STAT 612 **FINAL EXAM** **DECEMBER 13, 1996**

1. Let $\underline{Y} = \mathbf{X}\underline{\beta} + \underline{e}$, where $\underline{e} \sim \mathrm{N}(\underline{0},\, \sigma^2\mathbf{I})$, where $\underline{Y}$ is $n \times 1$ and $\mathbf{X}$ is $n \times p$ of rank p. Let S_1^2 be the usual estimator of σ^2. Let S_2^2 be another estimator that can be thought of as $S_1^2 + \underline{Y}'\mathbf{B}\underline{Y}/(n-p)$, where $\mathbf{B}$ is orthogonal to $\mathbf{X}$, has rank $n-p$ and $tr(\mathbf{B}) = 0$.

(a) Is S_2^2 unbiased?

(b) Compare the variance of S_1^2 to the variance of S_2^2 using first principles (do not just state a theorem).

2. Consider the three-factor model where the only interaction is between Factor A and Factor B. There are 2 levels of Factor A, 3 levels of Factor B, 4 levels of Factor C and $n_{ij} = 1$ observation per cell. Assume that Factor A is random, but Factor B and Factor C are fixed.

(a) Display the covariance matrix of $\underline{Y}$.

(b) Using Kronecker product notation, display the quadratic forms associated with the C and A×B sums-of-squares. Find the distributions and show that they are independent.

(c) Assuming $SS_a = 22$, $SS_{ab} = 15$ and $SS_e = 15$, find the methods-of-moments estimators of the three variance components (σ_a^2, σ_{ab}^2 and σ_e^2). How do these estimates compare to the REML estimates?

(d) Suppose someone asked you for the distribution of the methods-of-moments estimator of σ_a^2. How would you suggest approximating this distribution?

3. Consider a 2×3 experiment with $n_{12} = n_{13} = n_{21} = n_{22} = 1$ and $n_{11} = n_{23} = 0$. Let the constraints for the model be described by $G\underline{\mu} = \underline{0}$ and the null hypothesis by $H\underline{\mu} = 0$, where

$$G = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \text{ and } H = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Find the reduced model, under both the null and alternative hypotheses.

(b) Let $\underline{Y}' = (5,\, 7,\, 3,\, 1)$ and find an estimate of σ^2 under the null hypothesis.

4. Let $\underline{Y} \sim \mathrm{N}(\underline{\mu},\, \Sigma)$ where

$$\underline{\mu} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

(a) Find the distribution of $\underline{Y}'\mathbf{A}\underline{Y}$, where $\mathbf{A} = \mathbf{I_3} - \frac{1}{3}\mathbf{J}_3$?

(b) Let $U_1 = (Y_1 + Y_2 + Y_3 - 6)^2$. Find the distribution of U_1 and find a quadratic form, U_2, such that U_1 and U_2 are independent.