sin\lambda r) a_{j-m+r-1}}{[2\pi \sum_{r=1}^{l} h_r^2]^{1/2}}.$$

Then, by (5.1), for any $k \in \mathbb{Z}$ and $j \in \{1, \dots, N\}$, we can write

$$\begin{aligned} Y_{jn} &= \left(2\pi \sum_{r=1}^{l} h_r^2\right)^{-1} \left| d_j(\lambda) \right|^2 \\ &= \left| \sum_{m\in\mathbb{Z}} \epsilon_m (c_{jm} + \iota s_{jm}) \right|^2 \\ &= \left\{ \left(\sum_{m\in\mathbb{Z}} \epsilon_m c_{jm} \right)^2 + \left(\sum_{m\in\mathbb{Z}} \epsilon_m s_{jm} \right)^2 \right\} \\ &= \epsilon_k^2 (c_{jk}^2 + s_{jk}^2) + A_{n,j,-k} + 2\epsilon_k B_{n,j,-k}, \end{aligned} \tag{5.2}$$

where,

$$\begin{aligned} A_{n,j,-k} &\equiv \Big(\sum_{m\neq k} \epsilon_m c_{jm} \Big)^2 + \Big(\sum_{m\neq k} \epsilon_m s_{jm} \Big)^2, \quad \text{and} \\ B_{n,j,-k} &\equiv c_{jk} \sum_{m\neq k} \epsilon_m c_{jm} + s_{jk} \sum_{m\neq k} \epsilon_m s_{jm}. \end{aligned}$$

are *independent* of ϵ_k.

Now setting $k = j_0 l$ in (5.2), the sum in (2.7) is

$$
\begin{aligned}
&\sum_{j=j_0-m}^{j_0+m} W_{jn} \\
&= \sum_{j=j_0-m}^{j_0+m} \left(\frac{1}{l} \sum_{i=(j-1)l+1}^{jl} Y_{jn} \right) \\
&= \frac{1}{l} \sum_{j=(j_0-m-1)l+1}^{(j_0+m)l} Y_{jn} \\
&= \frac{1}{l} \sum_{j=(j_0-m-1)l+1}^{(j_0+m)l} \left[\epsilon_k^2 (c_{jk}^2 + s_{jk}^2) + 2\epsilon_k B_{n,j,-k} + A_{n,j,-k} \right] \\
&= \epsilon_k^2 \left[\frac{1}{l} \sum_{j=(j_0-m-1)l+1}^{(j_0+m)l} (c_{jk}^2 + s_{jk}^2) \right] + 2\epsilon_k \left[\frac{1}{l} \sum_{j=(j_0-m-1)l+1}^{(j_0+m)l} B_{n,j,-k} \right] + \left[\frac{1}{l} \sum_{j=(j_0-m-1)l+1}^{(j_0+m)l} A_{n,j,-k} \right] \\
&\equiv e_{n,k}\epsilon_k^2 + 2B_{n,-k}\epsilon_k + A_{n,-k}. \quad \text{(say)}, \qquad (5.3)
\end{aligned}
$$

where $e_{n,k} \equiv e_n$ is a constant (that does not depend on j_0 and hence, on k) and where $A_{n,-k}$ and $B_{n,-k}$ are random variables that are measurable with respect to (w.r.t) the σ-field $\mathcal{D}_{-k} \equiv \vee_{j \neq k} \mathcal{D}_j = \sigma\langle \epsilon_j : j \neq k \rangle$.

Next we consider the asymptotic behavior of $e_{n,k}$. Note that

$$
\begin{aligned}
&e_{n,k} \equiv e_n \\
&\quad = \left(2\pi l \sum_{r=1}^{l} h_r^2\right)^{-1} \sum_{j=-(m+1)l+1}^{ml} \left\{ \left[\sum_{r=1}^{l} h_r \cos(\lambda r) a_{j+r-1} \right]^2 + \left[\sum_{r=1}^{l} h_r \sin(\lambda r) a_{j+r-1} \right]^2 \right\} \\
&\quad = \left(2\pi l \sum_{r=1}^{l} h_r^2\right)^{-1} \sum_{j=-(m+1)l+1}^{ml} \left| \sum_{r=1}^{l} h_r \exp(\iota\lambda r) a_{j+r-1} \right|^2 \\
&\quad = \left(2\pi l \sum_{r=1}^{l} h_r^2\right)^{-1} \sum_{j=-(m+1)l+1}^{ml} \sum_{r=1}^{l}\sum_{s=1}^{l} h_r h_s e^{\iota\lambda(r-s)} a_{j+r-1} a_{j+s-1} \\
&\quad = \left(2\pi l \sum_{r=1}^{l} h_r^2\right)^{-1} \sum_{j=-(m+1)l+1}^{ml} \left[\sum_{p=-(l-1)}^{l-1} e^{\iota\lambda p} \sum_{s=1\vee(1-p)}^{l\wedge(l-p)} h_s h_{p+s} a_{j+s-1} a_{j+p+s-1} \right] \\
&\quad = \left(2\pi l \sum_{r=1}^{l} h_r^2\right)^{-1} \sum_{p=-(l-1)}^{l-1} e^{\iota\lambda p} \sum_{s=1\vee(1-p)}^{l\wedge(l-p)} h_s h_{s+p} \left[\sum_{j\in\mathbb{Z}} a_j a_{j+p} + R_n(s,p) \right]. \qquad (5.4)
\end{aligned}
$$

where, $R_n(s,p)$ is defined by subtraction:

$$\sum_{j=-(m+1)l+1}^{ml} a_{j+s-1}a_{j+p+s-1} = \sum_{j\in\mathbb{Z}} a_j a_{j+p} + R_n(s,p). \tag{5.5}$$

Using the geometric rate of decay of the a_j's for large $|j|$ and Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\sup_{\substack{s=1,\dots,l\\|p|\le(l-1)}} |R_n(s,p)| &\le \sup_{\substack{s=1,\dots,l\\|p|\le(l-1)}} \left[\sum_{j\le-(m+1)l+s} |a_j a_{j+p}| + \sum_{j>ml+s} |a_j a_{j+p}|\right]\\
&\le \sup_{\substack{s=1,\dots,l\\|p|\le(l-1)}} \left[\sum_{j\le -ml} |a_j||a_{j+p}| + \sum_{j\ge ml+1} |a_j||a_{j+p}|\right]\\
&\le 2\sum_{|j|>(m-1)l} a_j^2\\
&= O\Big(c_1^{2(m-1)l}\Big) \quad \text{as } n\to\infty.
\end{aligned} \tag{5.6}$$

Since the bound on $R_n(s,p)$ holds uniformly over all (s,p), by Cauchy-Schwarz inequality we get

$$\begin{aligned}
&\Big(2\pi l\sum_{r=1}^{l} h_r^2\Big)^{-1}\Big|\sum_{p=-(l-1)}^{l-1} e^{\iota\lambda p}\sum_{s=1\vee(1-p)}^{l\wedge(l-p)} h_s h_{s+p}\, R_n(s,p)\Big|\\
&\le \Big(2\pi l\sum_{r=1}^{l} h_r^2\Big)^{-1}\sum_{p=-(l-1)}^{l-1}\Big(\sum_{s=1}^{l} h_s^2\Big)\Big|R_n(s,p)\Big|\\
&\le \frac{2l+1}{2\pi l}\max_{|p|\le l-1}\Big|R_n(s,p)\Big|\\
&= O\Big(c_1^{2(m-1)l}\Big).
\end{aligned}$$

Thus, for $m\ge 2$,

$$e_n = \Big(2\pi l\sum_{r=1}^{l} h_r^2\Big)^{-1}\sum_{p=-(l-1)}^{l-1}\left\{\Big(\sum_{s=1\vee(1-p)}^{l\wedge(l-p)} h_s h_{s+p}\Big)e^{\iota\lambda p}\sum_{j\in\mathbb{Z}} a_j a_{j+p}\right\} + O(c_1^l). \tag{5.7}$$

Next let $\omega(\delta) = \sup\{|h(x)-h(y)| : |x-y|\le\delta,\ x,y\in[0,1]\}$, $\delta>0$. By the uniform continuity of $h(\cdot)$ on $[0,1]$,

$$\lim_{\delta\downarrow 0}\omega(\delta) = 0.$$

Hence, for $h_s = h\left(\frac{s}{l}\right)$, $1 \le s \le l$, by the bounded convergence theorem,

$$
\begin{aligned}
&\sup_{|p|^2 \le 4l} \left| l^{-1} \sum_{s=1\vee(1-p)}^{l\wedge(l-p)} h_s h_{s+p} - \int_0^1 h(x)^2 dx \right| \\
&\le \sup_{|p|^2 \le 4l} \left| l^{-1} \sum_{s=1\vee(1-p)}^{l\wedge(l-p)} h_s h_{s+p} - l^{-1} \sum_{s=1}^{l} h_s^2 \right| + \left| l^{-1} \sum_{s=1}^{l} h_s^2 - \int_0^1 h(x)^2 dx \right| \\
&\le \omega\left(\frac{2\sqrt{l}}{l}\right) \cdot l^{-1} \sum_{s=1}^{l} \left| h\left(\frac{s}{l}\right) \right| + \frac{4\sqrt{l}}{l} \left(\max_{x\in[0,1]} h(x)^2 \right) + \left| l^{-1} \sum_{s=1}^{l} \left(\frac{s}{l}\right)^2 - \int_0^1 h(x)^2 dx \right| \\
&= o(1) \quad \text{as} \quad n \to \infty. \qquad (5.8)
\end{aligned}
$$

Next, for any sequence $\{a_m\}_{m\in\mathbb{Z}}$, define the self-convolution sequence $a*a$ and the Fourier transform $\hat{a}(\cdot)$ of $\{a_m\}_{m\in\mathbb{Z}}$, respectively, as

$$
(a*a)(j) \equiv \sum_{p\in\mathbb{Z}} a_p a_{p+j},\ j \in \mathbb{Z} \quad \text{and} \quad \hat{a}(\lambda) = \sum_{j\in\mathbb{Z}} e^{\iota\lambda j} a_j,\ \lambda \in [-\pi, \pi]. \qquad (5.9)
$$

Then the Fourier transform of $a * a$ at frequency λ is given by

$$
\begin{aligned}
\widehat{a*a}(\lambda) &= \sum_{j\in\mathbb{Z}} e^{\iota\lambda j} (a*a)(j) \\
&= \sum_{j\in\mathbb{Z}} \sum_{p\in\mathbb{Z}} a_p a_{p+j} e^{\iota\lambda(j+p-p)} \\
&= \sum_{p\in\mathbb{Z}} a_p e^{-\iota\lambda p} \sum_{j\in\mathbb{Z}} a_{j+p} e^{\iota\lambda(j+p)} \\
&= |\hat{a}(\lambda)|^2. \qquad (5.10)
\end{aligned}
$$

Thus, it follows that

$$
\begin{aligned}
\sum_{|p|^2\le 4l} e^{\iota\lambda p} \sum_{j\in\mathbb{Z}} a_j a_{j+p} &= \sum_{p\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} a_j a_{j+p} e^{\iota\lambda p} - \sum_{|p|^2 > 4l} \sum_{j\in\mathbb{Z}} a_j a_{j+p} e^{\iota\lambda p} \\
&= \sum_{p\in\mathbb{Z}} (a*a)(p) e^{\iota\lambda p} + O\left(c_1^{\sqrt{l}}\right) \\
&= |\hat{a}(\lambda)|^2 + O\left(c_1^{\sqrt{l}}\right). \qquad (5.11)
\end{aligned}
$$

The last term is of the order $O\left(c_1^{\sqrt{l}}\right)$ because of the following:

$$\begin{aligned}\Big|\sum_{|p|^2>4l}\sum_{j\in\mathbb{Z}} a_j a_{j+p} e^{\iota\lambda p}\Big| &\leq \sum_{|p|^2>4l}\sum_{j\in\mathbb{Z}} |a_j a_{j+p}| \\ &\leq \sum_{|j|\leq\sqrt{l}}\sum_{|p|^2>4l} |a_j a_{j+p}| + \sum_{|j|>\sqrt{l}}\sum_{|p|^2\geq 4l} |a_j a_{j+p}| \\ &\leq 2\sum_{j\in\mathbb{Z}} |a_j| \sum_{|p|^2>l} |a_p| \\ &= O\left(c_1^{\sqrt{l}}\right) \qquad \text{as} \quad l\to\infty, \end{aligned} \tag{5.12}$$

Thus combining equations (5.7)-(5.12), we can write

$$\begin{aligned} &\Big|\sum_{|p|\leq(l-1)} e^{\iota\lambda p}\sum_{j\in\mathbb{Z}} a_j a_{j+p}\Big(\frac{1}{l}\sum_{s=1\vee(1-p)}^{l\wedge(l-p)} h_s h_{s+p}\Big) - |\hat{a}(\lambda)|^2\int_0^1 h(x)^2 dx\Big| \\ \leq &\Big|\sum_{|p|^2\leq 4l} e^{\iota\lambda p}\sum_{j\in\mathbb{Z}} a_j a_{j+p}\Big(\frac{1}{l}\sum_{s=1\vee(1-p)}^{l\wedge(l-p)} h_s h_{s+p}\Big) - |\hat{a}(\lambda)|^2\int_0^1 h(x)^2 dx\Big| \\ &\quad + \sum_{|p|^2>4l}\sum_{j\in\mathbb{Z}} |a_j a_{j+p}|\cdot\left(\frac{1}{l}\sum_{s=1}^{l} h_s^2\right) \\ = &\; o(1) \quad \text{as} \quad n\to\infty \end{aligned}$$

Consequently, it follows that

$$\lim_{n\to\infty} l\cdot e_n = \frac{|\hat{a}(\lambda)|^2}{2\pi} \neq 0 \quad (\text{as} \quad f(\lambda)\neq 0). \tag{5.13}$$

Now set $d_n = l$, $m_n = (\log n)^2, n \geq 2$. It is easy to verify that the requirements of condition (C.6) on these sequences of constants hold, provided $a \geq 1/2$. Now, by (5.13), the stationarity of $\{X_t\}$ and the Cramer's condition on $(\epsilon_1^2, \epsilon_1)$, there exists a $\kappa \in (0,1)$ such that

$$
\begin{aligned}
&\sup_{j_0\in J_n}\sup_{t\in A_n}\mathbf{E}\left|\mathbf{E}\left(\exp\left(\iota t\sum_{j=j_0-m}^{j_0+m}W_{jn}\right)\mid\tilde{\mathcal{D}}_{-j_0l}\right)\right|\\
\leq&\sup_{j_0\in J_n}\sup_{t\in A_n}\mathbf{E}\left|\mathbf{E}\left(\exp\left(\iota t\sum_{j=j_0-m}^{j_0+m}W_{jn}\right)\mid\mathcal{D}_{-j_0l}\right)\right|\\
\leq&\sup_{j_0\in J_n}\sup_{t\geq l}\mathbf{E}\Big|\mathbf{E}\Big(\exp\left(\iota t[e_{n,k}\epsilon_k^2+2B_{n,-k}\epsilon_k+A_{n,-k}]\right)\mid\{\epsilon_j:j\neq k\}\Big)\Big|\\
\leq&\sup_{t\geq l,u\in\mathbb{R}}\Big|\mathbf{E}\exp\Big(\iota[te_n\epsilon_1^2+u\epsilon_1]\Big)\Big|\\
\leq&\sup_{t\geq|\hat{a}(\lambda)|^2,u\in\mathbb{R}}\Big|\mathbf{E}\Big(\exp\Big(\iota[t\epsilon_1^2+u\epsilon_1]\Big)\Big|\\
\leq&1-\kappa,
\end{aligned}
$$

for n large. Hence, condition (C.6) holds for all $a\geq 1/2$. □

Proof of Theorem 3.1. For proving Theorem 3.1, we shall use Theorem 2.1 of Lahiri (2007), which gives conditions for valid EEs for the sum of block variables of the form

$$n^{-1}\sum_{j=1}^{n}\tilde{Y}_{jn}$$

for zero mean variables $\tilde{Y}_{jn}=f_{jn}(\mathcal{X}_{j,l})$, $j=1,\ldots,n$, where f_{jn}'s are Borel measurable functions from $\mathbb{R}^l\to\mathbb{R}$ and $\mathcal{X}_{j,l}=(X_j,\ldots,X_{j+l-1})$, $j\geq 1$. To this end, we set $\tilde{Y}_{jn}=Y_{j,n}-E(Y_{j,n})$ for $j=1,\ldots,N$ and $\tilde{Y}_{jn}=0$ for $j=N+1,\ldots,n$, where recall that $N=n-l+1$ and where $Y_{j,n}$'s are as defined in (1.2). Then, it is easy to see that all the conditions in Theorem 2.1 of Lahiri (2007) are satisfied, provided we show that:

$$\lim_{n\to\infty}\frac{\mathbf{V}(\sum_{j=1}^{n}\tilde{Y}_{jn})}{nl}\quad\text{exists and is nonzero}\tag{5.14}$$

and

$$\mathbf{E}|\tilde{Y}_{jn}-\tilde{Y}_{jn,m}^{\dagger}|\leq\kappa^{-1}l\exp(-\kappa m)\quad\text{for all}\quad m>\kappa^{-1},\tag{5.15}$$

for some $\kappa\in(0,1)$, where $\tilde{Y}_{jn,m}^{\dagger}$ is a random variable that is measurable w.r.t. $\sigma\langle\mathcal{D}_i:j-m\leq i\leq j+m+l\rangle$. Since $l=o(n)$, by (C.2)

$$\frac{\mathbf{V}(\sum_{j=1}^{n}\tilde{Y}_{jn})}{nl}=\frac{N^2\mathbf{V}(T_n(\lambda))}{[n/l]nl}\to\sigma_\infty^2$$

as $n \to \infty$. Thus, the first condition above holds.

As for the second, define $\tilde{Y}^{\dagger}_{jn,m}$ by replacing X_t's in the definition of $Y_{j,n}$ by $X^{\dagger}_{t,m}$'s, $j = 1, \ldots, N$ and let $\tilde{Y}^{\dagger}_{jn,m} = 0$ for $j = N+1, \ldots, n$. Then it follows that the $\tilde{Y}^{\dagger}_{jn,m}$ is measurable w.r.t. $\sigma\langle \mathcal{D}_i : j - m \leq i \leq j + m + l \rangle$ for all $j = 1, \ldots, n$. Further, by condition (C.3) and Cauchy-Schwarz and Jensen's inequalities,

$$
\begin{aligned}
&\sup_{j=1,\ldots,N} \mathbf{E}|\tilde{Y}_{jn} - \tilde{Y}^{\dagger}_{jn,m}| \\
&= \Big(\sum_{r=1}^{l} h_r^2\Big)^{-1} \mathbf{E}\left| \Big|\sum_{r=1}^{l} h_r X_r \exp(\iota\lambda r)\Big|^2 - \Big|\sum_{r=1}^{l} h_r X^{\dagger}_{r,m} \exp(\iota\lambda r)\Big|^2 \right| \\
&\leq \Big(\sum_{r=1}^{l} h_r^2\Big)^{-1} \times \left\{ \mathbf{E}\left(\Big|\sum_{r=1}^{l} h_r X_r \exp(\iota\lambda r)\Big| + \Big|\sum_{r=1}^{l} h_r X^{\dagger}_{r,m} \exp(\iota\lambda r)\Big| \right)^2 \right\}^{1/2} \\
&\qquad \times \left\{ \mathbf{E}\Big|\sum_{r=1}^{l} h_r (X_r - X^{\dagger}_{r,m})\Big|^2 \right\}^{1/2} \\
&\leq Cl \exp(-\kappa m)
\end{aligned}
$$

for m large, for some constant $C \in (0, \infty)$. Hence, the second requirement also holds and the result follows from Theorem 2.1 of Lahiri (2007). □

Proof of Corollary 3.2. Note that

$$\sup_{x \in \mathbb{R}} |\omega(I_{(-\infty,x]} : \delta)| = O(\delta)$$

as $\delta \downarrow 0$. Hence the result for the distribution function is implied by the Theorem 3.1 with $f = I_{(-\infty,x]}$, where I_A denotes the indicator function of a set A. □

Proof of Corollary 3.3. Follows from Theorem 2.4 of Lahiri (2007) and the proof of Theorem 3.1 above. □

Proof of Corollary 4.1. Follows from Theorem 3.1 and Corollary 3.2 above and using the fact $\sigma_n^2 \to \sigma_\infty^2$ as $n \to \infty$. □

Proof of Theorem 4.2. Recall that $Z_n(\lambda) = \sigma_n^{-1} b^{1/2} \left(\widehat{f}_n(\lambda) - \mathbf{E}\widehat{f}_n(\lambda) \right)$ and note that we can write

$$T_{1,n}(\lambda) = g_n(Z_n(\lambda)), \quad \text{where} \quad g_n(x) = (x + B_{1,n}) \left(1 + b^{-1/2} c_n x + B_{2,n} \right)^{-1},$$

where $B_{j,n}$, $j = 1, 2$, are as defined in (4.5). We can further expand $g_n(x)$ uniformly over $\{|x| \le \log n\}$ as follows

$$\begin{aligned} g_n(x) &= (x + B_{1,n}) \left[1 - \left(\frac{x c_n}{\sqrt{b}} + B_{2,n} \right) + \left(\frac{x c_n}{\sqrt{b}} + B_{2,n} \right)^2 + R_{1,n}(x) \right] \\ &= x \left[1 - \left(\frac{x c_n}{\sqrt{b}} + B_{2,n} \right) + \left(\frac{x c_n}{\sqrt{b}} + B_{2,n} \right)^2 \right] + B_{1,n} \left[1 - \left(\frac{x c_n}{\sqrt{b}} + B_{2,n} \right) \right] + R_{2,n}(x) \\ &= a_{0,n} + a_{1,n} x + a_{2,n} x^2 + a_{3,n} x^3 + R_{2,n}(x), \end{aligned}$$

where the $a_{j,n}$'s are as defined in (4.5), with

$$\begin{aligned} &R_{2,n}(x) = (x + B_{1,n}) R_{1,n}(x) + B_{1,n} \left(b^{-1/2} c_n x + B_{2,n} \right)^2, \quad \text{and,} \\ &R_{1,n}(x) = o\left(\left| b^{-1/2} c_n x + B_{2,n} \right|^2 \right). \end{aligned}$$

Going a step further, we can write

$$g_{2,n}(x) = a_{1,n}^{-1} \left(g_{1,n}(x) - a_{0,n} \right) = x + \tilde{a}_{2,n} x^2 + \tilde{a}_{3,n} x^3, \tag{5.16}$$

with $\tilde{a}_{j,n}$, $j = 2, 3$ defined as in (4.5). Hence we can represent $T_{1,n}(\lambda)$ as

$$T_{1,n}(\lambda) = a_{0,n} + a_{1,n} g_{2,n}(Z_n(\lambda)) + R_{2,n}(Z_n(\lambda)). \tag{5.17}$$

Using the polynomials $p_{j,n}$ (cf. (4.2)) define the new polynomials $r_{j,n}(t)$ by using the relation

$$p_{j,n}(u)\phi(u) = \int_{-\infty}^{u} r_{j,n}(t)\phi(t)dt, \quad j = 1, 2,$$

More precisely we will have,

$$\begin{aligned} r_{1,n}(t) &= \frac{\sqrt{b}\kappa_{3,n}}{6} H_3(t) \\ r_{2,n}(t) &= \frac{b\kappa_{4,n}}{24} H_4(t) + \frac{b\kappa_{3,n}^2}{72} H_6(t), \end{aligned}$$

where $H_k(x)$ is the k^{th} Hermite polynomial, $\kappa_{r,n}$ is the r^{th} cumulant of $Z_n(\lambda)$ and $\phi(t)$ is the density of standard normal random variable. Now we can write using Corollary 4.1 and (5.16),

$$\begin{aligned}
&\mathbf{P}\Big(g_{2,n}\left(Z_n(\lambda)\right) \le u\Big) \\
&= \int_{g_{2,n}(t)\le u, |t|\le \log n} \phi(t)\left[1 + b^{-1/2} r_{1,n}(t) + b^{-1} r_{2,n}(t)\right] dt + o\left(b^{-1}\right) \\
&= \int_{-\infty}^{u} \phi\left(g_{2,n}^{-1}(u)\right)\left[1 + b^{-1/2} r_{1,n}\left(g_{2,n}^{-1}(u)\right) + b^{-1} r_{2,n}\left(g_{2,n}^{-1}(u)\right)\right] \frac{dt}{\left|g_{2,n}^{'}\left(g_{2,n}^{-1}(u)\right)\right|} + o\left(b^{-1}\right).
\end{aligned} \tag{5.18}$$

It can be shown that the function $g_{2,n}(x)$ can be inverted inside a small neighborhood range of values of x and the inverse will be

$$g_{2,n}^{-1}(u) = u - \tilde{a}_{2,n}u^2 + \tilde{a}_{3,n}u^3 + 2\tilde{a}_{2,n}^2 u^3, \quad |u| \le \log n.$$

By plugging in the inverse function in the right side of (5.18) we can reduce $\mathbf{P}\big(g_{2,n}(Z_n(\lambda) \le u\big)$ to the following form:

$$\mathbf{P}\Big(g_{2,n}\left(Z_n(\lambda)\right) \le u\Big) = \int_{-\infty}^{u} \phi(t)\left[1 + s_{1,n}(t) + s_{2,n}(t)\right] dt + o\left(b^{-1}\right), \tag{5.19}$$

where, $s_{j,n}(t)$ are polynomials that can be expressed in terms of $r_{j,n}(t)$ as follows:

$$\begin{aligned}
s_{1,n}(t) &= \tilde{a}_{2,n}t^3 + \left(b^{-1/2} r_{1,n}(t) - 2\tilde{a}_{2,n}t\right) \\
s_{2,n}(t) &= \left(\tilde{a}_{3,n} - 2\tilde{a}_{2,n}^2\right) t^4 + \frac{\tilde{a}_{2,n}^2 t^4}{2} H_2(t) + \left[\frac{r_{1,n}(t)}{\sqrt{b}} - 2\tilde{a}_{2,n}t\right] \tilde{a}_{2,n}t^3 - \frac{\tilde{a}_{2,n}t^2}{b} r_{1,n}^{'}(t) \\
&\quad + \left(6\tilde{a}_{2,n}^2 - 3\tilde{a}_{3,n}\right) t^2 - \frac{2\tilde{a}_{2,n}t}{\sqrt{b}} r_{1,n}(t) + \frac{r_{2,n}(t)}{b}.
\end{aligned}$$

Define $q_{j,n}(.)$ by using the relation, $q_{j,n}(u)\phi(u) = \int_{-\infty}^{u} s_{j,n}(t)\phi(t)dt$, $j = 1, 2$. Also note the following recursion relation between consecutive Hermite polynomials:

$$\int H_j(t)\phi(t)dt = -H_{j-1}(t)\phi(t), \quad \text{for all } j \ge 1.$$

Using this we can rewrite (5.19) as

$$\mathbf{P}\Big(g_{2,n}\left(Z_n(\lambda)\right) \le u\Big) = \Phi(u) + q_{1,n}(u)\phi(u) + q_{2,n}(u)\phi(u) + o\left(b^{-1}\right), \tag{5.20}$$

with $q_{j,n}(u)$ defined as in (4.7). Note that, $R_{2,n}(Z_n(\lambda)) \xrightarrow{p} 0$, and

$$\mathbf{P}\Big(g_{1,n}(Z_n(\lambda)) \leq u\Big) = \mathbf{P}\Big(g_{2,n}(Z_n(\lambda)) \leq u_n\Big),$$

where $u_n = a_{1,n}^{-1}(u - a_{0,n})$. Using these along with (5.20) we can conclude (4.6). □

Proof of Corollary 4.3. The result follows by combining the EE result in Theorem 4.2 along with Theorem 2.4 of Hall (1992) (pp. 70) and after some simple algeabric simplifications. □

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