For each of these equalities, you must show containment in both directions.

a) \( x \in A \setminus B \iff x \in A \) and \( x \notin B \iff x \in A \) and \( x \notin A \cap B \iff x \in A \setminus (A \cap B) \).

Also, \( x \in A \) and \( x \notin B \iff x \in A \) and \( x \in B^c \iff x \in A \cap B^c \).

b) Suppose \( x \in B \). Then either \( x \in A \) or \( x \in A^c \). If \( x \in A \), then \( x \in B \cap A \), and, hence \( x \in (B \cap A) \cup (B \cap A^c) \). Thus \( B \subseteq (B \cap A) \cup (B \cap A^c) \). Now suppose \( x \in (B \cap A) \cup (B \cap A^c) \). Then either \( x \in (B \cap A) \) or \( x \in (B \cap A^c) \). If \( x \in (B \cap A) \), then \( x \in B \). If \( x \in (B \cap A^c) \), then \( x \in B \). Thus \( (B \cap A) \cup (B \cap A^c) \subseteq B \). Since the containment goes both ways, we have \( B = (B \cap A) \cup (B \cap A^c) \). (Note, a more straightforward argument for this part simply uses the Distributive Law to state that \( (B \cap A) \cup (B \cap A^c) = B \cap (A \cup A^c) = B \cap S = B \).)

c) Same as part a). Reverse the roles of \( A \) and \( B \).

d) \( B \cap A \subseteq A \), so \( A \cup (B \cap A) = A \). Hence, from part b),

\[
A \cup B = A \cup [(B \cap A) \cup (B \cap A^c)] = A \cup (B \cap A) \cup (B \cap A^c) = A \cup (B \cap A^c).
\]

1.9

For \( A_1, \ldots, A_n \)

i) \( \left( \bigcup_{i=1}^{n} A_i \right)^c = \bigcap_{i=1}^{n} A_i^c \)

ii) \( \left( \bigcap_{i=1}^{n} A_i \right)^c = \bigcup_{i=1}^{n} A_i^c \)

Proof of i): \( x \in (\bigcup A_i)^c \iff x \notin \bigcup A_i \iff x \notin A_i \) for any \( i \iff x \in A_i^c \) for every \( i \iff x \in \bigcap A_i^c \).

Proof of ii): Follows from part a) by replacing \( A_i \) with \( A_i^c \) and complementing both sides.

1.11

We must verify each of the three properties in Definition 1.2.1. (1) The empty set \( \emptyset \) is a subset of any set, in particular, \( \emptyset \subseteq S \). Thus \( \emptyset \in \mathcal{B} \). (2) If \( A \in \mathcal{B} \), then \( A \subseteq S \). By the definition of complementation, \( A^c \) is also a subset of \( S \), and hence \( A^c \in \mathcal{B} \). (3) If \( A_1, A_2, \ldots \in \mathcal{B} \), then, for each \( i \), \( A_i \subseteq S \). By the definition of union, \( \bigcup A_i \subseteq S \). Hence, \( \bigcup A_i \in \mathcal{B} \).
Let $B, C$ be Borel fields. (1) The empty set $\emptyset \in B$ and $\emptyset \in C$ by the definition of Borel field. Thus $\emptyset \in B \cap C$.

(2) If $A \in B \cap C$, then $A \in B$ and $A \in C$. By the definition of complementation, $A^c \in B$ and $A^c \in C$ and hence $A^c \in B \cap C$.

(3) If $A_1, A_2 \in B \cap C$, then $A_1, A_2 \in B$ and $A_1, A_2 \in C$. By the def of union, $\bigcup A_i \in B$ and $\bigcup A_i \in C$. Thus, $\bigcup A_i \in B \cap C$.

1.14

If $S = \{s_1, \ldots, s_n\}$, then any subset of $S$ can be constructed by either including or excluding $s_i$ for each $i$. Thus there are $2^n$ possible choices.

1.18

The probability is

$$\frac{n!}{n^n} = \frac{(n-1)(n-1)!}{2n^{n-2}}.$$}

There are many ways to obtain this. Here is one. The denominator is $n^n$ because this is the number of ways to place $n$ balls in $n$ cells. The numerator is the number of ways of placing the balls such that exactly one cell is empty. There are $n$ ways to specify the empty cell. There are $n-1$ ways of choosing the cell with two balls. There are $\binom{n}{2}$ ways of picking the $2$ balls to go into this cell. And there are $(n-2)!$ ways of placing the remaining $n-2$ balls into the $n-2$ cells, one ball in each cell. The product of these is the numerator:

$$n(n-1)\binom{n}{2}(n-2)! = \binom{n}{2}n!.$$}

1.24

a) $P(A \text{ wins}) = \sum_{i=0}^{\infty} P(A \text{ wins on } i^{th} \text{ toss}) = \frac{1}{2} + (\frac{1}{2})^2 \frac{1}{2} + (\frac{1}{2})^4 \frac{1}{2} + \cdots = \sum_{i=0}^{\infty} (\frac{1}{2})^{2i+1} = 2/3.$

b) $P(A \text{ wins}) = p + (1-p)^2p + (1-p)^4p + \cdots = \sum_{i=0}^{\infty} (1-p)^{2i}p = \frac{p}{1-(1-p)^2}.$

c) $\frac{d}{dp} \left( \frac{p}{1-(1-p)^2} \right) > 0$. Thus the probability is increasing in $p$, and the minimum is at zero. Using L'Hôpital's Rule we find $\lim_{p \to 0} \frac{p}{1-(1-p)^2} = 1/2.$
1.38

a) \( P(A) = P(A \cap B) + P(A \cap B^c) \) from Theorem 1.2.3a. But \((A \cap B^c) \subseteq B^c\) and \(P(B^c) = 1 - P(B) = 0\). So \(P(A \cap B^c) = 0\), and \(P(A) = P(A \cap B)\). Thus, \(P(A|B) = P(A \cap B)/P(B) = P(A)/1 = P(A)\).

b) \( A \subseteq B \) implies \( A \cap B = A \). Thus, \(P(B|A) = P(A \cap B)/P(A) = P(A)/P(A) = 1\). And also, \(P(A|B) = P(A \cap B)/P(B) = P(A)/P(B)\).

c) If \( A \) and \( B \) are mutually exclusive, then \( P(A \cup B) = P(A) + P(B) \) and \( A \cap (A \cup B) = A \).

Thus, \(P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A) + P(B)}\).

d) \( P(A \cap B \cap C) = P(A \cap (B \cap C)) = P(A|B \cap C)P(B \cap C) = P(A|B \cap C)P(B|C)P(C)\).

1.39

a) Suppose \( A \) and \( B \) are mutually exclusive. Then \( A \cap B = \emptyset \) and \( P(A \cap B) = 0 \). If \( A \) and \( B \) are independent, then \( 0 = P(A \cap B) = P(A)P(B) \). But this cannot be since \( P(A) > 0 \) and \( P(B) > 0 \). Thus \( A \) and \( B \) cannot be independent.

b) If \( A \) and \( B \) are independent and both have positive probability, then \( 0 < P(A)P(B) = P(A \cap B) \). This implies \( A \cap B \neq \emptyset \), that is, \( A \) and \( B \) are not mutually exclusive.

1.47

Functions a) through d) are continuous, hence right-continuous. Function e) is continuous everywhere except at 0 where it is right-continuous. Thus we need only to check the limit and see that they are nondecreasing.

a) \( \lim_{x \to -\infty} \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(x) = \frac{1}{2} + \frac{1}{\pi} \left( -\frac{\pi}{2} \right) = 0 \),

\( \lim_{x \to \infty} \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(x) = \frac{1}{2} + \frac{1}{\pi} \left( \frac{\pi}{2} \right) = 1 \),

\( \frac{d}{dx} \left( \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(x) \right) = \frac{1}{1 + x^2} > 0 \), so \( F(x) \) is increasing.

b)
c) \( \lim_{x \to \infty} e^{-x} = 0, \lim_{x \to \infty} e^{-\theta x} = 1, \frac{d}{dx} e^{-x} = e^{-x} e^{-\theta x} > 0. \)

d) \( F(x) = 0 \) for \( x < 0 \), so \( \lim_{x \to \infty} (1-e^{-x}) = 0, \lim_{x \to \infty} (1-e^{-\theta x}) = 1, F(x) \) is constant, hence nondecreasing for \( x \leq 0 \) and \( \frac{d}{dx} (1-e^{-x}) = e^{-x} > 0 \), for \( x > 0 \).

e) The limits are correct, and \( \frac{d}{dx} F(x) = (1-\epsilon) \frac{e^{-x}}{(1+e^{-\theta x})^2} > 0 \) for \( x \neq 0 \), so \( F \) is increasing.

If \( F(x) \) is a cdf, \( F(x) = P(X \leq x) \). Hence \( \lim_{x \to \infty} P(X \leq x) = 0 \) and \( \lim_{x \to \infty} P(X \leq x) = 1 \). \( F(x) \) is nondecreasing since the set \( \{ x : X \leq x \} \) is nondecreasing in \( x \). Lastly, as \( x \to x_0 \), \( P(X \leq x) \to P(X \leq x_0) \), so \( F(\cdot) \) is right-continuous. (This is merely a consequence of defining \( F(x) \) with "\( \leq \"."

1.49

For every \( t \), \( F_X(t) \leq F_Y(t) \). Thus we have

\[ P(X > t) = 1 - P(X \leq t) = 1 - F_X(t) \geq 1 - F_Y(t) = 1 - P(Y \leq t) = P(Y > t). \]

And for some \( t \), \( F_X(t) < F_Y(t) \). For this \( t \) we have

\[ P(X > t) = 1 - P(X \leq t) = 1 - F_X(t) > 1 - F_Y(t) = 1 - P(Y \leq t) = P(Y > t). \]

Proof by induction. For \( n = 2 \)

\[ \sum_{k=1}^{2} t^{k-1} = 1 + t = \frac{1-t^2}{1-t}. \]

Assume true for \( n \). For \( n + 1 \)

\[ \sum_{k=1}^{n+1} t^{k-1} = \sum_{k=1}^{n} t^{k-1} + t^n = \frac{1-t^n}{1-t} + t^n \quad \text{(induction hypothesis)} \]

\[ \frac{1-t^n+t^n(1-t)}{1-t} = \frac{1-t^{n+1}}{1-t}. \]

1.53

a) \( y \lim_{y \to \infty} F_Y(y) = y \lim_{y \to \infty} 0 = 0, \) and \( y \lim_{y \to \infty} F_Y(y) = y \lim_{y \to \infty} (1-1/y^2) = 1 - 0 = 1. \) For \( y \leq 1 \), \( F_Y(y) = 0 \) is constant. For \( y > 1 \), \( \frac{d}{dy} F_Y(y) = 2/y^3 > 0 \), so \( F_Y \) is increasing. Thus for all \( y \), \( F_Y \) is nondecreasing. These three facts verify that \( F_Y \) is a cdf.

b) The pdf is \( f_Y(y) = \frac{d}{dy} F_Y(y) = 2/y^3 \), for \( y > 1 \), and \( = 0 \), for \( y \leq 1 \).

c) \( F_Z(z) = P(Z \leq z) = P(10(V-1) \leq z) = P(V \leq (z/10)+1) = F_Y((z/10)+1). \) Thus, \( F_Z(z) = 0 \), for \( z \leq 0 \), and \( F_Z(z) = 1 - (1/[(z/10)+1])^2 \), for \( z > 0 \).
a) \( \int_0^{\pi/2} \sin x \, dx = 1 \). Thus, \( c = 1 \). \\

b) \( \int_{-\infty}^{\infty} e^{-|x|} \, dx = \int_{-\infty}^{0} e^x \, dx + \int_{0}^{\infty} e^{-x} \, dx = 1 + 1 = 2 \). Thus, \( c = 1/2 \).