STAT 612 **MIDTERM EXAM** **OCTOBER 23, 2001**

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1. Let $\mathbf{L}\underline{\beta}$ be a basis set of estimable functions. Show that $\mathbf{L}(\mathbf{X}'\mathbf{X})^c\mathbf{L}'$ is full rank.

2. Let $\underline{Y} \sim \mathrm{N}(\underline{1}_3,\, \sigma^2\mathbf{I_3})$. Let $U_1 = \underline{Y}'\mathbf{A_1}\underline{Y}$ and $U_2 = \underline{Y}'\mathbf{A_2}\underline{Y}$, where

$$\mathbf{A_1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{A_2} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Are U_1 and U_2 independent?

(b) Find the distribution of U_2/U_1. (Include specific values for any parameters of the distribution.)

3. Consider a two-factor cell means model where each factor has only two levels, and there is only one observation per cell. The model has the following constraints: $\mu_{11} = \mu_{22}$ and $\mu_{12} = \mu_{21}$. Let $\underline{Y}' = (8,\, 6,\, 4,\, 7)$.

(a) Display $\underline{Y}_R$, $\mathbf{W}_R$ and $\underline{\mu}_R$.

(b) Estimate $\underline{\mu}$ under the model with the additional constraint that $\mu_{11} - \mu_{12} = 1$.

4. Let $\underline{Y} = \mathbf{X}\underline{\beta} + \underline{e}$, where $\mathbf{X}$ is $n \times p$ of rank $k < p$, $\underline{e} \sim (\underline{0},\, \sigma^2\mathbf{V})$ and $\mathbf{V}$ is known.

(a) Derive the WLS estimator of the estimable function $\underline{\ell}'\underline{\beta}$.

(b) From first principles, prove that the estimator in part (a) is the best linear unbiased estimator of $\underline{\ell}'\underline{\beta}$. (This means that you should not appeal to some already proven version of the Gauss-Markov Theorem in your answer.)

5. Let $\underline{Y} = \mathbf{X}\underline{\beta} + \underline{e}$, where $\mathbf{X}$ is $n \times p$ of rank p and $\underline{e} \sim \mathbf{N}(\underline{0},\, \sigma^2\mathbf{I})$.

(a) Find two matrices $\mathbf{H_1}$ and $\mathbf{H_2}$ such that both $\mathbf{H_1}\underline{\beta} = \underline{0}$ and $\mathbf{H_2}\underline{\beta} = \underline{0}$ can be used to test the null hypothesis $\beta_1 = \beta_2 = 2\beta_3$, and show how they are related.

(b) Prove that the test statistic using $\mathbf{H_1}$ equals the test statistic using $\mathbf{H_2}$.