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1. Consider the linear model $\underline{Y} = \mathbf{X}\underline{\beta} + \underline{e}$ where $\underline{e} \sim N(0, \sigma^2\mathbf{I})$ and \mathbf{X} is $n \times p$ of rank p . Let $\hat{\sigma}_q^2 = \underline{Y}'(\mathbf{I} - \mathbf{X}\mathbf{X}^-)\underline{Y}/(n - q)$.

- (a) Find the limit of the mean square error of $\hat{\sigma}_q^2$ as $n \rightarrow \infty$ for fixed q .
 (b) Find the value of q which yields the smallest mean square error for fixed n .

2. Consider the model $\underline{Y} = \mathbf{X}\underline{\beta} + \underline{e}$ and constraint $\mathbf{G}\underline{\beta} = \underline{g}$ where

$$\underline{Y} = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 5 \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \underline{g} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad \& \quad \underline{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}.$$

- (a) Exhibit the specific values of \underline{Y}_R , \mathbf{X}_R and $\underline{\beta}_R$ for the reduced model.
 (b) Estimate $\underline{\beta}$ under $H_0 : \beta_1 = \beta_2$.

3. Consider $\underline{Y} = \mathbf{X}\underline{\beta} + \underline{e}$, where $\underline{e} \sim N(0, \sigma^2\mathbf{V})$ and $\mathbf{V} = \mathbf{I} + \mathbf{J}$. Let $\hat{\underline{\beta}}_{ols} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{Y}$ and $\hat{\underline{\beta}}_{wls} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\underline{Y}$.

- (a) What is the distribution of $Q(\hat{\underline{\beta}}_{ols}) = (\underline{Y} - \mathbf{X}\hat{\underline{\beta}}_{ols})'(\underline{Y} - \mathbf{X}\hat{\underline{\beta}}_{ols})$?
 (b) Are $\hat{\underline{\beta}}_{wls}$ and $Q(\hat{\underline{\beta}}_{ols})$ independent?

4. Consider the linear model

$$\underline{Y} = \mathbf{X}\underline{\beta} + \mathbf{Z}\underline{\gamma} + \underline{e}, \quad \underline{e} \sim N(\underline{0}, \sigma^2\mathbf{I}).$$

where \underline{Y} is $n \times 1$, $\underline{\beta}$ is $p \times 1$, $\underline{\gamma}$ is $q \times 1$, \mathbf{X} is $n \times p$, \mathbf{Z} is $n \times q$, $[\mathbf{X}, \mathbf{Z}]$ is of rank $p + q$ and $n > p + q$.

- (a) Show that $\mathbf{Z}'(\mathbf{I} - \mathbf{X}\mathbf{X}^-)\mathbf{Z}$ is positive definite.
 (b) Show that the maximum likelihood estimators of $\underline{\beta}$ and $\underline{\gamma}$ are:

$$\hat{\underline{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\underline{Y} - \mathbf{Z}\hat{\underline{\gamma}}) \quad \text{and} \quad \hat{\underline{\gamma}} = (\mathbf{Z}'(\mathbf{I} - \mathbf{X}\mathbf{X}^-)\mathbf{Z})^{-1}\mathbf{Z}'(\mathbf{I} - \mathbf{X}\mathbf{X}^-)\underline{Y}.$$

- (c) How would the estimators of $\underline{\beta}$ and $\underline{\gamma}$ change if \mathbf{X} and \mathbf{Z} were orthogonal? Find the joint distribution of the estimators.