

## SHOW YOUR WORK AND START EACH PROBLEM ON A NEW PAGE

1. The Gauss-Markov Theorem states that  $\underline{\ell}'\hat{\underline{\beta}}$  is the best linear unbiased estimator of  $\underline{\ell}'\underline{\beta}$ . Prove the Gauss-Markov Theorem, from first principles, for the model  $\underline{Y} = \underline{X}\underline{\beta} + \underline{e}$ , where  $\underline{X}$  is  $n \times p$  of rank  $q < p$  and  $\underline{e} \sim (\underline{0}, \sigma^2\mathbf{I})$ .
  
2. Consider a two-factor cell means model where each factor (Factor A & Factor B) has three levels, and there is only one observation per cell. Assume the cell means model has the following constraints:  $\mu_{11} = \mu_{22} = \mu_{33}$  and  $\mu_{12} = \mu_{21} = \mu_{13} = \mu_{31}$ . Let  $\underline{Y}' = (1, 3, 9, 8, 6, 4, 7, 4, 2)$ .
  - (a) Determine  $\underline{Y}_R$ ,  $\mathbf{W}_R$  and  $\underline{\mu}_R$ .
  - (b) Estimate  $\underline{\mu}$  and  $\sigma^2$  under the null hypothesis of no main effect for Factor A.
  - (c) How does the answer to part (a) change under the setting where cells (2,1), (1,3) and (2,2) are empty?
  
3. Consider the full rank model  $\underline{Y} = \underline{X}\underline{\beta} + \underline{e}$ , where  $\underline{X}$  is  $n \times p$  of rank  $p$  and  $\underline{e} \sim N(\underline{0}, \sigma^2\mathbf{I})$ . Let  $\hat{\sigma}^2 = SSE/(n - k)$ .
  - (a) Find the mean square error of  $\hat{\sigma}^2$ , for an arbitrary value of  $k$ .
  - (b) Find the value of  $k$  that yields the smallest mean square error estimator of  $\sigma^2$ .

RECALL: The mean square error of an estimator is often simplified to be *Variance* + *Bias*<sup>2</sup>.
  
4. Let  $\underline{Y} \sim N(\underline{1}_3, \sigma^2\mathbf{I}_3 + \delta\mathbf{J}_3)$ . Let  $U_1 = \underline{Y}'\mathbf{A}_1\underline{Y}$  and  $U_2 = \underline{Y}'\mathbf{A}_2\underline{Y}$ , where

$$\mathbf{A}_1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- (a) Are  $U_1$  and  $U_2$  independent?
- (b) Find the distribution of  $U_2/U_1$ . (Include specific values for any parameters of the distribution.)