STAT 612 **FINAL EXAM** **DECEMBER 14, 2004**

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1. Consider the following quadratic forms, $U_i = \underline{Y}'\mathbf{A}_i\underline{Y}$, $i = 1, 2, 3$, where $\underline{Y} \sim \mathrm{N}(\underline{0},\ \sigma^2\mathbf{I}_4 + \delta\mathbf{J}_4)$. Let $A_1 = \frac{1}{4}\mathbf{J}_4$ and $A_2 = \begin{pmatrix} 2 & 2 & -1 & -3 \\ 2 & 2 & -1 & -3 \\ -1 & -1 & 5 & -3 \\ -3 & -3 & -3 & 9 \end{pmatrix}$.

(a) Are U_1 and U_2 independent?

(b) Is there an $\mathbf{A}_3$ of rank 1 such that U_3 is independent of U_1 and U_2? If yes, find one. If no, explain why not.

(c) Is there an $\mathbf{A}_3$ of rank 2 such that U_3 is independent of U_1 and U_2? If yes, find one. If no, explain why not.

2. Consider the model $Y_{ijk} = \mu + \alpha_i + \tau_j + \alpha\tau_{ij} + c_k + e_{ijk}$, where $i = 1, 2$, $j = 1, 2, 3, 4$, $k = 1, 2, 3$ and all random terms (c and e) have the usual Normal distributions.

(a) Define the quadratic forms for $SS_{\alpha\tau}$ and SS_C and determine if they are independent.

(b) Find the REML estimates of the variance components when $SS_\alpha = 2$, $SS_\tau = 6$, $SS_{\alpha\tau} = 9$, $SS_c = 6$ and $SS_e = 35$.

(c) Show that the mean square error of the MOM estimator of σ^2_C, $\hat{\sigma}^2_C$, is greater than the mean square error of the *adjusted* MOM estimator $\tilde{\sigma}^2_C = max(0, \hat{\sigma}^2_C)$.
HINT: $\mathrm{MSE}(\hat{\sigma}^2_C) = \int_{-\infty}^{\infty}(\hat{\sigma}^2_C - \sigma^2_C)^2 f_{\hat{\sigma}^2_C}(\hat{\sigma}^2_C)d\hat{\sigma}^2_C$.

3. Let $\underline{Y} = \mathbf{X}\underline{\beta} + \underline{e}$, where $\underline{e} \sim (0,\ \mathbf{I}_3 + \mathbf{J}_3)$. (Note: $(\mathbf{I}_3 + \mathbf{J}_3)^{-1} = \mathbf{I}_3 - \frac{1}{4}\mathbf{J}_3$.) For this problem, assume that

$$\underline{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \text{ and that } \mathbf{X} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{pmatrix}.$$

Estimators of β_1 can be written as $\underline{a}'\underline{Y}$. One estimator of β_1 of that type is $\tilde{\beta}_{1A} = (Y_3 - Y_1)/3$. Two other estimators are the $\underline{a}'_{ols}\underline{Y}$ and the $\underline{a}'_{wls}\underline{Y}$ that produce the ols and the wls estimator of β_1.

(a) Are the estimators in part (a) unbiased?

(b) List the estimators in order from smallest to largest variance and explain how you determined the ordering.

4. Let $\underline{Y} = \mathbf{X}\underline{\beta} + \underline{e}$ where $\mathbf{X}$ is $n \times p$ of rank $q < p$ and $\underline{e} \sim \mathrm{N}(\underline{0},\ \sigma^2\mathbf{I})$. Let $\mathbf{H}$ be $q \times p$ of rank q where $\mathbf{H}\underline{\beta}$ is estimable.

(a) Is the covariance matrix of $\widehat{\mathbf{H}\underline{\beta}}$, $\sigma^2\mathbf{H}(\mathbf{X}'\mathbf{X})^c\mathbf{H}'$, positive definite? WHY?

(b) Show that $\mathbf{H}^* = \begin{bmatrix} \mathbf{H} \\ \underline{a}' \end{bmatrix}$ must either be of rank q or that $\underline{a}'\underline{\beta}$ is non-estimable.

(c) What happens to the test statistic to test $\mathbf{H}_*\underline{\beta} = \underline{0}$ if $\mathbf{H}_*\underline{\beta}$ is not estimable? (Note: $\mathbf{H}^* \neq \mathbf{H}_*$.)